Solution to Final Exam

1. The determinant of an $n \times n$ matrix is $(-1)^n$ times the product of the eigenvalues, each occurring with the appropriate multiplicity. In this case, we get a determinant of $-1 \cdot 2^2 \cdot 3 = -12$.

2. (a) A has two eigenvalues $\pm\frac{1}{2}$. The eigenvectors for $\frac{1}{2}$ are spanned by $v_1 = \begin{pmatrix} 2 & -1 \end{pmatrix}$, wile the eigenvectors for $-\frac{1}{2}$ are spanned by $v_2 = \begin{pmatrix} 1 & -1 \end{pmatrix}$.

(b) Notice that $v = v_1 - v_2$. Then $A^n v = A^n v_1 - A^n v_2 = (\frac{1}{2})^n v_1 - (-\frac{1}{2})^n v_2 = (\frac{1}{2})^n(v_1 - (-1)^n v_2)$. As n grows large, the length of this vector goes to zero, because the length of $v_1 - (-1)^n v_2$ can take on only two values, while the factor $(\frac{1}{2})^n$ goes to zero.

3. (a) The fact that $T^2 = T$ implies that T satisfies the polynomial $t^2 - t$. The minimal polynomial $m_T(t)$ of T must then divide this polynomial, so the only possible roots of the minimal polynomial are 0 and 1, hence these are the only possible eigenvalues.

(b) By the above observations, $m_T(t)$ divides $t^2 - t$, which has distinct linear factors. So $m_T(t)$ has distinct linear factors. By a theorem from class, this implies diagonalizability.

4. (a) Pick $u, w \in V$, and $c \in \mathbb{R}$. Then $\phi_v(cu + w) = \langle v, cu + w\rangle = c\langle v, u\rangle + \langle v, w\rangle = c\phi_v(u) + \phi_v(w)$, so ϕ_v is linear. We used here various properties of a (real) inner product.

(b) To show Φ is linear, pick $u, v \in V$ and $c \in \mathbb{R}$. Then $\Phi(cu + v)$ is the functional ϕ_{cu+v} which acts on a vector w by $\phi_{cu+v}(w) = \langle cu + v, w\rangle$. But we compute that $\langle cu + v, w\rangle = c\langle u, w\rangle + \langle v, w\rangle = c\phi_u(w) + \phi_v(w)$, and this latter is just the functional $c\Phi(u) + \Phi(v)$ applied to the vector w. So Φ is linear. We already know from class that $\dim V = \dim V^*$, so to check that Φ is an isomorphism, it suffices to check injectivity. So suppose $v \in N(\Phi)$, which means that ϕ_v is the zero map on V. Then $\langle v, w\rangle = 0$ for all vectors $w \in V$, which means v itself is the zero vector. Thus Φ has trivial nullspace, hence is injective, hence an isomorphism.

5. (a) S is linearly independent if $\sum_{i=1}^n c_i v_i = 0$ (for some $c_i \in \mathbb{R}$ and $v_i \in S$) implies that $c_1 = \cdots = c_n = 0$. S is an orthonormal set if for any two *distinct* vectors $u, v \in S$, we have $\langle v, w\rangle = \delta_{ij}$,i.e., it's 0 if $i \neq j$ and 1 if $i = j$.

(b) Suppose $\sum_{i=1}^n c_i v_i = 0$ for some $c_i \in \mathbb{R}$ and $v_i \in S$. Then for each $j = 1, \ldots, n$, we compute

$$0 = \langle \sum_{i=1}^n c_i v_i, v_j\rangle = \sum_{i=1}^n c_i\langle v_i, v_j\rangle = c_j\langle v_j, v_j\rangle,$$

since all the terms $\langle v_i, v_j\rangle$ when $i \neq j$ are zero. But $v_j \neq 0$ (because its length is one), so this implies $c_j = 0$. Since j was arbitrary amongst $1, \ldots, n$, this shows the independence.

6. (a) For a complex vector space, the diagonalizable operators are the normal ones. T is normal, because it's even self-adjoint: $T^* = (S^*S)^* = S^*S = T$. We've seen that self-adjoint implies normal.

(b) Suppose λ is an eigenvalue of T (T has eigenvalues, since V is complex), and v an eigenvector for λ. Then compute

$$\lambda\|v\|^2 = \langle \lambda v, v\rangle = \langle T(v), v\rangle = \langle S^*S(v), v\rangle = \langle S(v), S(v)\rangle = \|S(v)\|^2.$$

Since $\|v\|^2$ and $\|S(v)\|^2$ are both non-negative reals, so is λ.

7. Let $f(t) = t^n + \ldots + a_1 t + a_0$ be the characteristic polynomial of A (after possibly multiplying by -1 to remove the coefficient of t^n. By the Cayley-Hamilton theorem, A satisfies this polynomial. Moving A^n to one side, we have $A^n = -a_{n-1}A^{n-1} - \cdots - a_1 A - a_0 I$, so A^n is in the span of $I, A, \ldots, A^{n-1}$. This also shows that the span of $I, A, \ldots, A^{n-1}$ is A-invariant, so for any power A^k (with possibly $k > n$), we have $A^k \in span\{I, A, \ldots, A^{n-1}\}$. Thus W itself can be spanned by these n matrices, so its dimension is at most n.

8. (a) The polynomial 1 already has length one in this inner product. We replace x by the normalization of $x - (\int_0^1 1 \cdot x dx)1 = x - 1/2$. This polynomial has length $\sqrt{\int_0^1 (x - \frac{1}{2})^2 dx} = \sqrt{1/12}$, so the normalized vector is $\sqrt{12}(x - 1/2)$. Thus our orthonormal basis is $\{1, \sqrt{12}(x - 1/2)\}$. Let us denote this orthonormal basis by $p_1(x) = 1$, $p_2(x) = \sqrt{12}(x - 1/2)$.

(b) Define T by sending p_1 to e_1, and p_2 to e_2 (here e_1, e_2 comprise the standard basis for $\mathbb{R}^2$). By a theorem from class, this defines a unique linear map, which is an isomorphism since it sends a basis for $P_1(\mathbb{R})$ to a basis for $\mathbb{R}^2$. To check the equality on the inner products, note that it suffices to check that it holds when f and g are basis vectors, since the inner product is linear in both slots. But $\langle p_i, p_j \rangle = \delta_{ij}$ since they form an orthonormal basis, and $\langle T(p_i), T(p_j) \rangle = \langle e_i, e_j \rangle = \delta_{ij}$ since e_1, e_2 are an orthonormal basis for $\mathbb{R}^2$.

9. Since $\dim N(A - 2I) = 1$ but the algebraic multiplicity of the eigenvalue 2 is two, we must have $\dim N(A - 2I)^2 = 2$, so the eigenvalue 2 has a 2×2 Jordan block associated with it. For the eigenvalue 3, there are two independent eigenvectors and the algebraic multiplicity is also two, so we get two 1×1 Jordan blocks for the eigenvalue 3. For the eigenvalue 4, there is one independent eigenvector, so one cycle, which must therefore have length 3, so we get a 3×3 Jordan block for this eigenvalue. Thus a Jordan canonical form for A is given by the matrix

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix}.$$

10. The matrix has one eigenvalue, 0, so one generalized eigenspace, which is three dimensional. Observe that here $A - \lambda I$ is just A. To find a Jordan basis, we need a vector v which is in $N(A^3)$ (which is all of $\mathbb{C}^3$), but not in $N(A^2)$. But $N(A^2)$ is the span of e_2, e_3, so take v to be e_1. Then the next vector in our cycle is $Ae_1 = (0, 1, 1)$, and the final vector in the cycle is $A(0, 1, 1) = e_3$ (note it's an eigenvector, of course). Thus our Jordan basis is $\{(0, 0, 1), (0, 1, 1), (1, 0, 0)\}$. Since there is only one eigenvalue, with only one cycle, we don't need to compute anything to know that

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

is the Jordan form for A.