Solutions to Homework #10.

Section 5.2.

3 (f). Let $\beta = \{A_1, A_2, A_3, A_4\}$ be given by

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Note that $T(A_1) = A_1$, $T(A_2) = A_2$, $T(A_3) = -A_3$, and $T(A_4) = A_4$. Thus every element of β is an eigenvector for T, so by Theorem 5.4 we will have shown that T is diagonalizable once we show that β is a basis for $M_{2\times 2}(\mathbb{R})$. Since $\dim M_{2\times 2}(\mathbb{R}) = 4$, it suffices to show that β is a spanning set. Observe that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = aA_1 + \frac{1}{2}(b+c)A_2 + \frac{1}{2}(b-c)A_3 + dA_4.$$

Thus β spans $M_{2\times 2}(\mathbb{R})$, and T is diagonalizable.

Alternatively, one could let γ be the standard basis for $M_{2\times 2}(\mathbb{R})$ and obtain

$$[T]_\gamma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

One can now find that the characteristic polynomial of T is $(t-1)^3(t+1)$, and show that $\dim E_1 = 3$ by solving $([T]_\gamma - I)x = 0$.

7. The characteristic polynomial of A is

$$\det \begin{pmatrix} 1-t & 4 \\ 2 & 3-t \end{pmatrix} = t^2 - 4t - 5 = (t-5)(t+1).$$

The eigenvalues of A are -1 and 5. Solving $(A+I)x = 0$ gives

$$x = c(-2, 1), \quad c \in \mathbb{R}.$$

Similarly, solving $(A - 5I)x = 0$ gives

$$x = c(1, 1), \quad c \in \mathbb{R}.$$

Thus

$$\begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix},$$

and

$$A = \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix}^{-1}.$$

We now have

$$\begin{aligned} A^n &= \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} (-1)^n & 0 \\ 0 & 5^n \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix}^{-1} && (1) \\ &= -\frac{1}{3} \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} (-1)^n & 0 \\ 0 & 5^n \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & -2 \end{pmatrix} && (2) \\ &= \frac{1}{3} \begin{pmatrix} 2(-1)^n + 5^n & -2(-1)^n + 2 \cdot 5^n \\ -(-1)^n + 5^n & (-1)^n + 2 \cdot 5^n \end{pmatrix}. && (3) \end{aligned}$$

12. (a) Let $E_{T,\lambda}$ be the eigenspace of T corresponding to λ, and let $E_{T^{-1},\lambda^{-1}}$ be the eigenspace of T^{-1} corresponding to λ^{-1}. We show that these subspaces are equal by showing that they are contained in each other.

Suppose $x \in E_{T,\lambda}$. Then $T(x) = \lambda x$, and applying T^{-1} to both sides gives $x = \lambda T^{-1}(x)$. Since T is invertible, 0 is not an eigenvalue of T and consequently $\lambda \neq 0$. Thus dividing both sides by λ gives $T^{-1}(x) = \lambda^{-1}x$, and so $x \in E_{T^{-1},\lambda^{-1}}$.

Conversely, suppose $x \in E_{T^{-1},\lambda^{-1}}$. Then $T^{-1}(x) = \lambda^{-1}x$, and applying T to both sides gives $x = \lambda^{-1}T(x)$. Multiplying both sides by λ gives $T(x) = \lambda x$, and thus $x \in E_{T,\lambda}$ and $E_{T,\lambda} = E_{T^{-1},\lambda^{-1}}$.

(b) Suppose T is diagonalizable. Then there is a basis β for V such that

$$[T]_\beta = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}.$$

Since T is invertible, $\lambda_j \neq 0$. Then we have

$$[T^{-1}]_\beta = ([T]_\beta)^{-1} = \begin{pmatrix} \lambda_1^{-1} & & 0 \\ & \ddots & \\ 0 & & \lambda_n^{-1} \end{pmatrix}.$$

Since this matrix is diagonal, T^{-1} is diagonalizable.

13. (a) Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}.$$

Then the eigenvalues of A are 1 and 2 and the eigenspaces are 1-dimensional. One can check that the first standard basis vector is a basis for the eigenspace of A corresponding to 1. However, the first standard basis vector is not an eigenvector for

$$A^t = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}.$$

(b) Recall that $E_\lambda = \dim \mathrm{Null}(A - \lambda I)$ and similarly $E'_\lambda = \dim \mathrm{Null}(A^t - \lambda I)$. By the dimension theorem,

$$\dim \mathrm{Null}(A - \lambda I) = n - \mathrm{rank}(A - \lambda I), \qquad \dim \mathrm{Null}(A^t - \lambda I) = n - \mathrm{rank}(A^t - \lambda I).$$

Since $(A-\lambda I)^t = A^t - \lambda I$, and $\operatorname{rank} B = \operatorname{rank} B^t$ for all matrices B, we have

$$\operatorname{rank}(A-\lambda I) = \operatorname{rank}(A^t - \lambda I)$$

and thus

$$\dim \operatorname{Null}(A-\lambda I) = \dim \operatorname{Null}(A^t - \lambda I).$$

(c) Recall from the problem description that A and A^t have the same eigenvalues with the same multiplicities. Let m_λ be the multiplicity of λ as an eigenvalue of A (and as an eigenvalue of A^t). Suppose that A is diagonalizable. Then by 5.9, the characterististic polynomial of A (and thus of A^t) splits. If λ_j are the eigenvalues of A, then Theorem 5.9 says $\dim(E_\lambda) = m_\lambda$. But $\dim E'_\lambda = \dim E_\lambda = m_\lambda$, so by Theorem 5.9, A^t is diagonalizable.

18. (a) Note that if D_1 and D_2 are diagonal matrices, then $D_1 D_2 = D_2 D_1$. Using that fact and Theorem 2.11, we get

$$[TU]_\beta = [T]_\beta [U]_\beta = [U]_\beta [T]_\beta = [UT]_\beta.$$

By Theorem 2.20, we can conclude from $[TU]_\beta = [UT]_\beta$ that $TU = UT$.

(b) Let Q be a matrix such that $Q^{-1}AQ$ and $Q^{-1}BQ$ are diagonal. As noted above, this means that these matrices commute. Then

$$Q^{-1}ABQ = (Q^{-1}AQ)(Q^{-1}BQ) = (Q^{-1}BQ)(Q^{-1}AQ) = Q^{-1}BAQ.$$

Multiplying the above by Q on the left and Q^{-1} on the right gives $AB = BA$.

Section 5.4.

3. (a) Since $T(0) \in \{0\}$, the zero subspace is invariant. Since $T(V) \subseteq V$ by definition, V is also invariant.

(b) If $x \in N(T)$, then $T(x) = 0 \in N(T)$, so $T(N(T)) \subseteq N(T)$. For all $x \in V$, $T(x) \in R(T)$ by definition. Thus this holds if $x \in R(T)$, and so $T(R(T)) \subseteq R(T)$.

(c) If $x \in E_\lambda$, then $T(x) = \lambda x \in E_\lambda$ since E_λ is a subspace. Thus $T(E_\lambda) \subseteq E_\lambda$.

6. (a) We have $T(1,0,0,0) = (1,0,1,1)$, and then $T(1,0,1,1) = (1,-1,2,2)$. Applying T again we get $T(1,-1,2,2) = (0,-3,3,3)$. Row reducing

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

we can see that $\{e_1, T(e_1), T^2(e_1)\}$ is linearly independent. However

$$0(1,0,0,0) - 3(1,0,1,1) + 3(1,-1,2,2) = (0,-3,3,3),$$

so $T^3(e_1) \in \operatorname{span}\{e_1, T(e_1), T^2(e_1)\}$. Thus $\{(1,0,0,0),(1,0,1,1),(1,-1,2,2)\}$ is a basis for the T-cyclic subspace generated by $(1,0,0,0)$.

18. (a) If $a_0 \neq 0$, then $f(0) = a_0 \neq 0$ so 0 is not an eigenvalue of A. Thus A is invertible. Conversely, if A is invertible, then $f(0) \neq 0$. But $f(0) = a_0$, so $a_0 \neq 0$.

(b) By the Cayley-Hamilton Theorem, we have

$$(-1)^n A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I = 0.$$

Sine $a_0 \neq 0$, we can subtract $a_0 I$ from both sides and divide by $-a_0$ to get

$$-a_0^{-1}\left((-1)^n A^n + a_{n-1}A^{n-1} + \cdots + a_1 A\right) = I.$$

Factoring out an A gives

$$I = -a_0^{-1}\left((-1)^n A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_1 I\right)A.$$

Since A is square, we have

$$A^{-1} = -a_0^{-1}\left((-1)^n A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_1 I\right).$$

(c) The characteristic polynomial of A is $f(t) = (1-t)(2-t)(-1-t) = -t^3 + 2t^2 + t - 2$. Since 0 is not an eigenvalue of A, it is invertible. By part (b),

$$A^{-1} = \frac{1}{2}(-A^2 + 2A + I) = \frac{1}{2}\begin{pmatrix} 2 & -2 & -4 \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{pmatrix}.$$

19. We proceed by induction on k. If $k = 1$, then $A = (-a_0)$, and $\det(A - tI) = -a_0 - t = (-1)^1(a_0 + t^1)$.

Now assume the result for k, and we will prove it for $k+1$. We have

$$\det(A - tI) = -t\det\begin{pmatrix} 0 & \cdots & 0 & -a_1 \\ 1 & \cdots & 0 & -a_2 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & -a_{k-1} \\ 0 & \cdots & 1 & -a_k \end{pmatrix} + (-1)^{k+2}(-a_0)\det\begin{pmatrix} 1 & -t & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & -t \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Note that the factor $(-1)^{k+2}$ arises since $-a_0$ is in the $(1, k+1)$ position in the matrix, and that $(-1)^{k+2}(-a_0) = (-1)^{k+1}a_0$. We can compute the determinant of the first matrix above by the induction hypothesis, and the second matrix has determinant 1. Then

$$\begin{aligned} \det(A - tI) &= -t(-1)^k(a_1 + a_2 t + \cdots + a_k t^{k-1} + t^k) + (-1)^{k+1}a_0 \\ &= (-1)^{k+1}(a_1 t + a_2 t^2 + \cdots + a_k t^k + t^{k+1}) + (-1)^{k+1}a_0 \\ &= (-1)^{k+1}(a_0 + a_1 t + \cdots + a_k t^k + t^{k+1}). \end{aligned}$$

42. If $n = 1$, then $A = (1)$ and the characteristic polynomial is $t - 1$. We now assume $n > 1$.

Observe that the columns of A are identical, so their span is just $\operatorname{span}(1, 1, 1, 1)$, which has dimension 1. Thus $\operatorname{rank} A = 1$, so $\dim \operatorname{Null} A = n - 1$ by the dimension theorem. That is, 0 is an eigenvalue and $\dim E_0 = n - 1$. Also, note that

$$A \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} n \\ \vdots \\ n \end{pmatrix}$$

so $(1, \ldots, 1)$ is an eigenvector with eigenvalue n. If m_λ is the multiplicity of λ, then by Theorem 5.7 we have $1 \leq \dim(E_\lambda) \leq m_\lambda$. Thus $m_0 \geq n - 1$ and $m_n \geq 1$. However, the sum of the multiplicities of the eigenvalues can be at most the degree of the characteristic polynomial, which is n. Thus we must have $m_0 = n - 1$ and $m_n = 1$, and A cannot have any other eigenvalues as otherwise the multiplicities would sum to greater than n. Thus the characteristic polynomial of A is $(-1)^n t^{n-1}(t - n)$.