Math 110

Solutions to Homework 4

1 2.3 # 2

a) Let $A = \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & -3 \\ 4 & 1 & 2 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 1 & 4 \\ -1 & -2 & 0 \end{pmatrix}$ and $D = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$. Then

$$\begin{aligned} A(2B+3C) &= \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix} \left[2\begin{pmatrix} 1 & 0 & -3 \\ 4 & 1 & 2 \end{pmatrix} + 3\begin{pmatrix} 1 & 1 & 4 \\ -1 & -2 & 0 \end{pmatrix} \right] \\ &= \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix} \left[\begin{pmatrix} 2 & 0 & -6 \\ 8 & 2 & 4 \end{pmatrix} + \begin{pmatrix} 3 & 3 & 12 \\ -3 & -6 & 0 \end{pmatrix} \right] \\ &= \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 5 & 3 & 6 \\ 5 & -4 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 20 & -9 & 18 \\ 5 & 10 & 8 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} (AB)D &= \left[\begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -3 \\ 4 & 1 & 2 \end{pmatrix} \right] \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 13 & 3 & 3 \\ -2 & -1 & -8 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 29 \\ -26 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} A(BD) &= \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix} \left[\begin{pmatrix} 1 & 0 & -3 \\ 4 & 1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} \right] \\ &= \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} -7 \\ 12 \end{pmatrix} \\ &= \begin{pmatrix} 29 \\ -26 \end{pmatrix} \end{aligned}$$

b) Let $A = \begin{pmatrix} 2 & 5 \\ -3 & 1 \\ 4 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 3 & -2 & 0 \\ 1 & -1 & 4 \\ 5 & 5 & 3 \end{pmatrix}$, and $C = \begin{pmatrix} 4 & 0 & 3 \end{pmatrix}$.

Then

$$A^t = \begin{pmatrix} 2 & -3 & 4 \\ 5 & 1 & 2 \end{pmatrix}$$

and

$$A^tB = \begin{pmatrix} 2 & -3 & 4 \\ 5 & 1 & 2 \end{pmatrix} \begin{pmatrix} 3 & -2 & 0 \\ 1 & -1 & 4 \\ 5 & 5 & 3 \end{pmatrix}$$
$$= \begin{pmatrix} 23 & 19 & 0 \\ 26 & -1 & 10 \end{pmatrix}$$

and

$$BC^t = \begin{pmatrix} 3 & -2 & 0 \\ 1 & -1 & 4 \\ 5 & 5 & 3 \end{pmatrix} \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}$$
$$= \begin{pmatrix} 12 \\ 16 \\ 29 \end{pmatrix}$$

and

$$CB = \begin{pmatrix} 4 & 0 & 3 \end{pmatrix} \begin{pmatrix} 3 & -2 & 0 \\ 1 & -1 & 4 \\ 5 & 5 & 3 \end{pmatrix}$$
$$= \begin{pmatrix} 27 & 7 & 9 \end{pmatrix}$$

and, finally,

$$CA = \begin{pmatrix} 4 & 0 & 3 \end{pmatrix} \begin{pmatrix} 2 & 5 \\ -3 & 1 \\ 4 & 2 \end{pmatrix}$$
$$= \begin{pmatrix} 20 & 26 \end{pmatrix}$$

2 2.3 # 12

Let V, W, and Z be vector spaces, and let $T : V \to W$ and $U : W \to Z$ be linear transformations.

a) Suppose UT is one-to-one. Notice that $N(T) \subseteq N(UT)$, since if $v \in N(T)$ then $UT(v) = U(0) = 0$. Applying Theorem 2.4 twice, we see first that $N(UT) = (0)$ (implying $N(T) = (0)$), and second that T is one-to-one.

On the other hand, U does not need to be one-to-one. For example if $V = (0)$, $W \neq (0)$ and $U = x \mapsto 0$.

b) Suppose UT is onto, i.e. $R(UT) = Z$. Notice that $R(U) \supseteq R(UT)$ since every vector of the form $UT(v)$ is already of the form Uw, where $w = T(v)$. Then we have $Z \supseteq R(U) \supseteq R(UT) = Z$ so that $R(U) = Z$ and U is onto.

T need not be onto. Let $V, Z = (0)$ and $W \neq 0$.

Here's a less trivial counterexample as requested in **a)** and **b)**. Consider $\mathbf{R} \xrightarrow{T} \mathbf{R}^2 \xrightarrow{U} \mathbf{R}$ by $T(a) = (a, 0)$ and $U(a, b) = a$. The composition UT is just the identity map which is an isomorphism (a linear transformation which is both one-to-one and onto). Still, U is not one-to-one and T is not onto.

3 2.3 # 13

Let A, B be $n \times n$ matrices. As we have seen before we can define a linear map $\mathrm{tr} : M_{n\times n}(\mathbf{F}) \to \mathbf{F}$ by sending a matrix A to $\sum_{i=1}^{n} A_{ii} \in \mathbf{F}$. Recall that the ij-th coordinate of AB is $\sum_{k=1}^{n} A_{ik}B_{kj}$.

Then we have

$$
\begin{aligned}
\mathrm{tr}(AB) &= \sum_{i=1}^{n} (AB)_{ii} \\
&= \sum_{i=1}^{n} \sum_{k=1}^{n} A_{ik}B_{ki} \\
&= \sum_{k=1}^{n} \sum_{i=1}^{n} B_{ki}A_{ik} \\
&= \sum_{k=1}^{n} (BA)_{kk} \\
&= \mathrm{tr}(BA)
\end{aligned}
$$

and since $A^t_{ii} = A_{ii}$ the traces $\mathrm{tr}(A) = \sum A_{ii} = \sum A^t_{ii} = \mathrm{tr}(A^t)$ agree.