MATH 254 A: PROBLEM SET 7

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(1) Let $K = \mathbb{Q}(\alpha)$, where α satisfies $\alpha^5 - \alpha + 1 = 0$.

(a) Prove that $\mathcal{O}_K = \mathbb{Z}[\alpha]$.

(b) Using the Minkowski bound, show that the class number h_K of K is 1.

(2) Let $K = \mathbb{Q}(\sqrt{d})$ where d is a square free integer.

(a) Show that $\mathcal{O}_K$ is a PID when $d = 2, 3, 5, 13, -1, -2, -3, -7$.

(b) Show that $\mathcal{O}_K$ is a PID when $d = 21, 29, -11, -19$.

(c) Show that $\mathcal{O}_K$ is also a PID when $d = 6, 7, 33$.

(3) Let A be a Dedekind domain with fraction field K, and let $\mathrm{Pic}(A)$ denote the set of projective modules over A of rank 1 (the rank of a projective module M is defined to be the dimension of the K-vector space $M \otimes_A K$.

(a) Show that if M and N are projective modules of rank 1, then $M \otimes_A N$ is also projective of rank 1. Conclude that tensor product gives $\mathrm{Pic}(A)$ the structure of an abelian group.

(b) Show that if $\mathfrak{b} \subset K$ is a fractional ideal, then $\mathfrak{b}$ is a projective A-module of rank 1. Prove that the induced map

$$\{\text{frac. ideals of } A\} \to \mathrm{Pic}(A)$$

defines a group isomorphism $\mathrm{Cl}(A) \simeq \mathrm{Pic}(A)$.