Solutions to Homework #12.

Section 6.3

2. (b) Take $y = (1, -2)$. Then for $z = (z_1, z_2)$, we have $\langle z, y\rangle = z_1 - 2z_2 = g(z_1, z_2)$.

7. Take $V = \mathbb{R}^2$ with the standard inner product and

$$T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Then

$$T^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

So $N(T) = span\{(1, 0)\}$ is not equal to $N(T^*) = span\{(0, 1)\}$.

8. Suppose T is invertible and let $(T^{-1})^*$ denote the adjoint of the inverse.

Then we calculate $(T^{-1})^*T^* = (TT^{-1})^* = (Id)^* = Id$. Thus $(T^{-1})^*$ is the inverse of T^* or in other words $(T^{-1})^* = (T^*)^{-1}$.

12. (a) For any $x, y \in V$, we have $\langle Tx, y\rangle = \langle x, T^*y\rangle$.

If $x \in N(T)$, then for any $y \in V$, we have $0 = \langle 0, y\rangle = \langle x, T^*y\rangle$ and so $x \in R(T^*)^\perp$. Thus $N(T) \subset R(T^*)^\perp$.

If $x \in R(T^*)^\perp$, then for any $y \in V$, we have $0 = \langle x, T^*y\rangle = \langle Tx, y\rangle$ and so $Tx = 0$ and hence $x \in N(T)$. Thus $R(T^*)^\perp \subset N(T)$.

(b) Since V is finite-dimensional, all of its subspaces $W \subset V$ are finite-dimensional, hence we may apply Exercise 13 (c) of Section 6.2 to see $(W^\perp)^\perp = W$.

Applying this to the identity of part (a), we obtain $N(T)^\perp = (R(T^*)^\perp)^\perp = R(T^*)$.

Section 6.4

2. (a) Writing T as a matrix, we have

$$T = \begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix}$$

Thus T is real and symmetric so self-adjoint and thus normal with an orthonormal basis of eigenvectors.

Its characteristic polynomial is $chi_T(t) = (2-t)(5-t) - 4 = t^2 - 7t + 6 = (t-1)(t-6)$ and so its eigenvalues are $1, 6$.

The corresponding orthonormal eigenvectors are $\frac{1}{\sqrt{5}}(2, 1)$ and $\frac{1}{\sqrt{5}}(1, -2)$.

3. Let us take

$$T = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

Then T is symmetric so self-adjoint hence normal.

Let us choose the basis

$$\beta = \{(1, 0), (1, 1)\}$$

Then with respect to β, the matrix of T takes the form

$$[T]_\beta = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$$

Now this is a real matrix so its adjoint is its transpose

$$[T]_\beta^* = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$$

And we can check that it is not normal

$$[T]_\beta [T]_\beta^* = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -2 & 4 \end{pmatrix}$$

$$[T]_\beta^* [T]_\beta = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 5 \end{pmatrix}$$

6. (a)

$$T_1^* = \frac{1}{2}(T^* + (T^*)^*) = \frac{1}{2}(T + T^*) = T_1$$

$$T_2^* = \frac{1}{-2i}(T^* - (T^*)^*) = \frac{1}{2i}(T - T^*) = T_2$$

$$T_1 + iT_2 = \frac{1}{2}(T + T^*) + i\frac{1}{2i}(T - T^*) = T$$

(b) If $T = U_1 + iU_2$ with $U_1 = U_1^*, U_2 = U_2^*$, then

$$T_1 = \frac{1}{2}(T + T^*) = \frac{1}{2}((U_1 + iU_2) + (U_1^* - iU_2^*)) = \frac{1}{2}((U_1 + iU_2) + (U_1 - iU_2)) = U_1$$

$$T_2 = \frac{1}{2i}(T - T^*) = \frac{1}{2i}((U_1 + iU_2) - (U_1^* - iU_2^*)) = \frac{1}{2i}((U_1 + iU_2) - (U_1 - iU_2)) = U_2$$

(c) T is normal
$\iff TT^* = T^*T$
$\iff (T_1 + iT_2)(T_1 + iT_2)^* = (T_1 + iT_2)^*(T_1 + iT_2)$ (substituting from part (a))
$\iff (T_1 + iT_2)(T_1 - iT_2) = (T_1 - iT_2)(T_1 + iT_2)$ (using T_1, T_2 self-adjoint as proved in part (a))
$\iff T_1T_1 + iT_2T_1 - iT_1T_2 + T_2T_2 = T_1T_1 - iT_2T_1 + iT_1T_2 + T_2T_2$ (multiplying out)
$\iff T_1T_2 = T_2T_1$ (simplifying algebraically)

9. T normal means $TT^* = T^*T$. If $x \in N(T)$ then $TT^*x = T^*Tx = 0$ so $T^*x \in N(T)$.
But by Exercise 12 of Section 6.3, we know $T^*x \in R(T^*) = N(T)^\perp$.
Thus $T^*x \in N(T) \cap N(T)^\perp = \{0\}$ so $T^*x = 0$ and hence $x \in N(T^*)$.
Thus we have $N(T) \subset N(T^*)$.
Applying the same argument to T^* (and using the fact $(T^*)^* = T$) we see $N(T^*) \subset N(T)$.
Thus we have proved $N(T) = N(T^*)$.
Now taking orthogonal complements and using Exercise 12 of Section 6.3 again, we have $R(T^*) = N(T)^\perp = N(T^*)^\perp = R(T)$.