MATH 254 A: PROBLEM SET 6

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(1) Here is another proof of quadratic reciprocity (the exercise consists of filling in the details). Let p and q be distinct odd primes.

(a) Assume $p \equiv 1 \pmod 4$, and q arbitrary. Then

$$\begin{aligned} \left(\tfrac{p}{q}\right) = 1 \quad &\leftrightarrow \quad x^2 - x + \tfrac{1-p}{4} \equiv 0 \pmod q \text{ has a solution} \\ &\leftrightarrow \quad q \text{ splits in } \mathbb{Q}(\sqrt{p}) = \mathbb{Q}(\tfrac{1+\sqrt{p}}{2}) \\ &\leftrightarrow \quad \sigma_q \text{ fixes } \mathbb{Q}(\sqrt{p}) \\ &\leftrightarrow \quad q \equiv a^2 \pmod p \text{ has a solution} \\ &\leftrightarrow \quad \left(\tfrac{q}{p}\right) = 1. \end{aligned}$$

Here $\sigma_q : \mathbb{Q}(\omega_p) \to \mathbb{Q}(\omega_p)$ is the automorphism induced by $\omega_p \mapsto \omega_p^q$.

(b) Imitating part (a), show that if both p and q are congruent to 3 mod 4, then $(p/q) = -(q/p)$.

(c) Summarize the above with the law of quadratic reciprocity

$$(p/q) = (-1)^{\frac{(p-1)(q-1)}{4}}(q/p).$$

(2) Let $\widehat{\mathbb{Z}}$ denote $\varprojlim_n \mathbb{Z}/(n)$. A *Dirichlet character* is a continuous homomorphism $\chi : \widehat{\mathbb{Z}}^* \to \mathbb{C}^*$, where $\mathbb{C}^*$ is given the discrete topology.

(a) Show that any χ factors through the projection $\widehat{\mathbb{Z}}^* \to \mathbb{Z}/(n)^*$ for some n, and that conversely for any homomorphism $\chi : \mathbb{Z}/(n)^* \to \mathbb{C}^*$ the composite

$$\widehat{\mathbb{Z}}^* \longrightarrow \mathbb{Z}/(n)^* \xrightarrow{\chi} \mathbb{C}^*$$

is a Dirichlet character (i.e. continuous).

(b) Show that if $\chi : \widehat{\mathbb{Z}}^* \to \mathbb{C}^*$ is a Dirichlet character, then there exists a unique positive integer f_χ such that χ factors through an inclusion $(\mathbb{Z}/(f_\chi))^* \hookrightarrow \mathbb{C}^*$. The integer f_χ is called the *conductor* of χ.

(c) Show that the set of Dirichlet characters form a group $\mathcal{G}$ in a natural way.

(d) Let $\overline{\mathbb{Q}} \subset \mathbb{C}$ be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. Let $\mathbb{Q}^{\text{cycl}} \subset \overline{\mathbb{Q}}$ denote the compositum of the extensions $\mathbb{Q}(\omega_n)$, as n varies. Show that there is a natural isomorphism

$$\text{Gal}(\mathbb{Q}^{\text{cycl}}/\mathbb{Q}) \simeq \widehat{\mathbb{Z}}^*.$$

(e) If χ is a Dirichlet character, let $L_\chi \subset \mathbb{Q}^{\text{cycl}}$ denote the fixed field of the kernel $K_\chi \subset \widehat{\mathbb{Z}}^*$. Describe the ramification of a prime p in L_χ in terms of the character χ.