MATH 254 A: PROBLEM SET 4

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(1) Let K be a number field, and let $p \in \mathbb{Z}$ be a prime. Show that there is a natural isomorphism

$$K \otimes_{\mathbb{Q}} \mathbb{Q}_p \simeq \prod_{\mathfrak{p}} K_{\mathfrak{p}},$$

where on the right the product is taken over primes $\mathfrak{p} \subset \mathcal{O}_K$ and $K_{\mathfrak{p}}$ denotes the $\mathfrak{p}$-adic completion of K.

(2) Let K be a field, and let $Z \subset K$ be the image of the unique map $Z \to K$. Let $|\cdot|$ be an absolute value on K. Show that $|\cdot$ is non-archimedian if and only if $|\cdot|$ is bounded on Z. Hint: Show that if δ is a number such that $|m| < \delta$ for all $m \in Z$, then for any $a, b \in K$ we have

$$|(a+b)|^n < \delta(n+1)\max|a|, |b|.$$

Deduce that if K is a field of positive characteristic, then any absolute value on K is non-archimedian.

(3) Let a be an integer such that $a \equiv 1 \pmod 8$. Show that a is a square in $\mathbb{Q}_2$.

(4) Let $|\cdot|$ be an absolute value on $\mathbb{Q}$. Show that $|\cdot$ is equivalent to one of the standard absolute values as follows.

Case 1: There exists an integer $n \in \mathbb{Z}$ such that $n > 1$ by $|n| \leq 1$.

In this case show that for any integer m we have

$$|m| < \left(1 + \frac{\log m}{\log n}\right) n$$

by writing

$$m = a_0 + a_1 n + \cdots + a_r n^r, \quad 0 \leq a_i \leq n-1, \quad r \leq \frac{\log m}{\log n}.$$

Applying this with m^t deduce that for every positive integer t we have

$$|m| < \left(1 + t\frac{\log m}{\log n}\right)^{1/t} n^{1/t},$$

and therefore $|m| \leq 1$ for all $m \in \mathbb{Z}$. From this deduce that $|\cdot|$ is equivalent to $|\cdot|_p$ for some prime p.

Case 2: $|n| > 1$ for all $n > 1$.

Let $n, m > 1$ be positive integers. By an argument similar to the one in case 1, show that

$$|m| \leq n^{1/t}\left(1 + t\frac{\log m}{\log n}\right)^{1/t} |n|^{(\log m)/(\log n)}$$

for all positive integers t. Deduce from this that

$$|m|^{1/\log m} = |n|^{1/\log n} = e^c$$

for some positive real number c, and then use this to show that $|\cdot|$ is equivalent to the ordinary absolute value.