Solutions to Homework #3.

Section 2.1.

6. The map T is linear because if $A = (a_{ij})$ and $B = (b_{ij})$, then the trace of $A + B = (a_{ij} + b_{ij})$ is

$$\sum_{i=1}^{n}(a_{ii} + b_{ii}) = \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii} = \mathrm{tr}(A) + \mathrm{tr}(B)$$

and the trace of $cA = (ca_{ij})$, where $c \in F$, is $\sum_{i=1}^{n} ca_{ii} = c\sum_{i=1}^{n} a_{ii}$.

Since the codomain F is one-dimensional, the range can only be either the zero space or all of F. It's not the zero space since there are matrices with nonzero trace. So $R(T)$ is F, with basis any nonzero element of F.

Now we consider the null space $N(T)$. We use the standard basis $\{E^{ij}\}$ for the space of $n \times n$ matrices[1]. Thus any matrix $A = (a_{ij})$ can be written as $A = \sum_{i,j=1}^{n} a_{ij}E^{ij}$. Each of the E^{ij}, where $i \neq j$, has trace zero, so is in $N(T)$. Also, the matrices $E^{11} - E^{ii}$, where $i = 2, \ldots, n$ is in $N(T)$. We claim that this set, namely

$$\beta = \{E^{ij} \,|\, i \neq j\} \cup \{E^{11} - E^{ii} \,|\, i = 2, \ldots, n\}$$

is a basis for $N(T)$. To see why they form an independent set, note that the two subsets are each independent; their union is independent because their spans intersect in the zero space (the first set generates only matrices with zeroes on the diagonal, while the second set generates only matrices with zeroes off the diagonal). To see why β spans[2] $N(T)$, let $A = (a_{ij}) \in N(T)$, so $\sum_{i=1}^{n} a_{ii} = 0$. Then

$$A = \sum_{i \neq j} a_{ij}E^{ij} - a_{22}(E^{11} - E^{22}) - a_{33}(E^{11} - E^{33}) - \ldots - a_{nn}(E^{11} - E^{nn})$$

Note that the terms on the right give the correct coefficient (namely a_{11}) for E^{11} since $a_{11} = -\sum_{i=2}^{n} a_{ii}$. Thus β spans $N(T)$. Since there are $n^2 - 1$ vectors in β, the nulltiy of T is $n^2 - 1$, so the dimension theorem reads: $n^2 = (n^2 - 1) + 1$, which is true!

Finally, T is onto since $R(T)$ is all of F, but T is not one-to-one unless $n = 1$ since for $n > 1$ it has nontrivial nullspace.

11. Since $\{(1, 1), (2, 3)\}$ is a basis for $\mathbb{R}^2$, theorem 2.6 says that there is a uniques linear map $\mathbb{R}^2 \to \mathbb{R}^3$ sending $(1, 1)$ to $(1, 0, 2)$ and $(2, 3)$ to $(1, -1, 4)$. To compute $T(8, 11)$, we write $(8, 11)$ in terms of the above basis, namely $(8, 11) = 2 \cdot (1, 1) + 3 \cdot (2, 3)$, so

$$T(8, 11) = T(2 \cdot (1, 1) + 3 \cdot (2, 3)) = 2 \cdot T(1, 1) + 3 \cdot T(2, 3) = 2 \cdot (1, 0, 2) + 3 \cdot (1, -1, 4) = (5, -3, 16)$$

12. No, there is no such T: if there were, we would have

$$T(-2, 0, -6) = -2 \cdot T(1, 0, 3) = -2 \cdot (1, 1) = (-2, -2),$$

[1] See example 3 of section 1.6 in your text for the definition of E^{ij}.

[2] If we use the dimension theorem, then we don't need to check this: since the range is one-dimensional, and the domain n^2-dimensional, we know $N(T)$ is $(n^2 - 1)$-dimensional; β contains $n^2 - 1$ independent vectors, so it must be a basis for $N(T)$. But since we're supposed to be checking the dimension theorem directly, we can't use this shortcut.

which is impossible since we were given that $T(-2,0,-6) = (2,1)$.

20. To show that $T(V_1)$ is a subspace of W, we let $w_1, w_2 \in T(V_1)$, and $c \in F$. Then $w_1 = T(v_1)$ and $w_2 = T(v_2)$, for some $v_1, v_2 \in V_1$, so $w_1 + w_2 = T(v_1) + T(v_2) = T(v_1 + v_2)$ by linearity of T. But $v_1 + v_2 \in V_1$ since V_1 is a subspace of V, so this shows $w_1 + w_2 \in T(V_1)$. Next, $cw_1 = cT(v_1) = T(cv_1)$, and $cv_1 \in V_1$ since V_1 is a subspcace, so $cw_1 \in T(V_1)$. Finally the zero vector (of W) is in $T(V_1)$ since we can write $0_W = T(0_V)$.

Next let $U = \{x \in V \mid T(x) \in W_1\}$. To show U is a subspace of V, pick any $u_1, u_2 \in U$ and any $c \in F$. To see that $u_1 + u_2 \in U$, we must check that $T(u_1 + u_2) \in W_1$. This is true, because $T(u_1 + u_2) = T(u_1) + T(u_2)$ by linearity, and this is in W_1 since $T(u_1)$ and $T(u_2)$ are both in W_1 and W_1 is closed under addition. Similarly, $T(cu_1) = cT(u_1)$, which is in W_1 because $u_1 \in U$, so $T(u_1) \in W_1$, and W_1 is closed under addition. Finally, the zero vector (of V) is in U because $T(0_V) = 0_W \in W_1$ since W_1 is a subspace.

22. Given any linear map $T\colon \mathbb{R}^3 \to \mathbb{R}$, we have to find real numbers a, b, c such that for any vector (x, y, z) in $\mathbb{R}^3$, we have $T(x, y, z) = ax + by + cz$. To do this, we let $\{e_1, e_2, e_3\}$ be the standard basis for $\mathbb{R}^3$, and set $a = T(e_1)$, $b = T(e_2)$, and $c = T(e_3)$. Then given any vector (x, y, z), we can write it as $(x, y, z) = xe_1 + ye_2 + ze_3$, then compute

$$T(x, y, z) = T(xe_1 + ye_2 + ze_3) = xT(e_1) + yT(e_2) + zT(e_3) = ax + by + cz.$$

More generally, given any linear map $T\colon \mathbb{F}^n \to \mathbb{F}$, we use the standard basis $e_1, \dots, e_n$ for F^n and set $a_i = T(e_i)$ for $i = 1, \dots, n$. A similar computation to the above gives

$$T(x_1, \dots, x_n) = a_1 x_1 + \cdots + a_n x_n.$$

Even *more* generally, suppose now that we are given any linear map $T\colon F^n \to F^m$. Let $\{e_1, \dots, e_n\}$ be the standard basis for the domain F^n, and $\{f_1, \dots, f_m\}$ be the standard basis for the codomain F^m. Now for each j from 1 to n, $F(e_j)$ is some vector in F^m, hence it can be written as a linear combination of the f_is. So there are scalars $a_{1j}, a_{2j}, \dots, a_{mj}$ such that

$$T(e_j) = a_{1j} f_1 + a_{2j} f_2 + \cdots a_{mj} f_m = \sum_{i=1}^{m} a_{ij} f_i$$

Doing this for each $i = 1, \dots, n$, we get mn scalars a_{ij}, where i goes from 1 to m and j goes from 1 to n. Putting these numbers into an $m \times n$ matrix $A = (a_{ij})$, we have, for any vector $x = (x_1, x_2, \dots, x_n)$ in F^n, that $T(x) = Ax$. Here's a proof:

$$T(x) = T\left(\sum_{j=1}^{n} x_j e_j\right) = \sum_{j=1}^{n} x_j T(e_j) = \sum_{j=1}^{n} x_j \left(\sum_{i=1}^{m} a_{ij} f_i\right) = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} x_j a_{ij}\right) f_i$$

If you write this out as a column vector, it's just

$$\begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{pmatrix},$$

and this is exactly what you get when you multiply Ax.

24. (a) $T(a,b) = b$. (b) Any vector (x,y) can be written as $(0, y-x) + (x,x)$, where $(0, y-x)$ is on the y-axis and (x,x) is on the line L. Therefore $T(x,y) = (0, y-x)$.

35. (a) We only need to check that $N(T) \cap R(T)$ is the zero space. For this we use the dimension theorem (which is why we need the finite-dimensionality hypothesis). It says that $\dim V = \dim N(T) + \dim R(T)$. But $V = R(T) + N(T)$, and we proved on the last HW that

$$\dim(N(T) + R(T)) = \dim N(T) + \dim N(T) - \dim(N(T) \cap R(T)),$$

so $\dim(N(T) \cap R(T))$ must be zero, hence the intersectin is the zero subspace.

(b) Here we just need to show that $N(T) + R(T) = V$. By finite dimensionality, the dimension theorem applies and tells us that $\dim V = \dim N(T) + \dim R(T)$. By the formula from HW2 used in (a), and the hypothesis that $N(T) \cap R(T) = \{0\}$, we get $\dim N(T) + \dim R(T) = \dim(N(T) + R(T))$. Thus the subspace $N(T) + R(T)$ has the same dimension as V, so it must be all of V.

Section 2.2

2b. $\begin{pmatrix} 2 & 3 & -1 \\ 1 & 0 & 1 \end{pmatrix}$.

6. The main idea of this proof is to use as many parentheses as possible. The zero element is T_0, defined by $T_0(v) = 0_W$ for all $v \in V$. Now we check the axioms. Throughout, S, T, U will stand for three arbitrary transformations $V \to W$, and $a, b \in F$ will stand for arbitrary scalars. Note that to check an identity holds between transformations, one has to apply the transformations on both sides to an arbitrary vector in V. So throughout, v will stand for an arbitrary vector in V. Let us begin:

(VS1) $(S+T)(v) = S(v) + T(v) = T(v) + S(v) = (T+S)(v)$. The first and third equalities are from the definition of addition of transformations; the second is from commutativity of addition in W.

(VS2) $\big((S+T)+U\big)(v) = (S+T)(v) + U(v) = (S(v) + T(v)) + U(v) = S(v) + ((T(v) + U(v)) = S(v) + (T+U)(v) = \big(S + (T+U)\big)(v)$. The third equality uses associativity of addition in W; all others are from the definition of addition of linear maps.

(VS3) To check that T_0 is the additive identity, we compute $T + T_0$, for arbitrary T: $(T+T_0)(v) = T(v) + T_0(v) = T(v) + 0_W = T(v)$.

(VS4) Given T, its inverse is $-T$, defined by $(-T)(v) = -(T(v))$, where the minus sign on the right means additive inverse in W. Then we check that $(T + (-T))(v) = T(v) + (-T)(v) = T(v) - T(v) = 0_W$, as desired.

(VS5) $(1 \cdot T)(v) = 1 \cdot (T(v))$, where on the right it means scalar multiplication in W, giving just $T(v)$, as desired.

(VS6) $((ab)T)(v) = (ab)(T(v)) = a(bT(v))$, using VS6 for W, and this is the same as $a((bT)(v)) = (a(bT))(v)$.

(VS7) $(a(S+T))(v) = a((S+T)(v)) = a(S(v) + T(v)) = aS(v) + aT(v)$, using VS7 for W. This is then equal to $(aS + aT)(v)$, as desired.

(VS8) $((a+b)T)(v) = (a+b)T(v) = aT(v) + bT(v) = (aT + bT)(v)$, by VS8 for W; this shows that $(a+b)T = aT + bT$.

8. To make sure you understand the question, let's write out clearly what this map T is. Let our basis be $\beta = \{v_1, \dots, v_n\}$. Then given any vector $x \in V$, we can write it as $c_1v_1 + \cdots + c_nv_n$. Then all T does is take these n scalars $c_1, \dots, c_n$ and put them into a column vector, and that's something in F^n. This column vector is denoted $[x]_\beta$ for short. In other words,

$$T(x) = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = [x]_\beta$$

Now we prove this is linear. Let $x, y \in V$. Then we can write them as

$$x = a_1v_1 + \cdots + a_nv_n, \quad y = b_1v_1 + \cdots + b_nv_n$$

so

$$\begin{aligned} T(x+y) &= T((a_1+b_1)v_1 + \cdots + (a_n+b_n)v_n) \\ &= \begin{pmatrix} a_1+b_1 \\ \vdots \\ a_n+b_n \end{pmatrix} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \\ &= [x]_\beta + [y]_\beta = T(x) + T(y). \end{aligned}$$

Now let $c \in F$ be any scalar, and we have $T(cx) = \begin{pmatrix} ca_1 \\ \vdots \\ ca_n \end{pmatrix} = c \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = cT(x)$. Thus T is linear.

12. Note that for the question to make sense, we need to assume that $V = W \oplus W'$ (this is implicit in saying that T is a projection operator). Let $\{w_1, \dots, w_k\}$ be an (ordered) basis for W and $\{v_1, \dots, v_l\}$ be an (oredered) basis for W'. Then $\beta = \{w_1, \dots, w_k, v_1, \dots, v_l\}$ is an ordered basis for V since $V = W \oplus W'$. To compute the matrix $[T]_\beta$, we need to evaluate T on each of these basis vectors, and express the results as linear combinations of the elements of β. We have $T(w_i) = w_i$ for $i = 1, \dots, k$, and $T(v_i) = 0$ for $i = 1, \dots, l$. Thus the matrix of T with respect to the basis β is

$$\begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 \\ & \ddots & 0 & \cdots & \cdots & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ & \vdots & & & \vdots & \end{pmatrix},$$

In other words, it is the block matrix $\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$, where I is the $k \times k$ identity matrix. This is a diagonal matrix.

16. Let n be the common dimension of V and W. First pick any basis at all for $N(T)$, call it $\{v_1, \dots, v_k\}$. Now extend this to a basis $\beta = \{v_1, \dots, v_k, v_{k+1}, \dots, v_n\}$ for V. For $j = k+1, \dots, n$,

set $w_j = T(v_j)$. Then these w_j are independent; here's why. Suppose for contradiction that they're not; then some linear combination $\sum_{j=k+1}^n c_j w_j$ is zero, so we have

$$0 = \sum_{j=k+1}^{n} c_j T(v_j) = T(\sum_{j=k+1}^{n} c_j v_j),$$

so $\sum_{j=k+1}^n c_j v_j \in N(T)$. Therefore $\sum_{j=k+1}^n c_j v_j$ can be written as a linear combination of $v_1, \ldots, v_k$. This violates the independence of $\{v_1, \ldots, v_n\}$. Therefore since $\{w_{k+1}, \ldots, w_n\}$ is an independent set, it can be extended to a basis $\gamma = \{w_1, \ldots, w_k, w_{k+1}, \ldots, w_n\}$ for W. By a similar analysis to that of the previous problem, we find that the matrix $[T]_\beta^\gamma$ is the block-diagonal matrix $\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$, where here I stands for the $(n-k) \times (n-k)$ identity matrix.