MATH 254 A: PROBLEM SET 3

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(1) Show by an example that a prime $p \in \mathbb{Z}$ can be totally ramified in two different number fields K_1 and K_2 without being totally ramified in the compositum $K_1 \cdot K_2$.

(2) Let K be a field and $|\cdot|$ be an absolute value. Show that the set of Cauchy sequences form a ring, and that the set of null sequences form a maximal ideal.

(3) Let K be the number field $\mathbb{Q}[x]/(x^3 - 5)$. Show that the ring of integers of K is $\mathbb{Z}[x]/(x^3 - 5)$.

(4) For an integer N we can define the N-adic completion of $\mathbb{Z}$ to be the ring

$$\mathbb{Z}_N := \varprojlim_s \mathbb{Z}/(N^s).$$

Show that there is a natural isomorphism

$$\mathbb{Z}_N \to \prod_{p|N} \mathbb{Z}_p,$$

where on the right the product is taken over primes dividing N.

(5) (The *Teichmuller lifting*) Let R be a complete dvr whose fraction field is of characteristic 0 and whose residue field k is perfect of characteristic $p > 0$. Show that for every $x \in k$ there exists a unique lifting of x to R which has a p^n-th root in R for all positive integers n. This lifting is usually denoted $[x]$.