MACDONALD POLYNOMIALS AND GEOMETRY

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Contents

1. Introduction
2. Symmetric functions and Macdonald polynomials
3. The $n!$ conjecture
4. The Hilbert scheme and X_n
5. Frobenius series
6. The ideals J and J^m
7. Diagonal harmonics
8. The commuting variety

1. Introduction

This article is an explication of some remarkable connections between the two-parameter symmetric polynomials discovered in 1988 by Macdonald [**26**], and the geometry of certain algebraic varieties, notably the Hilbert scheme $\mathrm{Hilb}^n(\mathbf{C}^2)$ of points in the plane, and the variety C_n of pairs of commuting $n \times n$ matrices ("commuting variety," for short). The conjectures on diagonal harmonics introduced in [**19**] and [**12**] also relate to this geometric setting.

I have sought to give a reasonably self-contained treatment of these topics, by providing an introduction to the theory of Macdonald polynomials, to the "plethystic substitution" notation for symmetric functions which is invaluable in dealing with them, and to the conjectures and other phenomena relating to them that we aim to explain geometrically. The geometric discussion is less self-contained, as it is unavoidable to use scheme-theoretic language, constructions such as blowups, and some sheaf cohomological arguments. I do however give geometric descriptions in elementary terms of the various algebraic varieties encountered, and review whatever of their special features we might use, so as to orient the reader not previously familiar with them.

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The linchpin of the geometric connections we consider is the so-called "$n!$ conjecture" of Garsia and the author [**9**,**10**], which remains unproved at present. The $n!$ conjecture proposes a combinatorial interpretation of the famous Kostka-Macdonald coefficients $K_{\lambda\mu}(q,t)$, which relate the Macdonald polynomials to Schur functions and which were conjectured by Macdonald to be polynomials with non-negative integer coefficients[1] in the parameters q,t.

The $n!$ conjecture is really two conjectures: first, that certain simply defined spaces, quotient rings of the polynomial ring $\mathbf{C}[\mathbf{x},\mathbf{y}] = \mathbf{C}[x_1,y_1,\ldots,x_n,y_n]$, have dimension $n!$; and second, that these spaces, viewed as doubly graded representations of the symmetric group S_n, have Hilbert polynomials which are essentially the Kostka-Macdonald coefficients. It turns out, as we shall show, that the first (apparently weaker) part of the conjecture is equivalent to the Cohen-Macaulay property of a certain "iso-spectral" variety X_n over $\mathrm{Hilb}^n(\mathbf{C}^2)$. Using this fact we can prove that the first part of the conjecture actually implies the second part, and with it the Macdonald positivity conjecture for the $K_{\lambda\mu}(q,t)$.

As we shall see, the variety X_n is the blowup of $(\mathbf{C}^2)^n$ at the ideal J generated by those elements of its coordinate ring $\mathbf{C}[\mathbf{x},\mathbf{y}]$ which alternate in sign under the action of the symmetric group S_n. An obvious conjecture is that J is the ideal of the locus in $(\mathbf{C}^2)^n$ where two or more of the n points (x_i,y_i) coincide, that is,

$$J = \bigcap_{i\neq j}(x_i - x_j, y_i - y_j). \tag{1.1}$$

It is easy to see that J defines the coincidence locus set-theoretically, which is to say, the radical of J is the intersection on the right hand side above, but it is not obvious, and indeed it remains an open question, that J is a radical ideal. More generally, the geometry of the blowup X_n depends on module-theoretic properties of the powers J^m. We are led to extend (1.1) and conjecture that

$$J^m = \bigcap_{i\neq j}(x_i - x_j, y_i - y_j)^m, \tag{1.2}$$

that is, the powers of J are the symbolic powers of the ideal of the coincidence locus. In fact, we conjecture that the variables $x_1,\ldots,x_n$ form a regular sequence for the $\mathbf{C}[\mathbf{x},\mathbf{y}]$-module J^m, for all m. As we show, this conjecture implies (1.2). It further implies that the x_i's form a regular sequence on X_n (that is, on its structure sheaf $\mathcal{O}$, viewed as a sheaf of $\mathbf{C}[\mathbf{x},\mathbf{y}]$-algebras). Assuming this regular-sequence conjecture, we are able to give an inductive sheaf-cohomological argument to show that X_n is Cohen-Macaulay, and thus the $n!$ and Macdonald positivity conjectures follow.

An important point to remark on here is that the $n!$ conjecture and many of the related geometric conjectures have evident analogous statements in more than two sets of variables $X,Y,Z,\ldots$. For the most part, these analogs fail to hold.[2] However, the above conjectures

[1] Part of the conjecture is that the $K_{\lambda\mu}(q,t)$ are polynomials at all, which is not obvious from their definition and was only proved recently, in five independent papers [**14**,**15**,**23**,**24**,**32**].

[2] The $n!$ conjecture has acquired a minor history of exciting but unsuccessful ideas for simple proofs, by the author and others. A good reality check on a contemplated proof is to ask where the argument breaks down—as it must—in three sets of variables.

on the ideals J^m are an exception, as we expect them to hold in any number of sets of variables (the last conjecture then being that any one of the sets of variables forms a regular sequence). Of course the reasoning leading from there to the $n!$ conjecture makes essential use of having only two sets.

The iso-spectral Hilbert scheme X_n also provides the geometric setting for the study of *diagonal harmonics*, the subject of a series of conjectures by the author and others [**12**,**19**]. The space of diagonal harmonics may be identified with the quotient ring R_n of $\mathbf{C}[\mathbf{x},\mathbf{y}]$ by the ideal I generated by all S_n invariant polynomials with zero constant term. It is conjectured, among other things, that the dimension of R_n as a vector space is $(n+1)^{n-1}$. Further conjectures in [**19**] describe aspects of its structure as a graded S_n module in combinatorial terms. In [**12**] we conjectured a complete formula for the doubly graded character of R_n, in terms of Macdonald polynomials, and proved that this master formula implies all the earlier combinatorial conjectures.

In geometric terms R_n is the coordinate ring of the scheme-theoretic fiber over the origin under the natural map

$$(\mathbf{C}^2)^n \to S^n\mathbf{C}^2$$

from ordered n-tuples of points in the plane to unordered n-tuples. Now, there is a fiber square

$$\begin{array}{ccc} X_n & \longrightarrow & (\mathbf{C}^2)^n \\ {\scriptstyle\sigma}\downarrow & & \downarrow \\ \operatorname{Hilb}^n(\mathbf{C}^2) & \xrightarrow{\tau} & S^n\mathbf{C}^2, \end{array}$$

giving rise to a natural homorphism from R_n to the global sections of the structure sheaf on the fiber $(\tau\sigma)^{-1}(0) \subseteq X_n$. If X_n is Cohen-Macaulay, that is, if the $n!$ conjecture is true, these may be identified with global sections of a vector bundle on the zero-fiber $\tau^{-1}(0)$ in $\operatorname{Hilb}^n(\mathbf{C}^2)$. Under a suitable cohomology vanishing hypothesis, the homorphism from R_n to this space of global sections will be an isomorphism. Moreover, we can give its character explicitly, using a variant of the Atiyah-Bott Lefschetz formula. This yields the master formula for diagonal harmonics, on the assumption that the $n!$ conjecture and vanishing hypotheses hold. The agreement of this master formula with computational results for $n \leq 7$ is in my view striking evidence for the probable validity of the geometric conjectures which give rise to it.

Finally, the commuting variety C_n enters the picture because it contains a natural non-singular open set C_n^0 with a smooth map to $\operatorname{Hilb}^n(\mathbf{C}^2)$. The analog in this context of the iso-spectral Hilbert scheme X_n is the "iso-spectral commuting variety" IC_n of pairs (X,Y) of commuting matrices, together with $2n$-tuples $(a_1, b_1 \ldots, a_n, b_n)$ for which the (a_i, b_i) are the joint eigenvalues of the matrices X, Y, that is, they satisfy the equations

$$\det(I + rX + sY) = \prod_{i=1}^{n}(1 + ra_i + sb_i), \tag{1.3}$$

where r, s are indeterminates. The open set IC_n^0 of IC_n lying over C_n^0 is the fiber product of X_n with C_n^0 over $\mathrm{Hilb}^n(\mathbf{C}^2)$:

$$\begin{array}{ccc} IC_n^0 & \longrightarrow & X_n \\ \big\downarrow & & \big\downarrow \\ C_n^0 & \longrightarrow & \mathrm{Hilb}^n(\mathbf{C}^2). \end{array} \tag{1.4}$$

Thus IC_n^0 is smooth over X_n, and hence X_n is Cohen-Macaulay if and only if IC_n^0 is. In fact, as we shall see, if X_n is Cohen-Macaulay it is even Gorenstein, and thus the same is true of IC_n^0. We are led to conjecture that the *whole* iso-spectral commuting variety IC_n is Gorenstein, and not just the open subset IC_n^0. We have been able to verify this for small values of n. This conjecture not only implies the $n!$ conjecture, it also implies that the ordinary commuting variety C_n is Cohen-Macaulay, which is an open problem of long standing.

2. Symmetric functions and Macdonald polynomials

The general reference for material in this section is Macdonald's book [**27**], whose notation and terminology we follow, except as to the plethystic substitution, and as to the transformed Macdonald polynomials $\tilde{H}_\mu$ defined below. We give some definitions and derive some properties which are also in [**27**], both for completeness and to illustrate the utility of the plethystic notation. We also derive some additional facts that will be needed later.

We work throughout with symmetric functions in infinitely many indeterminates $x_1, x_2, \ldots$, with coefficients in the field $\mathbf{Q}(q,t)$ of rational functions of two variables q and t. The various classical bases of the ring of symmetric functions are indexed by integer partitions μ, and denoted as follows: the monomial symmetric functions by m_μ, the power-sums by p_μ, the elementary symmetric functions by e_μ, the complete homogeneous symmetric functins by h_μ, and the Schur functions by s_μ. In each basis, as μ ranges over partitions of a given integer d, we obtain a basis for the symmetric polynomials homogeneous of degree d.

The standard partial ordering on partitions of d is the *dominance order*, defined by $\lambda \leq \mu$ if $\lambda_1 + \cdots + \lambda_k \leq \mu_1 + \cdots + \mu_k$ for all k. The triangularity of transition matrices between certain bases of symmetric functions with respect to dominance plays a crucial role in the definition and development of Macdonald polynomials, as well as in the reasoning we will use later to deduce the Macdonald positivity conjecture from the $n!$ conjecture.

We now turn to the important device of plethystic substitution. The fact that the power-sums p_μ form a basis means, equivalently, that the ring of symmetric functions can be identified with the ring of polynomials in the power-sums $p_1, p_2, \ldots$. In particular, the p_k's may be specialized arbitrarily to elements of any algebra over the coefficient field, and the specialization extends uniquely to an algebra homomorphism on all symmetric functions.

Now let A be a formal Laurent series with rational coefficients in indeterminates $a_1, a_2, \ldots$, which may include our parameters q and t. We define $p_k[A]$ to be the result of replacing each indeterminate a_i in A by a_i^k. Extending the specialization $p_k \mapsto p_k[A]$ to arbitrary symmetric functions f, we obtain the *plethystic substitution* of A into f, denoted $f[A]$.

If A is merely a sum of indeterminates, $A = a_1 + \cdots + a_n$, then we see that $p_k[A] = p_k(a_1, a_2, \ldots, a_n)$, and hence for every f we have $f[A] = f(a_1, a_2, \ldots, a_n)$. This is why we view the operation as a kind of substitution. Similarly, if A has a series expansion as a sum of monomials, then $f[A]$ is f evaluated on these monomials, for example

$$f[1/(1-t)] = f(1, t, t^2, \ldots).$$

Our convention will be that in a plethystic expression X stands for the sum of the original indeterminates $x_1 + x_2 + \cdots$, so that $f[X]$ is the same as $f(X)$,

$$f[X/(1-t)] = f(x_1, x_2, \ldots, tx_1, tx_2, \ldots, t^2x_1, t^2x_2, \ldots),$$

and so forth. Among the virtues of this notation is that the substitution of $X/(1-t)$ for X as above has an explicit inverse, namely the plethystic substitution of $X(1-t)$ for X.

The one caution that must be observed with plethystic notation is that indeterminates must always be treated as formal symbols, never as variable numeric quantities. For instance, if f is homogeneous of degree d then it is true (and easy to see) that

$$f[tX] = t^d f[X],$$

but it is *false* that $f[-X] = (-1)^d f[X]$, that is, we cannot set $t = -1$ in the equation above. In fact $f[-X]$ is a very interesting quantity: it is actually equal to $(-1)^d \omega f(X)$, where ω is the classical involution on symmetric functions defined by $\omega p_k = (-1)^{k+1} p_k$, which interchanges the elementary and complete symmetric functions e_λ and h_λ, and more generally exchanges the Schur function s_λ with $s_{\lambda'}$, where λ' is the conjugate partition.

It is convenient when using plethystic notation to define

$$\Omega(X) = \exp(\sum_{k=1}^{\infty} p_k(X)/k). \tag{2.1}$$

Then since $p_k[A+B] = p_k[A] + p_k[B]$ and $p_k[-A] = -p_k[A]$ we have

$$\Omega[A+B] = \Omega[A]\Omega[B], \quad \Omega[-A] = 1/\Omega[A]. \tag{2.2}$$

From this and the single-variable evaluation $\Omega[x] = \exp(\sum_{k\geq 1} x^k/k) = 1/(1-x)$ we obtain

$$\Omega[X] = \prod_i \frac{1}{1-x_i} = \sum_{n=0}^{\infty} h_n(X) \tag{2.3}$$

$$\Omega[-X] = \prod_i (1-x_i) = \sum_{n=0}^{\infty} (-1)^n e_n(X). \tag{2.4}$$

Recall that the standard Hall inner product $\langle \cdot, \cdot \rangle$ is defined so that the Schur functions s_μ are an orthonormal basis, and the complete symmetric functions h_μ are dual to the monomials m_μ. The *Cauchy identity* is that for Hall-dual bases $\{u_\mu\}, \{v_\mu\}$ we have

$$\prod_{i,j} \frac{1}{1 - x_i y_j} = \sum_\mu u_\mu(X) u_\nu(Y), \quad \text{or plethystically,} \quad \Omega[XY] = \sum_\mu u_\mu[X] v_\mu[Y]. \tag{2.5}$$

This may be written in a basis-free way as

$$\langle \Omega[AX], f(X) \rangle = f[A], \tag{2.6}$$

which follows from (2.5) by taking $f = u_\mu$, and extending to arbitrary f by linearity. In particular we have

$$\langle \Omega[B(AX)], \Omega[CX] \rangle = \langle \Omega[BX], \Omega[C(AX)] \rangle = \Omega[ABC].$$

But since B and C are arbitrary, we may set $f(X) = \Omega[BX]$, $g(X) = \Omega[CX]$, to obtain the identity

$$\langle f[AX], g(X) \rangle = \langle f(X), g[AX] \rangle, \tag{2.7}$$

valid for all f, g. In other words, the plethystic subsitution of AX for X is self-adjoint.

Macdonald defines his polynomials by first introducing a q,t-analog of the Hall inner product $\langle \cdot, \cdot \rangle$, which in plethystic notation is simply

$$\langle f, g \rangle_{q,t} = \langle f(X), g[X \frac{1-q}{1-t}] \rangle.$$

In view of (2.7), this definition is symmetric in f and g. If $\{u_\mu\}$ and $\{v_\mu\}$ are $\langle \cdot, \cdot \rangle_{q,t}$-dual bases, then $\{u_\mu\}$ and $\{v_\mu[\frac{1-q}{1-t}]\}$ are Hall dual, so the Cauchy identity gives

$$\Omega[XY] = \sum_\mu u_\mu[X] v_\mu[X \frac{1-q}{1-t}], \quad \text{or} \quad \Omega[XY \frac{1-t}{1-q}] = \sum_\mu u_\mu[X] v_\mu[Y]. \tag{2.8}$$

Note that the non-plethystic expression for $\Omega[XY \frac{1-t}{1-q}]$, as in [**27**], is the rather mysterious product

$$\prod_{i,j} \frac{(t x_i y_j; q)_\infty}{(x_i y_j; q)_\infty}, \quad \text{where} \quad (a; q)_\infty = \prod_{k=0}^{\infty} (1 - a q^k).$$

As a particular case of (2.8), we see that the $\langle \cdot, \cdot \rangle_{q,t}$-dual basis to the monomials m_μ is the basis of transformed complete symmetric functions $h_\mu[X \frac{1-t}{1-q}]$ (denoted g_μ in [**27**]).

The Macdonald polynomials $P_\mu(X; q, t)$ may be defined by requiring that they are orthogonal with respect to $\langle \cdot, \cdot \rangle_{q,t}$, and lower-uni-triangular with respect to the monomials, that is,

$$P_\mu(X; q, t) = m_\mu(X) + \sum_{\lambda < \mu} c_{\lambda\mu}(q, t) m_\lambda, \tag{2.9}$$

for some coefficients $c_{\lambda\mu}$. Here, however, we shall define them directly as eigenfunctions of the plethystic operator

$$\Delta' f(X) = f[X - (1-q)/z] \Omega[zX(1 - t^{-1})]|_{z^0}, \tag{2.10}$$

where the vertical bar indicates we are to take the constant term with respect to z.

Before further examining the operator Δ' we define an important quantity B_μ which appears in the eigenvalues of the operator, and will turn out to have geometric significance later on.[3] First recall that the *diagram* of a partition μ is the array of lattice points

$$D(\mu) = \{(i,j) \in \mathbf{N} \times \mathbf{N} : j < \mu_{i+1}\}. \tag{2.11}$$

As is customary, we regard i as indexing rows and j as indexing columns, so that the rows of $D(\mu)$ have lengths equal to the parts of μ. We now set

$$B_\mu(q,t) = \sum_{(i,j)\in D(\mu)} t^i q^j, \tag{2.12}$$

a kind of generating function describing $D(\mu)$, with a term for each cell in the diagram, as illustrated here.

$$\mu : (4,2,1) \qquad D_\mu : \begin{array}{llll} \bullet & & & \\ \bullet & \bullet & & \\ \bullet & \bullet & \bullet & \bullet \end{array} \qquad B_\mu : \begin{array}{llll} t^2+ & & & \\ t+ & qt+ & & \\ 1+ & q+ & q^2+ & q^3 \end{array} . \tag{2.13}$$

Now let us return to the study of the operator Δ'.

Proposition 2.1. *A symmetric function $f(X;q,t)$ is an eigenfunction of Δ' with eigenvalue $\alpha(q,t^{-1})$ if and only if $f[X/(1-t^{-1});q,t^{-1}]$ is an eigenfunction of the operator*

$$\Delta f = f[X + (1-q)(1-t)/z]\Omega[-zX]|_{z^0},$$

with eigenvalue $\alpha(q,t)$.

Proof. We verify directly from the definitions that

$$\Delta(f[X/(1-t^{-1});q,t^{-1}]) = (\Delta' f)[X/(1-t^{-1});q,t^{-1}],$$

which implies the result. □

Proposition 2.2. *The operator Δ' is lower-triangular with respect to the basis of monomial symmetric functions. More precisely,*

$$\Delta' m_\mu = (1 - (1-q)(1-t^{-1})B_\mu(q,t^{-1}))m_\mu + \sum_{\lambda<\mu} b_{\lambda\mu} m_\lambda,$$

for some coefficients $b_{\lambda\mu}$.

Proof. Since the Schur functions are lower-uni-triangular with respect to the basis of monomials, it will do equally well to prove

$$\Delta' m_\mu = \sum_{\lambda\le\mu} a_{\lambda\mu} s_\lambda,$$

[3]The theory of Macdonald polynomials is rife with numerology. Quantities such as B_μ and various q,t-hook products crop up again and again—see [**11**,**12**] for many examples. In our geometric context, these quantities will turn out to have natural interpretations.

for some coefficients $a_{\lambda\mu}$, with $a_{\mu\mu} = 1-(1-q)(1-t^{-1})B_\mu(q,t^{-1})$. It is also sufficient to restrict to a finite set of variables $X = x_1+\cdots+x_n$. Then $\Omega[zX(1-t^{-1})]$ has the partial fraction expansion

$$\Omega[zX(1-t^{-1})] = \prod_{i=1}^{n} \frac{1-t^{-1}zx_i}{1-zx_i} = t^{-n} + \sum_{i=1}^{n} \frac{1}{1-zx_i} \frac{\prod_{j=1}^{n}(1-x_j/tx_i)}{\prod_{j\neq i}(1-x_j/x_i)}$$
$$= t^{-n} + t^{1-n}(1-t^{-1}) \sum_i \frac{1}{1-zx_i} \frac{v(X)_{(x_i\mapsto tx_i)}}{v(X)}, \qquad (2.14)$$

where $v(X) = \prod_{i<j}(x_i - x_j)$ is the Vandermonde determinant.

From (2.14), using the identity $f(1/z)/(1-zx)|_{z^0} = f(x)$, we see that for any function f,

$$f(1/z)\Omega[zX(1-t^{-1})]|_{z^0} = t^{-n} f(1/z)|_{z^0} + t^{1-n}(1-t^{-1}) \sum_i f(x_i) \frac{v(X)_{(x_i\mapsto tx_i)}}{v(X)},$$

and therefore, since $m_\mu[X - x_i(1-q)] = m_\mu(X)_{(x_i\mapsto qx_i)}$, we have

$$\Delta' m_\mu(X) = t^{-n} m_\mu(X) + t^{1-n}(1-t^{-1}) \sum_i m_\mu(X)_{(x_i\mapsto qx_i)} \frac{v(X)_{(x_i\mapsto tx_i)}}{v(X)}.$$

Note that the substitution of x_i for z^{-1} inside the plethysm is permissible, since we are substituting one indeterminate for another.

Recall the Jacobi formula

$$s_\lambda(X)v(X) = \det\left[x_i^{\lambda_j+n-j}\right]_{i,j=1}^{n}.$$

From this we see that the coefficient of s_λ in the Schur function expansion of any symmetric function $f(X)$ is the coefficient of $\mathbf{x}^{\lambda+\delta}$ in $f(X)v(X)$, where $\delta = (n-1, n-2, \ldots, 1, 0)$. In particular the coefficient $a_{\lambda\mu}$ of s_λ in $\Delta' m_\mu$ is given by

$$t^{-n} k_{\lambda\mu} + t^{1-n}(1-t^{-1}) \sum_i m_\mu(X)_{(x_i\mapsto qx_i)} v(X)_{(x_i\mapsto tx_i)} \Big|_{\mathbf{x}^{\lambda+\delta}}, \qquad (2.15)$$

where $k_{\lambda\mu}$ is the coefficient of s_λ in m_μ, so $k_{\mu\mu} = 1$. Now in each summand above, the leading term in dominance order is clearly $\mathbf{x}^{\mu+\delta}$, establishing the triangularity. This term arises from the term $\mathbf{x}^\mu$ in m_μ, multiplied by the term $\mathbf{x}^\delta$ in $v(X)$. In the i-th summand the indicated substitutions multiply it by $q^{\mu_i}t^{n-i}$. Thus we find

$$a_{\mu\mu} = t^{-n} + t^{1-n}(1-t^{-1}) \sum_{i=1}^{n} q^{\mu_i} t^{n-i}.$$

With the understanding that μ_i is zero for i exceeding the number of parts $l(\mu)$ of μ, we readily verify that the above expression is independent of n for $n > l(\mu)$, as it must be, and reduces to

$$(1-t^{-1}) \sum_{i=1}^{\infty} q^{\mu_i} t^{1-i} = 1-(1-q)(1-t^{-1})B_\mu(q,t^{-1}).$$

□

Corollary 2.3. *The operator* Δ' *has distinct eigenvalues* $1-(1-q)(1-t^{-1})B_\mu(q,t^{-1})$ *and its corresponding eigenfunction is a linear combination of the monomial symmetric functions* $m_\lambda : \lambda \leq \mu$*, with non-zero coefficient of* m_μ*.*

Definition. The *Macdonald polynomial* $P_\mu(X;q,t)$ is the eigenfunction of the operator Δ',

$$\Delta' P_\mu = (1-(1-q)(1-t^{-1})B_\mu(q,t^{-1}))P_\mu,$$

normalized so that

$$P_\mu = m_\mu + \sum_{\lambda<\mu} c_{\lambda\mu} m_\lambda.$$

The coefficients $c_{\lambda\mu}(q,t)$ are rational functions with non-trivial denominators. Macdonald proposed an alternate normalization, called the *integral form*

$$J_\mu = \prod_{s\in D(\mu)} (1-q^{a(s)}t^{1+l(s)})P_\mu, \tag{2.16}$$

and conjectured that its coefficients are *polynomials* in q and t, *i.e.*, the above product clears the denominators. Here $a(s)$ and $l(s)$ are the *arm* and *leg* of the cell s in the diagram of μ, defined to be the number of cells strictly east and north of s, respectively.

This *integrality conjecture* has recently been proven [**14**,**15**,**23**,**24**,**32**]. Macdonald made a further remarkable conjecture, that if we write the integral forms as

$$J_\mu(X;q,t) = \sum_\lambda K_{\lambda\mu}(q,t) s_\lambda[X(1-t)] \tag{2.17}$$

then the coefficients $K_{\lambda\mu}(q,t)$ are not only polynomials in q and t, but they have *non-negative integer coefficients*. The search for an algebraic-combinatorial proof of this *Macdonald positivity conjecture*, which remains open, has been the moving force behind the work described in this article. It is pertinent to mention here that $J_\mu(X;0,t)$ specializes to the Hall-Littlewood function $Q_\mu(X;t)$, and therefore the coefficients $K_{\lambda\mu}(0,t)$ specialize to the famous *t-Kostka coefficients* $K_{\lambda\mu}(t)$ which have been central to much beautiful work in combinatorics, geometry and representation theory. This is one of many reasons for the great interest in Macdonald polynomials in the decade since their discovery.

For our purposes it is convenient to work with the following variant. In all that follows we fix $|\mu| = n$ (not to be confused with $n(\mu)$).

Definition. The *transformed Macdonald polynomials* are

$$\tilde{H}_\mu(X;q,t) = t^{n(\mu)} J_\mu[\frac{X}{1-t^{-1}};q,t^{-1}], \tag{2.18}$$

where $n(\mu) = \sum_i (i-1)\mu_i = \sum_{s\in D(\mu)} l(s)$.

From Proposition 2.1 and equation (2.17) we immediately obtain

Proposition 2.4. *The transformed polynomial* $\tilde{H}_\mu$ *is an eigenfunction of the operator* Δ*,*

$$\Delta \tilde{H}_\mu = (1-(1-q)(1-t)B_\mu)\tilde{H}_\mu,$$

and its Schur function expansion is

$$\tilde{H}_\mu = \sum_\lambda \tilde{K}_{\lambda\mu}(q,t) s_\lambda,$$

where $\tilde{K}_{\lambda\mu}(q,t) = t^{n(\mu)} K_{\lambda\mu}(q,t^{-1})$*. In particular (since it is known that* $K_{(n),\mu} = t^{n(\mu)}$ [**27**]*),* $\tilde{H}_\mu$ *is normalized so that its coefficient of* $s_{(n)}$ *is equal to* 1*.*

Now the operator Δ is symmetric in q and t, while $B_\mu(t,q) = B_{\mu'}(q,t)$. Hence we obtain

Proposition 2.5. *For all* μ *we have* $\tilde{H}_{\mu'}(X;q,t) = \tilde{H}_\mu(X;t,q)$ *and, consequently,* $\tilde{K}_{\lambda\mu'}(q,t) = \tilde{K}_{\lambda\mu}(t,q)$*.*

As mentioned earlier, the functions $\tilde{H}_\mu$ can be characterized by certain triangularity relations.

Proposition 2.6. *The transformed Macdonald polynomials* $\tilde{H}_\mu$ *satisfy, and are uniquely characterized by:*

1. $\tilde{H}_\mu[X(1-q);q,t] \in \mathbf{Q}(q,t)\{s_\lambda : \lambda \geq \mu\}$,
2. $\tilde{H}_\mu[X(1-t);q,t] \in \mathbf{Q}(q,t)\{s_\lambda : \lambda \geq \mu'\}$,
3. $\langle \tilde{H}_\mu, s_{(n)} \rangle = 1$.

Proof. Note that $\tilde{H}_\mu[X(1-t);q,t] = t^{|\mu|}\tilde{H}_\mu[-X(1-t^{-1});q,t]$ is a scalar multiple of $P_\mu[-X;q,t^{-1}]$ and thus of $\omega P_\mu(X;q,t^{-1})$. Since P_μ belongs to the space $\mathbf{Q}(q,t)\{s_\lambda : \lambda \leq \mu\}$, and conjugation reverses the dominance order, ωP_μ belongs to $\mathbf{Q}(q,t)\{s_\lambda : \lambda \geq \mu'\}$, which is (2) above. From the symmetry given by Proposition 2.5 we then obtain (1). The normalization from Proposition 2.4 is (3).

For uniqueness, suppose $H'_\mu(X)$ is another solution of (1) and (2). Then (1) implies that $H'_\mu[X(1-q)] \in \mathbf{Q}(q,t)\{\tilde{H}_\lambda[X(1-q)] : \lambda \geq \mu\}$ and hence that $H'_\mu \in \mathbf{Q}(q,t)\{\tilde{H}_\lambda : \lambda \geq \mu\}$. Similarly, (2) implies that $H'_\mu \in \mathbf{Q}(q,t)\{\tilde{H}_\lambda : \lambda \leq \mu\}$. Together these mean that H'_μ is a scalar multiple of $\tilde{H}_\mu$, and (3) fixes the scalar factor as 1. □

Corollary 2.7. *For all* μ *we have* $\omega\tilde{H}_\mu(X;q,t) = t^{n(\mu)}q^{n(\mu')}\tilde{H}_\mu(X;q^{-1},t^{-1})$ *and, consequently,* $\tilde{K}_{\lambda'\mu}(q,t) = t^{n(\mu)}q^{n(\mu')}\tilde{K}_{\lambda\mu}(q^{-1},t^{-1})$

Proof. One verifies easily that $\omega t^{n(\mu)}q^{n(\mu')}\tilde{H}_\mu(X;q^{-1},t^{-1})$ satisfies (1) and (2) of Proposition 2.6, and hence is a scalar multiple of $\tilde{H}_\mu$. To fix the scalar as 1 requires that $\tilde{K}_{(1^n),\mu} = t^{n(\mu)}q^{n(\mu')}$. But this is known [**27**], as it is equivalent to $K_{(1^n),\mu} = q^{n(\mu')}$. □

To conclude, let us recover the orthogonality of the P_μ's with respect to $\langle\cdot,\cdot\rangle_{q,t}$, as in Macdonald's original definition. Replacing t by t^{-1}, we are to show that

$$\langle P_\mu(X;q,t^{-1}), P_\nu(X\frac{1-q}{1-t^{-1}};q,t^{-1})\rangle = 0$$

for $\mu \neq \nu$, or equivalently that

$$\langle P_\mu(X;q,t^{-1}), P_\nu(-X\frac{1-q}{1-t};q,t^{-1})\rangle = 0$$

Since $P_\mu(X;q,t^{-1})$ is a scalar multiple of $\tilde{H}_\mu[X(1-t)]$, we are to show that

$$\langle \tilde{H}_\mu[X(1-t)], \tilde{H}_\nu[-X(1-q)]\rangle = 0.$$

Now from Proposition 2.6 and the orthogonality of Schur functions it is clear that this last inner product vanishes unless $\nu \le \mu$. But then by symmetry it also vanishes unless $\mu \le \nu$, that is, unless $\mu = \nu$.

3. The $n!$ conjecture

Let $D = \{(p_1, q_1), \ldots, (p_n, q_n)\}$ be an n-element subset of $\mathbf{N}\times\mathbf{N}$. We define a polynomial in $2n$ variables $x_1, y_1, \ldots, x_n, y_n$ as follows:

$$\Delta_D(\mathbf{x},\mathbf{y}) = \det\left[x_i^{p_j} y_i^{q_j}\right]_{i,j=1}^{n}. \tag{3.1}$$

Note that Δ_D is well defined, up to a change of sign, independent of the ordering chosen for the elements of D, and that it alternates in sign under the action of the symmetric group S_n permuting the $\mathbf{x}$ and the $\mathbf{y}$ variables simultaneously. That is,

$$w\Delta_\mu = \epsilon(w)\Delta_\mu \quad \text{for all} \quad w \in S_n,$$

where $\epsilon(w)$ is the sign of the permutation w. For a partition diagram, we set

$$\Delta_\mu = \Delta_{D(\mu)}.$$

Then Δ_μ is doubly homogeneous, of degree $n(\mu)$ in the $\mathbf{x}$ variables and $n(\mu')$ in the $\mathbf{y}$ variables. In the cases $\mu = (1^n)$ and $\mu = (n)$, Δ_μ is the usual Vandermonde determinant in the $\mathbf{x}$ and $\mathbf{y}$ variables, respectively.

Our conjectures concern the space of all derivatives of Δ_μ,

$$D_\mu = \{p(\partial x_1, \partial y_1, \ldots, \partial x_n, \partial y_n)\Delta_\mu : p \in \mathbf{Q}[\mathbf{x},\mathbf{y}]\}.$$

Since Δ_μ is doubly homogeneous, this space is doubly graded:

$$D_\mu = \bigoplus_{r,s} (D_\mu)_{r,s},$$

where $(D_\mu)_{r,s}$ consists of those elements of D_μ which are doubly homogeneous of degree r in the $\mathbf{x}$ variables and s in the $\mathbf{y}$ variables. Since Δ_μ is alternating, the space D_μ is stable under the action of S_n. Thus it affords a doubly graded representation of the symmetric group S_n. We denote by $\operatorname{mult}(\chi^\lambda, \chi)$ the multiplicity of the irreducible S_n-character χ^λ in a given character χ.

Conjecture 3.1. *($n!$ Conjecture) The dimension of the space D_μ is $n!$.*

Conjecture 3.2. *The bivariate character multiplicity Hilbert series*

$$\sum_{r,s} t^r q^s \operatorname{mult}(\chi^\lambda, \operatorname{ch}(D_\mu)_{r,s}) \tag{3.2}$$

is equal to $\tilde{K}_{\lambda\mu}(q,t)$. In particular, the latter is a polynomial with non-negative integer coefficients.

It is known [**27**] that $\tilde{K}_{\lambda\mu}(1,1) = \chi^\lambda(1)$, the degree of the character χ^λ or the number of standard Young tableaux of shape λ. Hence, according to Conjecture 3.2, we must have $\mathrm{mult}(\chi^\lambda, \mathrm{ch}(D_\mu)) = \chi^\lambda(1)$, so that when we ignore the grading, D_μ affords the regular representation of S_n, and hence has dimension $n!$. Thus Conjecture 3.2 implies Conjecture 3.1. One of the chief things we will achieve in the geometric setting of Sections 4 and 5 is to prove the converse implication.

It will be helpful to reformulate the conjectures in two ways. The first is to introduce a more convenient notation for (3.2). Recall that the *Frobenius map* from S_n characters to symmetric functions[4] homogeneous of degree n is defined by

$$\Phi(\chi) = \frac{1}{n!} \sum_{w \in S_n} \chi(w) p_{\tau(w)}(X), \tag{3.3}$$

where $\tau(w)$ is the partition whose parts are the lengths of the cycles of the permutation w. For the irreducible characters we have the symmetric function identity $\Phi(\chi^\lambda) = s_\lambda(X)$, from which follows, for any character,

$$\Phi(\chi) = \sum_\lambda \mathrm{mult}(\chi^\lambda, \chi) s_\lambda.$$

Now, by analogy to the Hilbert series, we define the *Frobenius series* of a doubly graded S_n representation D to be

$$\mathcal{F}_D(X; q, t) = \sum_{r,s} t^r q^s \Phi\, \mathrm{ch}(D)_{r,s}.$$

This given, Conjecture 3.2 takes the simple form

$$\mathcal{F}_{D_\mu} = \tilde{H}_\mu. \tag{3.4}$$

In Section 5 we extend the notion of Frobenius series to S_n actions on modules over a geometric regular local ring with an equivariant two-dimensional torus action, providing the basic tool to link Conjecture 3.2 with the geometry.

The second reformulation we need is of the definition of D_μ itself—the definition in terms of derivatives is simple, but geometrically misleading, and we need a derivative-free version. This is given by the next propositions. We will need the following definition here and later.

Definition. The *alternation* operator over the symmetric group is

$$\mathrm{Alt}\, f = \sum_{w \in S_n} \epsilon(w) w(f).$$

Since the polynomials Δ_D form a basis of all the S_n-alternating polynomials in $\mathbf{Q}[\mathbf{x}, \mathbf{y}]$, it makes sense to speak of the coefficient of Δ_D in $\mathrm{Alt}\, f$. Indeed, it is merely the coefficient of the monomial $x_1^{p_1} y_1^{q_1} \cdots x_n^{p_n} y_n^{q_n}$.

Proposition 3.1. *The ideal J_μ of polynomials $p(\mathbf{x}, \mathbf{y}) \in \mathbf{Q}[\mathbf{x}, \mathbf{y}]$ for which the differential operator $p(\partial\mathbf{x}, \partial\mathbf{y})$ annihilates Δ_μ can be characterized as follows: $p \in J_\mu$ if and only if for all $g \in \mathbf{Q}[\mathbf{x}, \mathbf{y}]$, the coefficient of Δ_μ in $\mathrm{Alt}\, gp$ is zero.*

[4]The symmetric function indeterminates X are not to be confused with the coordinates $\mathbf{x}$.

Proof. Observe that the constant term of $g(\partial\mathbf{x}, \partial\mathbf{y})p(\partial\mathbf{x}, \partial\mathbf{y})\Delta_\mu$ is, apart from a constant factor, the coefficient of Δ_μ in $\operatorname{Alt} gp$. Hence if $p(\partial\mathbf{x}, \partial\mathbf{y})\Delta_\mu = 0$, the characterization certainly holds. Conversely, if the characterization holds, then $p(\partial\mathbf{x}, \partial\mathbf{y})\Delta_\mu$ has the property that it and all its partial derivatives of all orders have zero constant term. By Taylor's theorem this implies that $p(\partial\mathbf{x}, \partial\mathbf{y})\Delta_\mu = 0$. □

Proposition 3.2. *The quotient ring* $\mathbf{Q}[\mathbf{x}, \mathbf{y}]/J_\mu$ *is isomorphic as a doubly graded* S_n *representation to* D_μ.

Proof. Define an inner product $(\cdot, \cdot)$ on $\mathbf{Q}[\mathbf{x}, \mathbf{y}]$ by

$$(f, g) = f(\partial\mathbf{x}, \partial\mathbf{y})g(\mathbf{x}, \mathbf{y})|_{\mathbf{x},\mathbf{y}=0}. \tag{3.5}$$

One sees immediately that monomials are mutually orthogonal under $(\cdot, \cdot)$. Hence the form is symmetric and non-degenerate, separately on each doubly homogeneous subspace $(\mathbf{Q}[\mathbf{x}, \mathbf{y}])_{r,s}$. It follows that $\mathbf{Q}[\mathbf{x}, \mathbf{y}]/J_\mu$ is isomorphic as a doubly graded S_n representation to the orthogonal complement $J_\mu^\perp$.

Now I claim that $J_\mu^\perp = D_\mu$. Taking $g = \Delta_\mu$ and $f \in J_\mu$ in (3.5), we see that $\Delta_\mu \in J_\mu^\perp$. From the definition we have $(p(\mathbf{x}, \mathbf{y})f, g) = (f, p(\partial\mathbf{x}, \partial\mathbf{y})g)$ for all $p(\mathbf{x}, \mathbf{y})$, *i.e.*, multiplication is adjoint to differentiation. Since J_μ is an ideal it follows that $J_\mu^\perp$ is closed under differentiation and hence contains D_μ.

Now both $J_\mu^\perp$ and D_μ are finite-dimensional, and the map $p \mapsto p(\partial\mathbf{x}, \partial\mathbf{y})\Delta_\mu$ is an isomorphism of vector spaces from $\mathbf{Q}[\mathbf{x}, \mathbf{y}]/J_\mu$ onto D_μ. Therefore $J_\mu^\perp$ and D_μ have the same dimension and hence are equal. □

For geometric purposes we will replace $\mathbf{Q}$ by $\mathbf{C}$, extending J_μ to $J_\mu \otimes_{\mathbf{Q}} \mathbf{C}$, which we again denote J_μ, and for which the characterization in Proposition 3.1 still holds.

Definition. The ring R_μ is $\mathbf{C}[\mathbf{x}, \mathbf{y}]/J_\mu$.

Of course R_μ has the same Frobenius series as $\mathbf{Q}[\mathbf{x}, \mathbf{y}]/J_\mu$, and also, by Proposition 3.2, as D_μ.

The evidence for Conjecture 3.2 includes the fact that various known symmetries and specializations of $\tilde{H}_\mu$ can also be established for $\mathcal{F}_{D_\mu}$. We conclude this section by demonstrating a few of these.

Proposition 3.3. *We have the identity* $\mathcal{F}_{D_{\mu'}}(X; q, t) = \mathcal{F}_{D_\mu}(X; t, q)$ *(compare Proposition 2.5).*

Proof. Obvious, since $\Delta_{\mu'}(\mathbf{x}, \mathbf{y}) = \Delta_\mu(\mathbf{y}, \mathbf{x})$. □

Proposition 3.4. *We have the identity* $\omega\mathcal{F}_{D_\mu}(X; q, t) = t^{n(\mu)}q^{n(\mu')}\mathcal{F}_{D_\mu}(X; q^{-1}, t^{-1})$ *(compare Corollary 2.7).*

Proof. In the proof of Proposition 3.2 there are two isomorphisms of $\mathbf{Q}[\mathbf{x}, \mathbf{y}]/J_\mu$ with D_μ. The first is the inclusion of D_μ in $\mathbf{Q}[\mathbf{x}, \mathbf{y}]$, followed by projection mod J_μ, which is an isomorphism of graded S_n representations. The second is the map $p \mapsto p(\partial\mathbf{x}, \partial\mathbf{y})\Delta_\mu$. As Δ_μ is S_n-alternating and homogeneous of degrees $(n(\mu), n(\mu'))$, this map reverses degrees and tensors S_n characters by the sign character. Recalling that the Frobenius map satifies $\Phi(\epsilon \otimes \chi) = \omega\Phi(\chi)$, we see that reversing degrees and tensoring with the sign character on D_μ yields a space with Frobenius series $\omega t^{n(\mu)} q^{n(\mu')} \mathcal{F}_{D_\mu}(X; q^{-1}, t^{-1})$. Combining the two isomorphisms shows this is equal to $\mathcal{F}_{D_\mu}$. □

Remark: The algebraic correlate of this symmetry of D_μ is that the ring R_μ is *Gorenstein.* More generally, for any homogeneous ideal $J \subseteq \mathbf{Q}[\mathbf{x}]$, the quotient $\mathbf{Q}[\mathbf{x}]/J$ is finite-dimensional (as a vector space) and Gorenstein if and only if J is the ideal of differential operators annihilating some homogeneous polynomial.

Proposition 3.5. *We have the identity* $\mathcal{F}_{D_\mu}(X; 0, t) = \tilde{H}_\mu(X; 0, t)$.

Proof. The left-hand side is the Frobenius series of the subspace $\mathbf{Q}[\mathbf{x}] \cap D_\mu$. The *Garnir polynomial* $g_\mu(\mathbf{x})$ is defined as the product of the Vandermonde determinants in the first μ'_1 variables, the next μ'_2 variables, and so on. It is not hard to see that the space $\mathbf{Q}[\mathbf{x}] \cap D_\mu$ is spanned by all derivatives of g_μ and its images under permutation of the variables.

This space has been well-studied [**3**,**6**,**13**,**21**,**25**,**33**], and its Frobenius series (in one variable t) is known to be the transformed Hall-Littlewood polynomial $t^{n(\mu)} Q_\mu[X/(1-t^{-1}); t^{-1}]$. Since $J_\mu(X; 0, t) = Q_\mu(X; t)$, the result follows. □

It is also possible to give an elementary proof (see [**10**]) that if the $n!$ conjecuture holds then $\mathcal{F}_{D_\mu}(X; 1, t) = \tilde{H}_\mu(X; 1, t)$. Since $\mathcal{F}_{D_\mu}(X; 1, 1)$ determines the character and thus the dimension of D_μ, one must of course assume the $n!$ conjecture for this. But as we are going to prove that the $n!$ conjecture implies $\mathcal{F}_{D_\mu} = \tilde{H}_\mu$, the proof for the $q = 1$ specialization would be redundant here.

4. The Hilbert scheme and X_n

Let $\mathbf{C}^2 = \operatorname{Spec} \mathbf{C}[\mathbf{x}, \mathbf{y}]$ be the affine plane over $\mathbf{C}$. By definition, closed subschemes $S \subseteq \mathbf{C}^2$ correspond to ideals $I \subseteq \mathbf{C}[x, y]$. The subscheme S is 0-*dimensional*, of *length* n, if $\mathbf{C}[x, y]/I$ has dimension n as a vector space. The generic example of such a subscheme S is a set of n points in $\mathbf{C}^2$, with the reduced subscheme structure. In this case I is a radical ideal and $\mathbf{C}[x, y]/I$ can be identified with the ring of complex-valued functions on the finite set S, which is clearly n-dimensional.

Other such subschemes S have fewer than n points, with non-reduced subscheme structures at these points. If we define the multiplicity of each point $P \in S$ as the length of the local ring $\mathcal{O}_{S,P}$ then the total length, n, is the sum of the multiplicites. Associated to each partition μ of n is a non-reduced subscheme S_μ whose ideal I_μ is spanned by the monomials $\{x^r y^s : (r, s) \notin D(\mu)\}$. Note that this is indeed an ideal. Since the remaining monomials

$$\mathcal{B}_\mu = \{x^r y^s : (r, s) \in D(\mu)\} \tag{4.1}$$

form a basis of $\mathbf{C}[x, y]/I_\mu$, S_μ has length n, and the Hilbert series of $\mathbf{C}[x, y]/I_\mu$ as a doubly graded algebra is the quantity $B_\mu(q, t)$ defined in (2.12). We readily see that in fact, the S_μ's are the only 0-dimensional length-n subschemes whose ideals are spanned by monomials. As a set, S_μ has only one point, the origin, with multiplicity n.

The set of all 0-dimensional length-n subschemes of $\mathbf{C}^2$, or equivalently, the set of ideals I such that $\dim_{\mathbf{C}} \mathbf{C}[x,y]/I = n$, has the structure of an algebraic variety, the *Hilbert scheme of n points in the plane*, or $\operatorname{Hilb}^n(\mathbf{C}^2)$. We will prefer the point of view that the (closed) points of $\operatorname{Hilb}^n(\mathbf{C}^2)$ are the ideals I. Its variety structure may be described in either of two ways.

First, for every $I \in \operatorname{Hilb}^n(\mathbf{C}^2)$, at least one of the sets $\mathcal{B}_\mu$ spans modulo I [**16**]. In particular, the set $M_N = \{x^r y^s : r + s < N\}$ always spans mod I, for $N \geq n$. Thus $\mathbf{C}[x,y]/I$ may be identified with an element of the Grassmann variety $G_n(W)$ of n-dimensional quotients of the space $W = \mathbf{C}M_N$, giving a map from $\operatorname{Hilb}^n(\mathbf{C}^2)$ into $G_n(W)$. For N sufficiently large (in fact, for $N \geq n+1$ [**17**]) this map is injective, its image is a locally closed subvariety of $G_n(W)$, and the induced variety structure on $\operatorname{Hilb}^n(\mathbf{C}^2)$ is independent of N.

The second description is via Grothendieck's universal property [**18**], as follows. There exists a subscheme

$$U \subseteq \operatorname{Hilb}^n(\mathbf{C}^2) \times \mathbf{C}^2, \tag{4.2}$$

$$\begin{array}{ccc} U & \longrightarrow & \mathbf{C}^2 \\ {\scriptstyle\pi}\big\downarrow & & \\ \operatorname{Hilb}^n(\mathbf{C}^2), & & \end{array} \tag{4.3}$$

called the *universal family*, whose scheme-theoretic fiber over a point $I \in \operatorname{Hilb}^n(\mathbf{C}^2)$ is the corresponding subscheme $S \subseteq \mathbf{C}^2$. In particular, U is flat and finite of degree n over $\operatorname{Hilb}^n(\mathbf{C}^2)$. The universal property is that for any scheme T and any family $F \subseteq T \times \mathbf{C}^2$ of subschemes of $\mathbf{C}^2$,

$$\begin{array}{ccc} F & \longrightarrow & \mathbf{C}^2 \\ \big\downarrow & & \\ T, & & \end{array} \tag{4.4}$$

flat and finite of degree n over T, there is a unique morphism $\phi : T \to \operatorname{Hilb}^n(\mathbf{C}^2)$, such that F is the fiber product $F = T \times_{\operatorname{Hilb}^n(\mathbf{C}^2)} U$. The universal property, as usual, characterizes $\operatorname{Hilb}^n(\mathbf{C}^2)$ and the universal family U up to canonical isomorphism.

The following result, which is unique to $\mathbf{C}^2$ and fails in more than two dimensions, is one of the true miracles of the mathematical universe.

Theorem 1. [**8**] *The Hilbert scheme* $\operatorname{Hilb}^n(\mathbf{C}^2)$ *is irreducible and non-singular, of dimension* $2n$.

Given $I \in \operatorname{Hilb}^n(\mathbf{C}^2)$, the operators X and Y of multiplication by x and y, respectively, are commuting endomorphisms of the vector space

$$\mathbf{C}[x,y]/I \cong \prod_{P \in S} \mathcal{O}_{S,P}$$

which preserve the direct factors $\mathcal{O}_{S,P}$. For $P = (x_0, y_0)$, $X - x_0$ and $Y - y_0$ are nilpotent on $\mathcal{O}_{S,P}$, so that $\mathcal{O}_{S,P}$ is the characteristic subspace associated with the joint eigenvalue (x_0, y_0) of (X, Y). Thus the points $(x_1, y_1), \ldots, (x_n, y_n)$ of S, each included with its multiplicity in S, are the joint spectrum of (X, Y). In particular, the *polarized power sums* $p_{h,k}(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} x_i^h y_i^k$ satisfy the identity

$$p_{h,k}(\mathbf{x}, \mathbf{y}) = \operatorname{tr} X^h Y^k. \tag{4.5}$$

Note that trace map associated to the finite morphism $\pi : U \to \operatorname{Hilb}^n(\mathbf{C}^2)$ sends the regular function $x^h y^k$ on U (coming from $U \to \mathbf{C}^2$) to $\operatorname{tr} X^h Y^k$, so the latter is a regular function on $\operatorname{Hilb}^n(\mathbf{C}^2)$.

Now the symmetric group S_n acts on the variety $(\mathbf{C}^2)^n = \operatorname{Spec} \mathbf{C}[x_1, y_1, \ldots, x_n, y_n]$ of ordered n-tuples of points in the plane, and we may identify Spec of the ring of invariants as the variety of unordered n-tuples, or n-element multisets, $S^n\mathbf{C}^2 = \operatorname{Spec} \mathbf{C}[\mathbf{x}, \mathbf{y}]^{S_n}$. By a theorem of Weyl [**35**], the polarized power-sums $p_{h,k}$ generate this ring of invariants. It follows from this and the preceding paragraph that the map

$$\tau : \operatorname{Hilb}^n(\mathbf{C}^2) \to S^n\mathbf{C}^2$$

sending I to the multiset of points of the corresponding subscheme S, with their multiplicities in S, is a morphism. It extends to a morphism $\hat{\tau} : \operatorname{Hilb}^n(\mathbf{P}^2) \to S^n\mathbf{P}^2$, with τ being the restriction to $\hat{\tau}^{-1}(S^n\mathbf{C}^2)$. Since $\operatorname{Hilb}^n(\mathbf{P}^2)$ is a projective variety, τ is a projective morphism, called the *Chow morphism*. Note that τ restricts to an isomorphism of the open sets where the n points are distinct, and so is birational.

Definition. The *iso-spectral Hilbert scheme* X_n is the reduced fiber product

$$\begin{array}{ccc} X_n & \longrightarrow & (\mathbf{C}^2)^n \\ {\scriptstyle\sigma}\big\downarrow & & \big\downarrow \\ \operatorname{Hilb}^n(\mathbf{C}^2) & \xrightarrow{\ \tau\ } & S^n\mathbf{C}^2. \end{array} \tag{4.6}$$

In other words, a point of X_n is a point I of the Hilbert scheme, together with an ordered n-tuple of points $(x_1, y_1), \ldots, (x_n, y_n)$ whose underlying unordered multiset is $\tau(I)$, that is, the joint spectrum of the operators (X, Y) on $\mathbf{C}[x, y]/I$. We should stress here that the scheme-theoretic fiber product indicated by the above diagram is *not* reduced, but our definition is that X_n is the underlying reduced subscheme. What this reflects is that the equations $p_{h,k}(x_1, y_1, \ldots, x_n, y_n) = \operatorname{tr} X^h Y^k$, which define X_n set-theoretically, do not generate its ideal. Indeed we do not know a fully explicit set of generators for the ideal of X_n. Such a set of generators will necessarily be complicated, since it must specialize to give generators of all the ideals J_μ of Proposition 3.1.

It is possible to describe the ideal of X_n implicitly, as we shall do next.

Let $U^{\times n}$ denote the n-fold fiber product of the universal family U over the Hilbert scheme $\operatorname{Hilb}^n(\mathbf{C}^2)$. It is a subscheme of $\operatorname{Hilb}^n(\mathbf{C}^2) \times (\mathbf{C}^2)^n$, flat and finite of degree n^n, and generically reduced, since it has reduced fibers whenever S consists of n distinct points. As the Hilbert scheme is irreducible, this implies that $U^{\times n}$ is reduced.[5] Thus $U^{\times n}$ is just the set of tuples $I, (x_1, y_1), \ldots, (x_n, y_n)$ with all $(x_i, y_i) \in S = V(I)$, irrespective of multiplicities. X_n is the subscheme of $U^{\times n}$ consisting of tuples for which the points occur with their correct multiplicities in S. Generically, when S has n distinct points, this means the points (x_i, y_i) are a permutation of S.

Now let $B = \pi_* \mathcal{O}_U$, the sheaf of $\mathcal{O}$-algebras on $\mathrm{Hilb}^n(\mathbf{C}^2)$ such that $U = \operatorname{Spec} B$ as a scheme over $\mathrm{Hilb}^n(\mathbf{C}^2)$. Since π is flat of degree n, B is locally free of rank n, that is, it is the sheaf of sections of a rank-n vector bundle over the Hilbert scheme. Indeed, B is the *tautlogical bundle* whose fiber over a point $I \in \mathrm{Hilb}^n(\mathbf{C}^2)$ is $\mathbf{C}[x, y]/I$. We have $U^{\times n} = \operatorname{Spec} B^{\otimes n}$, and we seek to identify the ideal sheaf of X_n in $U^{\times n}$ as a submodule sheaf of the sheaf of $\mathcal{O}$-algebras $B^{\otimes n}$. Note that the S_n action here permutes the tensor factors.

Proposition 4.1. *Let*

$$B^{\otimes n} \otimes B^{\otimes n} \to B^{\otimes n} \to \bigwedge^n B \tag{4.7}$$

be the map given by multiplication, followed by operator Alt. *Then the ideal sheaf of X_n is the kernel of the map*

$$\phi : B^{\otimes n} \to (B^{\otimes n})^* \otimes \bigwedge^n B \tag{4.8}$$

induced by (4.7).

Proof. Since X_n is reduced, a section of $B^{\otimes n}$ belongs to the ideal sheaf of X_n if and only if, regarded as a regular function on X_n, it vanishes on any dense open subset. Hence it is enough to check the proposition generically, over the locus where S consists of n distinct points. Suppose s is a section which vanishes on X_n. Then so do gs and $\operatorname{Alt} gs$, for any section g of $B^{\otimes n}$. But since it is alternating, $\operatorname{Alt} gs$ also vanishes at any point $I, P_1, \ldots, P_n$ of $U^{\times n}$ for which two of the points P_i, P_j coincide. Hence it vanishes on $U^{\times n} \setminus X_n$ and thus on all of $U^{\times n}$, which means it is zero in $B^{\otimes n}$. The condition that s belongs to the kernel of (4.8) is precisely that $\operatorname{Alt} gs = 0$ for all g.

Conversely, suppose s does not vanish on X_n, and choose a point $Q = (I, P_1, \ldots, P_n) \in X_n$ outside the vanishing locus $V(s)$, with the P_i all distinct. After multiplying by a suitable g we can arrange that gs vanishes at all points of the S_n orbit of Q, except Q. Then $\operatorname{Alt} gs$ does not vanish at Q, so $\operatorname{Alt} gs \neq 0$. □

Now consider the situation over one of the distinguished points I_μ. The fiber $B(I_\mu)$ of the vector bundle B is $\mathbf{C}[x, y]/I_\mu$, and that of $B^{\otimes n}$ is $\mathbf{C}[\mathbf{x}, \mathbf{y}]/(I_\mu(x_1, y_1) + \cdots + I_\mu(x_n, y_n))$. Notice that every alternating polynomial Δ_D as in (3.1) vanishes modulo $(I_\mu(x_1, y_1) + \cdots + I_\mu(x_n, y_n))$, except for Δ_μ. In other words, the 1-dimensional space $\bigwedge^n B(I_\mu)$ is spanned by the image of Δ_μ, and the linear functional $\operatorname{Alt} : B^{\otimes n}(I_\mu) \to \bigwedge^n B(I_\mu) \cong \mathbf{C}\{\Delta_\mu\}$, composed with the natural projection $\mathbf{C}[\mathbf{x}, \mathbf{y}] \to B^{\otimes n}(I_\mu)$, is just the map sending f to the coefficient of Δ_μ in $\operatorname{Alt} f$. Together with Proposition 3.1, this proves

[5]By the same reasoning, the universal family U itself is reduced. This fails in higher dimension, when the Hilbert scheme is not irreducible.

Proposition 4.2. *The ideal J_μ of polynomials which as differential operators annihilate Δ_μ is the kernel of the map*

$$\mathbf{C}[\mathbf{x},\mathbf{y}] \to B^{\otimes n}(I_\mu) \to (B^{\otimes n})^*(I_\mu) \otimes \bigwedge^n B(I_\mu)$$

induced by the map ϕ in (4.8)*.*

Comparing this with Proposition 4.1 we might well expect the fiber of the ideal sheaf of X_n over I_μ to be J_μ, and the scheme-theoretic fiber of X_n over I_μ to be $\operatorname{Spec} R_\mu$. We cannot yet draw this conclusion, however, since the map ϕ is only a sheaf homomorphism, not necessarily a homomorphism of vector bundles, and thus the fiber of its kernel at I_μ need not equal the kernel of its fiber. What we can say is is that the fiber of the kernel factors through the kernel of the fiber, which gives the following result.

Proposition 4.3. *The image of the ideal J_μ in $B^{\otimes n}(I_\mu)$ contains the image of the fiber map $\mathcal{J}(I_\mu) \to B^{\otimes n}(I_\mu)$, where $\mathcal{J}(I_\mu)$ is the fiber of the ideal sheaf $\mathcal{J}$ of X_n at I_μ. As a consequence $\dim R_\mu \leq n!$, and R_μ is isomorphic to a submodule of the regular representation of S_n.*

Proof. The first part is clear, by the previous propositions. It only remains to prove the consequence.

By Proposition 4.1, we have an exact sequence of sheaves on $\operatorname{Hilb}^n(\mathbf{C}^2)$,

$$0 \to \mathcal{J} \to B^{\otimes n} \to \sigma_* \mathcal{O}_{X_n} \to 0, \tag{4.9}$$

in which $\sigma_*\mathcal{O}_{X_n}$ is the image of the sheaf homomorphism ϕ given by (4.8). At generic ideals $I \in \operatorname{Hilb}^n(\mathbf{C}^2)$ corresponding to sets of n distinct points $S \subseteq \mathbf{C}^2$, the fibers $\sigma_*\mathcal{O}_{X_n}(I)$ have constant dimension $n!$. This implies that the rank of the fiber map

$$\phi(I)\colon B^{\otimes n}(I) \to (B^{\otimes n})^*(I) \otimes \bigwedge^n B(I)$$

is generically $n!$.

At the special ideal I_μ the rank of $\phi(I_\mu)$ cannot exceed the generic rank, and by Proposition 4.2 this implies $\dim R_\mu \leq n!$. Since S_n acts equivariantly on everything, the same considerations apply to the isotypic components corresponding to each irreducible character of S_n, to show that the character multiplicities in R_μ cannot exceed those in a generic fiber $\sigma_*\mathcal{O}_{X_n}(I)$. When $I = I(S)$, this fiber is the coordinate ring of the set of all permutations of S, and thus affords the regular representation of S_n. □

Continuing with this argument, we see that if the $n!$ conjecture holds for μ then the rank of $\phi(I)$ does not decrease at I_μ, and so is constant on a neighborhood of I_μ. This is the criterion for ϕ to be locally a homomorphism of vector bundles on a neighborhood of I_μ. When this holds, the sheaves in (4.9) are locally free, and the scheme X_n is flat of degree $n!$ over $\operatorname{Hilb}^n(\mathbf{C}^2)$ at I_μ.

Conversely, if X_n is flat over $\operatorname{Hilb}^n(\mathbf{C}^2)$ at I_μ, then $B^{\otimes n}$ and $B^{\otimes n}/\mathcal{J}$ are both locally free, which implies that $\mathcal{J}$ is locally free and ϕ is a homorphism of vector bundles. By Propositions 4.1 and 4.2 it then follows that $\mathcal{J}(I_\mu) = J_\mu$ and $\dim R_\mu = n!$. Recall that a finite, surjective morphism $X \to H$ with H non-singular is flat if and only if X is Cohen-Macaulay. We have proved

Theorem 2. *The following are equivalent:*

1. *The $n!$ conjecture holds for the partition μ,*
2. *The map $X_n \to \mathrm{Hilb}^n(\mathbf{C}^2)$ is flat over a neighborhood of I_μ,*
3. *The iso-spectral Hilbert scheme X_n is Cohen-Macaulay in a neighborhood of the point $Q_\mu = (I_\mu, 0, \dots, 0)$.*

This theorem has an interesting connection with the Hilbert scheme of regular S_n orbits in $(\mathbf{C}^2)^n$ which we shall discuss briefly. Given a set S of n distinct points in the plane, its $n!$ permutations describe a regular orbit of S_n in $(\mathbf{C}^2)^n$. The ideal J of the orbit is a point of $\mathrm{Hilb}^{n!}((\mathbf{C}^2)^n)$. Let Z_n be the closure in $\mathrm{Hilb}^{n!}((\mathbf{C}^2)^n)$ of the set of such points. It is not hard to show that the set of ideals J which are S_n stable and for which $\mathbf{C}[\mathbf{x}, \mathbf{y}]/J$ has a given character is closed in $\mathrm{Hilb}^{n!}((\mathbf{C}^2)^n)$, so for every $J \in Z_n$, $\mathbf{C}[\mathbf{x}, \mathbf{y}]/J$ affords the regular representation of S_n.

Now there is a natural map from Z_n to $\mathrm{Hilb}^n(\mathbf{C}^2)$ which may be described as follows. Since $\mathbf{C}[\mathbf{x}, \mathbf{y}]/J$ affords the regular representation, its only invariants are the constants. This means that modulo J we have $p_{h,k}(\mathbf{x}, \mathbf{y}) = c_{h,k}$ for some constant $c_{h,k}$, for all h, k. By Weyl's theorem, the S_{n-1}-invariants in $\mathbf{C}[\mathbf{x}, \mathbf{y}]$, for the action of S_{n-1} on x_2, y_2 through x_n, y_n, are generated by x_1, y_1, and the polarized power-sums $p_{h,k}(x_2, y_2, \dots, x_n, y_n)$. Modulo J, the latter are congruent to $c_{h,k} - x_1^h y_1^k$, so the S_{n-1} invariants of $\mathbf{C}[\mathbf{x}, \mathbf{y}]/J$ are generated by x_1 and y_1. In other words, $\mathbf{C}[x_1, y_1]/(J \cap \mathbf{C}[x_1, y_1]) = (\mathbf{C}[\mathbf{x}, \mathbf{y}]/J)^{S_{n-1}}$. It follows that $J \cap \mathbf{C}[x_1, y_1]$ belongs to $\mathrm{Hilb}^n(\mathbf{C}^2)$, after identifying x_1, y_1 with x, y.

The above construction defines the map $Z_n \to \mathrm{Hilb}^n(\mathbf{C}^2)$, which also has the follow geometric description. Let W be the universal family over Z_n. It has a natural S_n-action, in which every fiber affords the regular representation. Then W/S_{n-1} is flat and finite of degree n over Z_n, and by the above calculation can be identified with a family of subschemes of $\mathbf{C}^2$. The map $Z_n \to \mathrm{Hilb}^n(\mathbf{C}^2)$ is the one given by the universal property of $\mathrm{Hilb}^n(\mathbf{C}^2)$, for the family W/S_{n-1}.

If the equivalent conditions of Theorem 2 hold, then X_n is a flat family of subschemes of $(\mathbf{C}^2)^n$, of degree $n!$ over $\mathrm{Hilb}^n(\mathbf{C}^2)$. The universal property of $\mathrm{Hilb}^{n!}((\mathbf{C}^2)^n)$ then yields a map $\mathrm{Hilb}^n(\mathbf{C}^2) \to \mathrm{Hilb}^{n!}((\mathbf{C}^2)^n)$, whose image lies in Z_n. Generically, for sets S of n distinct points in $\mathbf{C}^2$ and their corresponding regular orbits in $(\mathbf{C}^2)^n$, these two maps are mutually inverse. Hence, assuming the $n!$ conjecture, they are inverse everywhere, and the natural map $Z_n \to \mathrm{Hilb}^n(\mathbf{C}^2)$ is an isomorphism.

Conversely, if $Z_n \to \mathrm{Hilb}^n(\mathbf{C}^2)$ is an isomorphism, then its inverse defines a family $X' \subseteq \mathrm{Hilb}^n(\mathbf{C}^2) \times (\mathbf{C}^2)^n$ which is flat over $\mathrm{Hilb}^n(\mathbf{C}^2)$ and coincides with X_n generically. But then X' is reduced and hence equal to X_n, so X_n is flat, and the $n!$ conjecture holds.

It would even suffice to show that $Z_n \to \mathrm{Hilb}^n(\mathbf{C}^2)$ is injective. For it is proper and birational, hence surjective, and a bijective morphism onto a non-singular variety (or any normal variety) is an isomorphism, by Zariski's theorem. To summarize, we have

Proposition 4.4. *The equivalent conditions of Theorem 2, for all partitions μ of n, are also equivalent to*

(4) *The natural map $Z_n \to \mathrm{Hilb}^n(\mathbf{C}^2)$ is injective,*
(5) *This map is an isomorphism.*

Proposition 4.4 implies that the $n!$ conjecture is equivalent to an instance of a conjecture of Nakamura [**29**], cited in [**30**], connected with the *McKay correspondence*, as we now explain.

Let G be a finite subgroup of $SL(V)$, where $V = \mathbf{C}^n$. For any finite linear group action on V, $V/G = \operatorname{Spec} \mathcal{O}(V)^G$ is Cohen-Macaulay, and its canonical module is the isotypic component $\mathcal{O}(V)^{\delta}$ of $\mathcal{O}(V)$, where δ is the determinant representation of G on $\bigwedge^n(V)$. For $G \subseteq SL(V)$ the determinant is trivial and the canonical sheaf $\omega_{V/G}$ is equal to $\mathcal{O}_{V/G}$, that is, V/G is *Gorenstein*.

A resolution of singularities $H \to V/G$ is said to be *crepant* [6] if $\omega_H = \mathcal{O}_H$. The Gorenstein condition on V/G is necessary, but not sufficient, for a crepant resolution to exist. When they do exist, crepant resolutions need not be unique, but they do enjoy a good minimality property: given

$$Z \underset{f}{\to} H \to V/G,$$

if Z and H are both crepant resolutions, then f is an isomorphism.

The McKay correspondence is a conjecture to the effect that if H is a crepant resolution of V/G, then the dimension of the cohomology ring of the complex analytic manifold H is equal to the number of conjugacy classes (or irreducible representations) of G. A refinement of this given in [**2**] specifies the dimension of each cohomology group separately. For $V = (\mathbf{C}^2)^n$ and $G = S_n$, the Hilbert scheme $\mathrm{Hilb}^n(\mathbf{C}^2)$ is a crepant resolution, by Lemma 6.13, and the computation of its cohomology in [**7**] (see Lemma 6.4) verfies that the McKay correspondence holds in this case.

Returning to the general G, if $x \in V$ is chosen generically, then its G-orbit has $N = |G|$ elements, and thereby defines a point of the Hilbert scheme $\mathrm{Hilb}^N(V)$. Nakamura defines the G-Hilbert scheme $\mathrm{Hilb}^G(V)$ to be the closure of the locus in $\mathrm{Hilb}^N(V)$ consisting of such orbits. There is a canonical morphism $\mathrm{Hilb}^G(V) \to V/G$. Nakamura's conjecture is as follows.

Conjecture 4.1. $\mathrm{Hilb}^G(V)$ *is a crepant resolution of V/G whenever one exists.*

The relevance of this to the McKay correspondence is that there are vector bundles on $\mathrm{Hilb}^G(V)$ canonically associated to the irreducible representations of G. It is expected that when the conjecture applies, they will form a basis of the Grothendieck group, and the latter will be isomorphic to the cohomology ring, establishing the McKay correspondence for the resolution $\mathrm{Hilb}^G(V)$.

In our situation, Nakamura's $\mathrm{Hilb}^G(V)$ is our Z_n, and we have already established the factorization

$$Z_n \to \mathrm{Hilb}^n(\mathbf{C}^2) \to S^n\mathbf{C}^2 = V/G.$$

[6] The etymology of this term appears to be "not discrepant."

If Nakamura's conjecture holds in this case, then both Z_n and $\mathrm{Hilb}^n(\mathbf{C}^2)$ are crepant resolutions, hence they are isomorphic. Conversely if $Z_n \cong \mathrm{Hilb}^n(\mathbf{C}^2)$, then obviously Z_n is a crepant resolution. The equivalent conditions of Theorem 2 and Proposition 4.4 are therefore also equivalent to Conjecture 4.1 in the case $V = (\mathbf{C}^2)^n$, $G = S_n$.

5. Frobenius series

Let $\mathbf{T}$ denote the two dimensional algebraic torus group, *i.e.*, the multiplicative group $\mathbf{C}^* \times \mathbf{C}^*$. It acts algebraically on $\mathbf{C}^2$ by the rule

$$(t,q)\cdot(x,y) = (tx, qy), \quad (t,q) \in \mathbf{T},\ (x,y) \in \mathbf{C}^2.$$

The universal construction of the Hilbert scheme is functorial with respect to automorphisms of $\mathbf{C}^2$, so $\mathbf{T}$ also acts on $\mathrm{Hilb}^n(\mathbf{C}^2)$. This action sends a subscheme $S \subseteq \mathbf{C}^2$ to $(t,q)\cdot S$, so on ideals $I \subseteq \mathbf{C}[x,y]$ it is given by the pullback through the ring endomorphism (t,q), that is,

$$(t,q)\cdot I = I(x/t, y/q).$$

Similarly $\mathbf{T}$ acts on the the schemes U, $U^{\times n}$, $(\mathbf{C}^2)^n$ and X_n, and the various maps between these schemes are $\mathbf{T}$-equivariant.

Observe that a polynomial $p(\mathbf{x},\mathbf{y})$ is doubly homogeneous of degree (r,s) if and only if it is an eigenfunction for the $\mathbf{T}$ action with eigenvalue $t^r q^s$. Hence the bivariate Hilbert series of a finite-dimensional doubly graded space of polynomials D is simply the character of the $\mathbf{T}$-action, as a function of q and t. Similar considerations apply to the Frobenius series when S_n acts, as it is just a generating function for the Hilbert series of the the various S_n-isotypic subspaces.

In the geometric situation we have to deal with $\mathbf{T}$ actions on local rings and sheaves of modules which may be neither graded nor finite-dimensional. For this we need recourse to a "formal" Hilbert series which captures the naive $\mathbf{T}$ character in the finite dimensional case, and extends to the more general setting.

Definition. Let R be the local ring of a scheme X of finite type over $\mathbf{C}$ at a closed point x with maximal ideal $\mathfrak{m}$. Assume X is non-singular at x and that x is an isolated fixed point for an algebraic action of $\mathbf{T}$ on X. Let M be a finitely generated R-module with an equivariant $\mathbf{T}$ action. Then the *formal Hilbert series* of M is given by

$$\mathcal{H}_M(q,t) = \frac{\sum_i (-1)^i \operatorname{tr}(\operatorname{Tor}_i^R(M,\mathbf{C}),\lambda)}{\det(\mathfrak{m}/\mathfrak{m}^2, 1-\lambda)}, \quad \lambda = (t,q) \in \mathbf{T}. \tag{5.1}$$

To see that the definition is sound, observe first that the modules $\operatorname{Tor}_i^R(M,\mathbf{C})$ and the cotangent space $\mathfrak{m}/\mathfrak{m}^2$ are finite dimensional representations of $\mathbf{T}$, so the trace and determinant in the formula make sense. Since R is regular, the syzygy theorem implies that the sum in the numerator is finite. Since x is an isolated fixed point, the $\mathbf{T}$ action on the cotangent space does not have 1 as an eigenvalue, so the denominator does not vanish identically, but is a product of factors of the form $(1 - t^r q^s)$ with r, s not both zero. Note that $\mathcal{H}_M(q,t)$ is a rational function of q and t.

Proposition 5.1. *We have*

(1) *If* $0 \to M \to N \to P \to 0$ *is an exact sequence, then* $\mathcal{H}_N = \mathcal{H}_M + \mathcal{H}_P$.
(2) *If* M *has finite length then* $\mathcal{H}_M$ *is its ordinary bivariate Hilbert series as a doubly graded space.*

Proof. (1) follows from the long exact sequence for Tor and the additivity of the trace on exact sequences. In view of (1), (2) reduces to the case $M = \mathbf{C}$. Since R is regular, we have in this case from the Koszul resolution of $\mathbf{C}$ that $\operatorname{Tor}_i(M, \mathbf{C}) \cong M \otimes \bigwedge^i T_x^*$, where $T_x^* = \mathfrak{m}/\mathfrak{m}^2$ is the cotangent space. Here we have kept the one-dimensional factor M explicit, since $\mathbf{T}$ might not act trivially on it. Let $t^{r_1}q^{s_1}, \ldots, t^{r_d}q^{s_d}$ be the eigenvalues of $\mathbf{T}$ on T_x^*, repeated according to their multiplicities. Then the denominator is

$$\prod_j (1 - t^{r_j} q^{s_j}) = \sum_i (-1)^i e_i(t^{r_1}q^{s_1}, \ldots, t^{r_d}q^{s_d}),$$

while the numerator is the same thing multiplied by the $\mathbf{T}$ character of M. □

When M has an S_n action commuting with the $\mathbf{T}$ action, we can define a formal Frobenius series in an analogous manner. The character of a finite dimensional $S_n \times \mathbf{T}$ module V is now a function $\operatorname{tr}(V, (w, \lambda))$ of both $w \in S_n$ and $\lambda = (t, q) \in \mathbf{T}$, which we can regard as a $\mathbf{Q}(q,t)$-valued character on S_n. With the same definition of the Frobenius map Φ as in (3.3), $\Phi \operatorname{ch} V$ is now a symmetric function with coefficients in $\mathbf{Q}(q,t)$. Indeed if we identify the $\mathbf{T}$ action with a double grading of V then $\Phi \operatorname{ch} V$ in this sense is our earlier Frobenius series $\mathcal{F}_V(X; q, t)$.

Definition. With R, X, x, and M as in the definition of formal Hilbert series, and an action of S_n by R-module automorphisms of M commuting with the $\mathbf{T}$ action, the *formal Frobenius series* of M is given by

$$\mathcal{F}_M(X; q, t) = \frac{\sum_i (-1)^i \Phi \operatorname{ch}(\operatorname{Tor}_i^R(M, \mathbf{C}))}{\det(\mathfrak{m}/\mathfrak{m}^2, 1 - \lambda)}, \quad \lambda = (t, q) \in \mathbf{T}.$$

The formula makes sense for the same reasons as in the Hilbert series case and the analog of Proposition 5.1 holds.

Proposition 5.2. *We have*

(1) *If* $0 \to M \to N \to P \to 0$ *is an exact sequence, then* $\mathcal{F}_N = \mathcal{F}_M + \mathcal{F}_P$.
(2) *If* M *has finite length then* $\mathcal{F}_M$ *is its ordinary Frobenius series as a doubly graded* S_n *representation.*

Proof. The same as the previous Proposition, except that for (2) we reduce to the case that M is an irreducible $S_n \times \mathbf{T}$-equivariant R module. This means M is an irreducible representation of S_n (with trivial $\mathbf{T}$ action) tensored over $\mathbf{C}$ by a trivial R module $\mathbf{C}$ (with some one-dimensional $\mathbf{T}$ action). □

Further properties of the formal Frobenius series are given by the next proposition. The last one is especially important, as it provides a geometric interpretation of plethystic substiution.

Proposition 5.3. *The formal Frobenius series $\mathcal{F}_M$ satisfies the following identities:*

(1) *Suppose the S_n character χ_λ has multiplicity zero in $M/\mathfrak{m}M$. Then $\langle s_\lambda, \mathcal{F}_M\rangle = 0$.*
(2) *Let V be a finite-dimensional doubly graded S_n module. Then (tensoring over $\mathbf{C}$) $\mathcal{F}_{V\otimes M} = \mathcal{F}_V * \mathcal{F}_M$, where $*$ is the internal product of symmetric functions.*
(3) *Suppose M is an S_n-equivariant S-module, where S is a finite R-algebra with an S_n action. Suppose $x_1, \dots, x_n \in S$ is an M-regular sequence, that $\mathbf{T}$ acts on the elements x_i by $(t,q)\cdot x_i = tx_i$, and that S_n acts on these elements by permuting them. Then $\mathcal{F}_{M/(\mathbf{x})M}(X;q,t) = \mathcal{F}_M[X(1-t);q,t]$.*

Proof. (1) Let Θ_λ be the Reynolds operator, the central idempotent in the group algebra of S_n which acts in each representation as the projection on the S_n isotypic component with character χ^λ. If M is a finitely generated R-module with $S_n\times\mathbf{T}$ action then it has a canonical isotypic decomposition

$$M = \bigoplus_{|\lambda|=n} \Theta_\lambda M,$$

and each $\Theta_\lambda M$ is also a finitely generated R module with $S_n\times\mathbf{T}$ action. The decomposition means that the identity functor is naturally isomorphic to the direct sum of the functors $M\mapsto \Theta_\lambda M$. In particular the functors $M\mapsto\Theta_\lambda M$ are exact, and commute with Tor, since the S_n action does.

Now, comparing the definitions of the formal Hilbert and Frobenius series, and using the fact that $\Phi\chi^\lambda = s_\lambda$, we see that

$$\langle s_\lambda, \mathcal{F}_M\rangle = \frac{1}{\chi_\lambda(1)}\mathcal{H}_{\Theta_\lambda M}.$$

By Nakayama's Lemma, if $\Theta_\lambda(M/\mathfrak{m}M) = \Theta_\lambda M/\mathfrak{m}\Theta_\lambda M = 0$, then $\Theta_\lambda M = 0$.

(2) Recall that the internal product $*$ is defined by $p_\lambda * p_\mu = \langle p_\lambda, p_\mu\rangle p_\lambda$, and satisfies the identity

$$\Phi(\chi\otimes\mu) = \Phi(\chi) * \Phi(\mu)$$

for all characters χ, μ. From this it is clear that (2) holds in the case where M has finite length, by Proposition 5.2. For the general case we have $\mathrm{Tor}_i(V\otimes M,\mathbf{C}) = V\otimes\mathrm{Tor}_i(M,\mathbf{C})$, which reduces the identity to the corresponding one with the finite-length modules $\mathrm{Tor}_i(M,\mathbf{C})$ in place of M.

(3) Let V denote the space spanned by elements of the regular sequence $\mathbf{x}$; as an $S_n\times\mathbf{T}$ module it affords the permutation representation of S_n tensored by the one-dimensional representation of $\mathbf{T}$ with character t. Since $\mathbf{x}$ is a regular sequence we have the following exact sequence, the Koszul resolution of $M/(\mathbf{x})M$:

$$0\to M\otimes\bigwedge^n V\to\cdots\to M\otimes\bigwedge^1 V\to M\otimes\bigwedge^0 V\to M/(\mathbf{x})M\to 0, \tag{5.2}$$

where the tensor products are over $\mathbf{C}$. The maps in the Koszul complex are $S_n\times\mathbf{T}$ equivariant if we take the $\mathbf{T}$ action on $\bigwedge^k V$ to be multiplication by t^k. From part (1) of Proposition 5.2, together with (2) above, we deduce that $\mathcal{F}_{M/(\mathbf{x})M} = \mathcal{F}_M * g(X;t)$, where $g(X;t) = \sum_k (-1)^k t^k \Phi\,\mathrm{ch}\bigwedge^k V$.

Now I claim that $g(X;t) = h_n[X(1-t)]$. To prove this it suffices to show that for each power-sum p_λ, $|\lambda| = n$, we have

$$\langle g(X;t), p_\lambda \rangle = \langle h_n[X(1-t)], p_\lambda \rangle = p_\lambda[1-t].$$

The second equality here comes from (2.6). For any character χ, it follows from the definition of the Frobenius map that $\langle \Phi\chi, p_\lambda \rangle = \chi(w)$, where w is a permutation whose cycle lengths are the parts of λ. The symmetric group acts on $\bigwedge^k V$ by signed permutations of the basis of monomials $x_{i_1} \wedge \cdots \wedge x_{i_k}$, and such a monomial is stabilized by w if and only if, for each cycle of w, either all or none of the corresponding variables appear in the monomial. In that case $w(x_{i_1} \wedge \cdots \wedge x_{i_k}) = \pm x_{i_1} \wedge \cdots \wedge x_{i_k}$, the sign being $(-1)^{k+r}$ if the variables for r of the cycles appear. Hence the trace of w on $\bigwedge^k V$ is the sum of $(-1)^{k+r}$ over all collections of the parts of λ which add up to k, where r is the number of parts included. It follows that

$$\sum_k (-1)^k t^k (\operatorname{ch} \bigwedge^k V)(w) = \prod_i (1 - t^{\lambda_i}) = p_\lambda[1-t].$$

Finally, we have

$$h_n[X(1-t)] * p_\lambda = \langle h_n[X(1-t)], p_\lambda \rangle p_\lambda = p_\lambda[1-t] p_\lambda = p_\lambda[X(1-t)],$$

where the last equality holds because of the identity $p_k[AB] = p_k[A]p_k[B]$. Since both sides are linear in p_λ it follows that

$$h_n[X(1-t)] * f = f[X(1-t)]$$

for all symmetric functions f. □

The remainder of this section is devoted to the proof of the following theorem.

Theorem 3. *If the $n!$ conjecture holds for μ, then the Frobenius series of R_μ is given by*

$$\mathcal{F}_{R_\mu} = \tilde{H}_\mu,$$

so the Macdonald positivity conjecture holds for $K_{\lambda\mu}(q,t)$, for all λ.

For the rest of the discussion we *assume the $n!$ conjecture holds for μ*, so that X_n is locally Cohen-Macaulay at the point Q_μ.

We take R to be the local ring of $\operatorname{Hilb}^n(\mathbf{C}^2)$ at I_μ. Note that $\operatorname{Hilb}^n(\mathbf{C}^2)$ is non-singular, and I_μ is a $\mathbf{T}$ fixed point, since this is equivalent to $I_\mu \subseteq \mathbf{C}[x,y]$ being doubly homogeneous, and thus spanned by monomials. As remarked earlier, the ideals I_μ are the only such ideals, so I_μ is an isolated fixed point.

The local ring S of X_n at Q_μ is a finite R-algebra on which $\mathbf{T}$ acts equivariantly. The symmetric group S_n also acts on S by R-algebra automorphisms, commuting with the $\mathbf{T}$ action.

Via the map $X_n \to (\mathbf{C}^2)^n$, the coordinates $x_1, y_1, \ldots, x_n, y_n$ on $(\mathbf{C}^2)^n$ define global regular functions on X_n and thus elements of the local ring S.

Lemma 5.4. *If X_n is Cohen-Macaulay at Q_μ then $y_1, \ldots, y_n$ is a regular sequence.*

Proof. We are to show that $\mathbf{y} = y_1, \dots, y_n$ cut out a complete intersection in X_n. Since $\dim(X_n) = 2n$ we must show that $\dim V(\mathbf{y}) = n$. Now $V(\mathbf{y})$ consists of those points $(I, P_1, \dots, P_n) \in X_n$ for which all the points P_i lie on the x-axis, which is the same as saying that $V(I)$ lies on the x-axis.

Ellingsrud and Strömme [**7**], studying the cohomology of $\mathrm{Hilb}^n(\mathbf{P}^2)$, constructed a cell decomposition which contains within it a cell decomposition of the subset $H_X \subseteq \mathrm{Hilb}^n(\mathbf{C}^2)$ consisting of points I for which $V(I)$ lies on the x-axis. Every cell in their decomposition of H_X has dimension n.

Since the locus $V(\mathbf{y}) \subseteq X_n$ is finite over H_X its dimension is n as well. □

The local ring S of X_n at Q_μ is not only finite over R, it is free, and $S/\mathfrak{m}S \cong R_\mu$, by the proof of Theorem 2, where $\mathfrak{m}$ is the maximal ideal of R. By Nakayama's lemma, S is freely generated as an R module by any subspace D complementary to the ideal $\mathfrak{m}S$. We may choose D to be $S_n \times \mathbf{T}$ stable (in fact, we can choose D to be the space of derivatives D_μ), and then D will have the same Frobenius series as R_μ. This shows that

$$\mathcal{F}_S(X; q, t) = \mathcal{F}_{R_\mu}(X; q, t)\mathcal{H}_R(q, t). \tag{5.3}$$

Incidentally, the quantity $\mathcal{H}_R(q, t)$ has a tantalizing explicit value. In [**20**] we constructed an explicit system of doubly homogeneous regular local parameters for R. From their degrees one obtains

$$\mathcal{H}_R(q, t) = \frac{1}{\prod_{s \in D(\mu)}(1 - q^{-a(s)}t^{1+l(s)}) \prod_{s \in D(\mu)}(1 - q^{1+a(s)}t^{-l(s)})},$$

where the *arms* and *legs* $a(s)$, $l(s)$ are as in (2.16). After the replacement $q \mapsto q^{-1}$ the first factor in the denominator is exactly the normalizing factor for the definition of Macdonald's integral forms. This coincidence is a typical example of the links between the numerology associated with Macdonald polynomials and geometrically significant quantities attached to the Hilbert scheme.

Now let us consider the ring $S/(\mathbf{y})$. Just as S is generated as an R module by any subspace representing $S/\mathfrak{m}S$, $S/(\mathbf{y})$ is generated (no longer freely, however) by representatives of $S/((\mathbf{y}) + \mathfrak{m}S) = R_\mu/(\mathbf{y})$. This last space can be identified with the component of R_μ, or of D_μ, homogeneous of degree zero in y. As mentioned in the proof of Proposition 3.5, this space is well-understood, and its Frobenius series is

$$t^{n(\mu)}Q_\mu[X/(1 - t^{-1}); t^{-1}] = \sum_\lambda t^{n(\mu)} K_{\lambda\mu}(t^{-1}) s_\lambda,$$

where Q_μ is a Hall-Littlewood polynomial and $K_{\lambda\mu}(t)$ denotes the classical one-variable Kostka coefficient. In particular, since $K_{\lambda\mu}(t) = 0$ unless $\lambda \geq \mu$, the space $R_\mu/(\mathbf{y})$ contains only those S_n representations χ^λ with $\lambda \geq \mu$. It follows, by Proposition 5.3, part (1), that

$$\mathcal{F}_{S/(\mathbf{y})} \in \mathbf{Q}(q, t)\{s_\lambda : \lambda \geq \mu\}.$$

Now using Proposition 5.3, part (3), with q in place of t, this implies

$$\mathcal{F}_S[X(1 - q)] \in \mathbf{Q}(q, t)\{s_\lambda : \lambda \geq \mu\},$$

and hence, by (5.3),
$$\mathcal{F}_{R_\mu}[X(1-q)] \in \mathbf{Q}(q,t)\{s_\lambda : \lambda \geq \mu\}.$$
Everything we have done applies symmetrically, with $\mathbf{x}$ in place of $\mathbf{y}$ and t in place of q, to show that also
$$\mathcal{F}_{R_\mu}[X(1-t)] \in \mathbf{Q}(q,t)\{s_\lambda : \lambda \geq \mu'\}.$$
Finally, since R_μ affords the regular representation (this follows from the $n!$ conjecture by Proposition 4.3), its only S_n invariants are the constants, so
$$\langle \mathcal{F}_{R_\mu}, s_{(n)} \rangle = 1.$$
By Proposition 2.6 these three conditions imply that $\mathcal{F}_{R_\mu} = \tilde{H}_\mu$, and the proof of Theorem 3 is complete.

6. The ideals J and J^m

Let $R = \mathbf{C}[\mathbf{x},\mathbf{y}] = \mathbf{C}[x_1, y_1, \ldots, x_n, y_n]$, and let J denote the ideal in R generated by all S_n-alternating polynomials, that is, by the polynomials $\Delta_D(\mathbf{x},\mathbf{y})$ of (3.1). Since any alternating polynomial must vanish whenever two of the points (x_i, y_i) and (x_j, y_j) coincide, it follows that
$$J \subseteq \bigcap_{i<j} (x_i - x_j, y_i - y_j), \tag{6.1}$$
and more generally
$$J^m \subseteq \bigcap_{i<j} (x_i - x_j, y_i - y_j)^m. \tag{6.2}$$
We shall denote the ideal on the right hand side of (6.2) by $J^{(m)}$; it is the m-th *symbolic power* of $J^{(1)}$.

Conjecture 6.1. *We have $J^m = J^{(m)}$ for all m, i.e., we have equality in (6.2).*

Here is a result which at least gives the impression of reducing the conjecture to something simpler.

Proposition 6.1. *Suppose that for all $n \geq 3$, $(x_1 - x_2, x_2 - x_3)$ is a regular sequence for the R module J^m. Then $J^m = J^{(m)}$.*

Proof. The proof is by induction on n. Note that for $n = 1$ and $n = 2$ we trivially have $J^m = J^{(m)}$, and we have the remaining cases through $n - 1$ by induction.

First consider the situation locally at a point $P \in \mathbf{C}[\mathbf{x},\mathbf{y}]$ where the (x_i, y_i) are not all equal. Without loss of generality we can assume that none of $(x_1, y_1), \ldots, (x_r, y_r)$ is equal to any of $(x_{r+1}, y_{r+1}), \ldots, (x_n, y_n)$. In the local ring R_P, the differences $x_i - x_j$, $y_i - y_j$ are invertible whenever i is in the first group and j in the second. Hence $J^{(m)}$ reduces locally to the product of the ideals $J^{(m)}$ in the first r indices and the last $n - r$ separately.

Less obvious, but still true, is that J, and hence J^m, decomposes similarly. To see this, let g be a generator of $J' = J(x_1, y_1, \ldots, x_r, y_r)J(x_{r+1}, y_{r+1}, \ldots, x_n, y_n)$, alternating in the first r and last $n - r$ indices, *i.e.*, the subgroup $S_r \times S_{n-r} \subseteq S_n$ acts on g by the sign character.

Let h be any polynomial which belongs to the localization J_Q at every point $Q \neq P$ in the S_n orbit of P, but does not vanish at P. Now $f = \operatorname{Alt} gh$ belongs to J. The terms in the alternation corresponding to elements $w \in S_n$ which do not stablize P belong to J_P, by construction of h. Since g is already alternating with respect to the stabilizer of P, the remaining terms sum to $g \sum_{wP=P} wh$, and the sum here is invertible in R_P. This shows $g \in J_P$, and so $J'_P \subseteq J_P$. The reverse inclusion $J \subseteq J'$ is clear.

Using the induction hypothesis, we conclude that $J^m = J^{(m)}$ locally outside the locus V where all n points coincide. Now since $x_1 - x_2$ and $x_2 - x_3$ belong the ideal of V, our hypothesis implies $\operatorname{depth}_V J^m \geq 2$, and the local cohomology exact sequence for the sheaf of ideals $\tilde{J}^m$ associated to J^m gives

$$0 = H^0_V(\tilde{J}^m) \to H^0(\mathbf{C}^2, \tilde{J}^m) = J^m \to H^0(U, \tilde{J}^m) \to H^1_V(\tilde{J}^m) = 0, \tag{6.3}$$

where $U = \mathbf{C}^2 \setminus V$. Thus $J^m = H^0(U, \tilde{J}^m) = H^0(U, \widetilde{J^{(m)}})$. The latter is the ideal of all polynomials whose restrictions to U belong locally to $J^{(m)}$, so we have shown $J^m \supseteq J^{(m)}$. As we had $J^m \subseteq J^{(m)}$ to begin with, we must have $J^m = J^{(m)}$. □

There is of couse nothing special about the choice of $x_1 - x_2$, $x_2 - x_3$ in the above Proposition; it's just an explicit way to guarantee that $\operatorname{depth}_V J^m \geq 2$. This would also follow if $\operatorname{depth}_V(R/J^m) \geq 1$, which means the ideal of V contains an element which is a non-zero-divisor modulo J^m. Note, by the way, that the proof of Proposition 6.1 works equally well with more than two sets of variables.

Some explorations we have done for small values of n using the computer algebra system MACAULAY [**1**] suggest the following conjecture.

Conjecture 6.2. *If J denotes the ideal generated by the S_n alternants in $\mathbf{C}[\mathbf{x}, \mathbf{y}, \ldots, \mathbf{z}]$, for any number of sets of n variables, and V is the locus where all the points $(x_i, y_i, \ldots, z_i)$, $(x_j, y_j, \ldots, z_j)$ coincide, then $x_1 - x_2, x_2 - x_3, \ldots, x_{n-1} - x_n$ is a maximal J^m-regular sequence in the ideal of V, for all m. In particular,* $\operatorname{depth}_V J^m = n - 1$.

We remark that if the sequence $\mathbf{x}$ in question is regular then it is maximal: modulo $(\mathbf{x})J$, the Vandermonde determinant $v(\mathbf{x})$ is annihiliated by the ideal of V, so $\operatorname{depth}_V J/(\mathbf{x})J = 0$.

The relevance of all this to X_n and the $n!$ conjecture is given by the following proposition.

Proposition 6.2. *The iso-spectral Hilbert scheme X_n is the blowup* $\operatorname{Proj} R[tJ]$ *of $(\mathbf{C}^2)^n$ at the ideal J.*

Proof. We only outline the proof, as the analogous result for the ordinary Hilbert scheme $\operatorname{Hilb}^n(\mathbf{C}^2)$ was given in [**20**], and the proof carries over to X_n with only superficial modifications.

First, one shows that the pullback of the ideal J to X_n becomes locally principal. In fact, it can be identified with $\bigwedge^n B$, where B is the tautological bundle (of $\operatorname{Hilb}^n(\mathbf{C}^2)$, lifted to X_n). By the universal property of the blowup, this gives a morphism from X_n to $\operatorname{Proj} R[tJ]$. This map is projective and generically an isomorphism, so it's surjective. To show it's also a closed embedding, we have to show that all regular functions on X_n are pulled back from $\operatorname{Proj} R[tJ]$.

The regular functions on X_n are the coodinates x_i, y_i, which come from R, and the lifts of regular functions on the Hilbert scheme. But the proof of the result for $\mathrm{Hilb}^n(\mathbf{C}^2)$ shows the latter are generated by fractions of the form Δ_D/Δ_μ, which are (local) regular functions on $\operatorname{Proj} R[tJ]$. □

In d sets of variables, the above proposition applies as well to the iso-spectral Hilbert scheme of points in $\mathbf{C}^d$, with an important qualification. For general d these Hilbert schemes are not irreducible and even have components of dimension greater than dn [**22**]. What the blowup construction gives is the component which is the closure of the locus corresponding to reduced subschemes of n distinct points in $\mathbf{C}^d$. We suspect that this *generic component* may have good geometric properties, indeed may be Cohen-Macaulay and even Gorenstein for all d. Note that this would not imply the $n!$ conjecture in more sets of variables, since the generic component of $\mathrm{Hilb}^n(\mathbf{C}^d)$ may be singular, and thus the iso-spectral Hilbert scheme need not be flat over it.

In the remainder of this section we prove the following reduction of the $n!$ conjecture to Conjecture 6.2.

Theorem 4. *If Conjecture 6.2 holds in two sets of variables for all m and n, then X_n is Cohen-Macaulay and normal for all n.*

The proof is by induction on n, using geometric properties of the *nested Hilbert scheme* $H^{n-1,n}$ to be defined shortly. We employ a local cohomology argument in the same spirit as the proof of Proposition 6.1. For this we need a large open set where we may assume the result by induction, which the next lemma provides.

Lemma 6.3. *Let $P = (I, P_1, \dots, P_n)$ be a point of X_n. Let the distinct points among $P_1, \dots, P_n$ be $Q_1, \dots, Q_k$, with multiplicities $r_1, \dots, r_k$. Then in X_n there is a neigborhood of P isomorphic to an open set in the product $X_{r_1} \times \cdots \times X_{r_k}$.*

Proof. Without loss of generality we can assume $Q_1 = P_1 = \cdots = P_{r_1}$, $Q_2 = P_{r_1+1} = \cdots = P_{r_1+r_2}$, and so on. For our neighborhood of P we can take the preimage in X_n of the open set $U \subseteq (\mathbf{C}^2)^n$ of points where the only coincidences $P_i = P_j$ that occur have i, j within one of these k consecutive blocks. Then the result is clear from Proposition 6.2, together with the product decomposition, valid on U, of the ideal J as $J_{1,\dots,r_1} J_{r_1+1,\dots,r_1+r_2} \cdots$ from the proof of Proposition 6.1. □

For some dimension arguments below we will need the following results.

Lemma 6.4. *There is a decomposition of* $\mathrm{Hilb}^n(\mathbf{C}^2)$ *into locally closed affine cells C_μ, such that every point of C_μ contains I_μ in the closure of its* $\mathbf{T}$ *orbit, and* $\dim C_\mu = n + l(\mu)$*, where $l(\mu)$ is the number of parts of μ.*

Lemma 6.5. *There is a decomposition of the zero-fiber $H_0^n = \tau^{-1}(0) \subseteq$* $\mathrm{Hilb}^n(\mathbf{C}^2)$ *into locally closed affine cells C'_μ, such that every point of C'_μ contains I_μ in the closure of its* $\mathbf{T}$ *orbit, and* $\dim C'_\mu = l(\mu) - 1$.

Proof. See [**7**]. □

Lemma 6.6. *Let G_r be the (closed) locus of ideals $I \in \mathrm{Hilb}^n(\mathbf{C}^2)$ for which some point of $V(I)$ has multiplicity at least r. Then G_r has codimension $r-1$, and has only one irreducible component of maximal dimension.*

Proof. It is known [**4**] that the zero-fiber $H_0^n = \tau^{-1}(0)$ is irreducible of dimension $n-1$. The locus where all the points coincide is just the product of $\mathbf{C}^2$ (for the choice of origin) by H_0^n, so it is irreducible of dimension $n+1$.

By Lemma 6.3, it follows that the (locally closed) locus where the multiplicities are $r_1, \ldots, r_k$ has dimension $\sum_i (r_i + 1) = n + k$ and codimension $n - k$. If one multiplicity is at least r, this codimension is at least $r-1$, with equality only for multiplicities $r, 1, \ldots, 1$. Again by Lemma 6.3, the locus in X_n where $P_1 = \cdots = P_r$, and the other points are distinct from P_1 and each other, is irreducible. It surjects on the locus in $\mathrm{Hilb}^n(\mathbf{C}^2)$ where the multiplicities are $r, 1, \ldots, 1$, so the latter is irreducible as well. □

As a step toward the Cohen-Macaulay property we need normality results for X_n and U_n.

Definition. An ideal $I \in \mathrm{Hilb}^n(\mathbf{C}^2)$ is *curvilinear* if the local rings $\mathcal{O}_{S,P}$ have embedding dimension 1, *i.e.*, their maximal ideals are principal.

This is equivalent to $S = V(I)$ being a subscheme of a smooth curve in $\mathbf{C}^2$, whence the name.

Lemma 6.7. *The locus W of curvilinear ideals $I \in \mathrm{Hilb}^n(\mathbf{C}^2)$ is open and equal to $\bigcup_z W_z$, where $z = ax+by$ is a linear form, and W_z is the open set of ideals I such that $\{1, z, \ldots, z^{n-1}\}$ is a basis of $\mathbf{C}[x,y]/I$.*

Proof. If I is curvilinear, then for a generically chosen linear form z, the values $z(P)$ at distinct points $P \in V(I)$ will be distinct, and for each P, $z - z(P)$ will be a local parameter generating the maximal ideal $\mathfrak{m}_P \subseteq \mathcal{O}_{S,P}$. This implies that, as a $\mathbf{C}[z]$ module and hence as a ring,

$$\mathbf{C}[x,y]/I \cong \mathbf{C}[z]/\prod_P (z - z(P))^{r_P}, \tag{6.4}$$

where r_P is the multiplicity of P, and therefore $\{1, z, \ldots, z^{n-1}\}$ is a basis of $\mathbf{C}[x,y]/I$. Conversely, if $\{1, z, \ldots, z^{n-1}\}$ is a basis, then z generates $\mathbf{C}[x,y]/I$, and I contains a monic polynomial of degree n in z, so (6.4) holds and I is curvilinear. □

Lemma 6.8. *The universal scheme U over $\mathrm{Hilb}^n(\mathbf{C}^2)$ is Cohen-Macaulay and normal.*

Proof. U is Cohen-Macaulay because it is flat over $\mathrm{Hilb}^n(\mathbf{C}^2)$. Hence it is normal if its singular locus has codimension at least 2.

Now I claim that over the curvilinear locus W, U is non-singular. After a linear transformation of $\mathbf{C}^2$, we can restrict to W_x. For $I \in W_x$, we have y and x^n congruent mod I to unique polynomials of degree at most $n-1$ in x, so I contains elements

$$(x^n - e_1 x^{n-1} + e_2 x^{n-2} - \cdots + (-1)^n e_n,\, y - (a_{n-1}x^{n-1} + \cdots + a_1 x + a_0)), \tag{6.5}$$

where the parameters e_i and a_i are regular functions of I on W_x. On the other hand these two equations clearly generate a complete intersection ideal $I \subseteq \mathbf{C}[x, y]$ modulo which $1, x, \ldots, x^{n-1}$ are a basis, so they determine I. This exhibits W_x explicitly as an affine cell with coordinates e_i, a_i.[7] Moreover, regarded as equations on $W_x \times \mathbf{C}^2 = \operatorname{Spec} \mathbf{C}[\mathbf{e}, \mathbf{a}, x, y]$, equations (6.5) define the universal family U.

Viewing the first equation as eliminating e_n and the second as eliminating y we conclude that the open subset of U lying over W_x is $\operatorname{Spec} \mathbf{C}[x, e_1, \ldots, e_{n-1}, a_0, \ldots, a_{n-1}]$ and in particular is non-singular. It follows that U is non-singular over the whole curvilinear locus.

Finally, if I is not curvilinear, then some point of $V(I)$ has to have multiplicity at least 3. By Lemma 6.6 this occurs only on a locus of codimension 2. Since U is finite over $\operatorname{Hilb}^n(\mathbf{C}^2)$, its singular locus also has codimension at least 2. ☐

Lemma 6.9. *If Conjecture 6.2 holds for all m and any given n, then X_n is normal.*

Proof. Recall that an ideal J in a normal domain R is said to be *integrally closed* if every element $x \in R$ satisfying

$$x^n \in Jx^{n-1} + J^2 x^{n-1} + \cdots + J^n \tag{6.6}$$

already belongs to J. The above condition for J^m is equivalent to saying that $t^m x$ belongs to the integral closure of $R[tJ]$ in $R[t]$, so all the ideals J^m are integrally closed if and only if $R[tJ]$ is normal.

For our R and J, Conjecture 6.2 implies $J^m = J^{(m)}$, by Proposition 6.1. It is well-known that the powers of an ideal generated by a regular sequence are integrally closed, and it is obvious that an intersection of integrally closed ideals is integrally closed, so $J^{(m)}$ is integrally closed.

This shows that $R[tJ]$ is normal, so $X_n = \operatorname{Proj} R[tJ]$ is—by definition—*arithmetically normal* in the given projective embedding. In particular it is normal. ☐

Now we come to the geometric construction that supplies the inductive machinery.

Definition. The *nested Hilbert scheme* $H^{n-1,n}$ is the subvariety of pairs

$$H^{n-1,n} = \{(I_{n-1}, I_n) : I_{n-1} \supseteq I_n\} \subseteq \operatorname{Hilb}^{n-1}(\mathbf{C}^2) \times \operatorname{Hilb}^n(\mathbf{C}^2).$$

Proposition 6.10. [**5,34**] *The nested Hilbert scheme $H^{n-1,n}$ is irreducible of dimension $2n$ and non-singular.*

If (I_{n-1}, I_n) is a point of $H_{n-1,n}$ then the corresponding subscheme $V(I_{n-1}) \subseteq \mathbf{C}^2$ is a subscheme of $V(I_n)$, so the multiset $\tau(I_n)$ contains $\tau(I_{n-1})$ along with one additional point, or else with the multiplicity of one of the original points increased by 1. So if the specturm of I_{n-1} is $(x_1, y_1), \ldots, (x_{n-1}, y_{n-1})$ then that of I_n is $(x_1, y_1), \ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$ for a *distinguished point* (x_n, y_n). Now both the S_{n-1} invariants $p_{h,k}(x_1, y_1, \ldots, x_{n-1}, y_{n-1})$ and the S_n invariants $p_{h,k}(x_1, y_1, \ldots, x_n, y_n)$ are regular functions on $H^{n-1,n}$, hence so are $x_n = p_1(x_1, \ldots, x_n) - p_1(x_1, \ldots, x_{n-1})$ and similarly y_n. This means we have a morphism

$$H^{n-1,n} \to \mathbf{C}^2 = \operatorname{Spec} \mathbf{C}[x_n, y_n],$$

[7]As a matter of fact W_x is the cell $C_{(1^n)}$ in Lemma 6.4.

mapping a pair to its distinguished point. Of course $(x_n, y_n) \in V(I_n)$, and by suitable choice of I_{n-1}, given I_n, the distinguished point can be any point of $V(I_n)$. Hence the combined map $H^{n-1,n} \to \mathbf{C}^2 \times \mathrm{Hilb}^n(\mathbf{C}^2)$ factors $H^{n-1,n} \to \mathrm{Hilb}^n(\mathbf{C}^2)$ through a surjective morphism

$$\alpha\colon H^{n-1,n} \to U \tag{6.7}$$

to the universal scheme U over $\mathrm{Hilb}^n(\mathbf{C}^2)$. Where the n points are distinct, this map is locally an isomorphism, so it is birational.

The above map and the map $H^{n-1,n} \to \mathrm{Hilb}^n(\mathbf{C}^2)$ are projective. In fact, given I_n, I_{n-1} is determined by its single generator mod I_n, so $H^{n-1,n}$ is a subvariety of the projective space bundle $\mathbf{P}(B)$, where B is the tautological bundle over $\mathrm{Hilb}^n(\mathbf{C}^2)$. More precisely, given I_n and $P = (x_n, y_n)$, the possible ideals I_{n-1} correspond one-to-one with length-1 ideals in the local ring $\mathcal{O}_{S,P}$ of $V(I_n)$ at P. Such ideals are simply the 1-dimensional subspaces of the *socle*, $\operatorname{soc}\mathcal{O}_{S,P} = (0 : \mathfrak{m}_P)$. Thus each fiber of the map (6.7) is a projective space $\mathbf{P}(\operatorname{soc}\mathcal{O}_{S,P})$, of dimension $\dim\operatorname{soc}\mathcal{O}_{S,P} - 1$.

Lemma 6.11. *If the dimension of the fiber of $\alpha\colon H^{n-1,n} \to U$ over a point $(I,P) \in U$ is d, then the multiplicity of P is at least $\binom{d+2}{2}$.*

Proof. We are to show that if $T = \mathbf{C}[x,y]/I$ is a local ring of finite length, with $d + 1 = \dim\operatorname{soc} T$, then the length of T is at least $\binom{d+2}{2}$. This is equivalent to showing that if there exists a fiber of the map $H^{n-1,n} \to U$ with dimension at least d, then $n \geq \binom{d+2}{2}$. Now by the upper-semicontinuity of fiber dimension, and the fact (Lemma 6.4) that every I has one of the ideals I_μ in the closure of its $\mathbf{T}$ orbit, the maximal fiber dimension must occur at some I_μ. There we see immediately that the dimension of the socle is the number of corners of μ, so the result reduces to the fact that if the diagram of a partition of n has k corners then $n \geq \binom{k+1}{2}$. □

Lemma 6.12. *The map $\alpha\colon H^{n-1,n} \to U$ restricts to an isomorphism outside a locus of codimension 2 in $H^{n-1,n}$.*

Proof. First note that the 2-dimensional and higher fibers of α form a locus of codimension at least 3, by Lemmas 6.6 and 6.11, since for $d \geq 2$ we have $\binom{d+2}{2} - 1 - d \geq 3$.

For I_n curvilinear, $\operatorname{soc}\mathcal{O}_{S,P}$ is always 1-dimensional, so α restricts to an bijective morphism on the curvilinear locus, which is then an isomorphism by Zariski's theorem and Lemma 6.8.

As noted in the proof of Lemma 6.8, the non-curvilinear locus in $\mathrm{Hilb}^n(\mathbf{C}^2)$ is contained in G_3, so its codimension is at least 2. If it had codimension exactly 2 then it would contain the whole codimension 2 component of G_3, and in particular, every point where the multiplicities are $3, 1, \ldots, 1$. But there are clearly curvilinear subschemes with these multiplicities, so the co-dimension of the non-curvilinear locus is at least 3. If its preimage in $H^{n-1,n}$ had a component of codimension 1, then every fiber in that component would have dimension at least 2, contradicting the observation made at the outset. Hence the locus where I_n is non-curvilinear has codimension at least 2 in $H^{n-1,n}$, and α restricts to an isomorphism outside it. □

Lemma 6.13. *The canonical sheaf ω of regular $2n$-forms on $\operatorname{Hilb}^n(\mathbf{C}^2)$ is trivial, i.e., isomorphic to the structure sheaf $\mathcal{O}$.*

Proof. We again use the description in the proof of Lemma 6.8 of the open set W_x of ideals I modulo which $1, x, \ldots, x^{n-1}$ is a basis: W_x is an affine $2n$-cell with coordinates $e_1, \ldots, e_n, a_0, \ldots, a_{n-1}$, in terms of which I is generated at each point by equations (6.5). On the locus where I is the ideal of a reduced subscheme $S = \{(x_1, y_1), \ldots, (x_n, y_n)\}$, the first equation

$$x^n - e_1 x^{n-1} + e_2 x^{n-2} - \cdots + (-1)^n e_n$$

must be the polynomial $\prod_i (x - x_i)$, so the parameters e_i are the elementary symmetric functions $e_i(\mathbf{x})$. From the second equation,

$$y - (a_{n-1} x^{n-1} + \cdots + a_1 x + a_0),$$

the a_i are the coefficients of the interpolating polynomial $\phi_a(\mathbf{x})$ which satisfies $y_i = \phi_a(x_i)$ when the x_i's are all distinct.

It is well-known that the elementary symmetric functions satisfy $de_1 \wedge \cdots \wedge de_n = v(\mathbf{x}) dx_1 \wedge \cdots \wedge dx_n$, where $v(\mathbf{x})$ is the Vandermonde determinant. In particular, since the $e_i(\mathbf{x})$ and $e_i(\mathbf{y})$ are global regular functions on $\operatorname{Hilb}^n(\mathbf{C}^2)$, and $v(\mathbf{x})v(\mathbf{y})$ is S_n invariant, $d\mathbf{x}d\mathbf{y} = dx_1 \wedge \cdots \wedge dx_n \wedge dy_1 \wedge \cdots \wedge dy_n = v(\mathbf{x})^{-1} v(\mathbf{y})^{-1} de_1(\mathbf{x}) \wedge \cdots de_n(\mathbf{x}) de_1(\mathbf{y}) \wedge \cdots de_n(\mathbf{y})$ makes sense as a rational $2n$-form on $\operatorname{Hilb}^n(\mathbf{C}^2)$.

Moreover, the equations $y_i = \phi_a(x_i)$ say that the vector $(y_1, \ldots, y_n)$ is the product of $(a_{n-1}, \ldots, a_0)$ by the Vandermonde matrix, so $da_0 \wedge \cdots \wedge da_{n-1} = dy_1 \wedge \cdots \wedge dy_n / v(\mathbf{x})$, and hence

$$de_1 \wedge \cdots \wedge de_n \wedge da_{n-1} \wedge \cdots \wedge da_0 = d\mathbf{x}d\mathbf{y}.$$

In particular, the rational $2n$-form $d\mathbf{x}d\mathbf{y}$ is regular and has no zeroes on W_x. But $d\mathbf{x}d\mathbf{y}$ is invariant under the action of SL_2 on $\mathbf{C}^2$, so it follows that $d\mathbf{x}d\mathbf{y}$ is regular and nowhere vanishing on every W_z. Since we have already seen that the complement of $\bigcup_z W_z$ has codimension greater than 1, it follows that $d\mathbf{x}d\mathbf{y}$ is regular everywhere and vanishes nowhere, which shows that $\omega = \mathcal{O}$. □

Lemma 6.14. *The canonical sheaf ω of regular $2n$-forms on $H^{n-1,n}$ is isomorphic to L^{-1}, where L is the line bundle defined by the exact sequence*

$$0 \to L \to B_n \to B_{n-1} \to 0 \tag{6.8}$$

induced on the tautological bundles over $H^{n-1,n}$ by the containment $I_{n-1} \supseteq I_n$.

Proof. We are to show that the line bundle $L\omega$ is trivial, and it suffices to do this on an open set whose complement has codimension ≥ 2. By Lemma 6.12, we can use the open set where I_n is curvilinear and the map $H^{n-1,n} \to U$ restricts to an isomorphism, which means we can verify it on the curvilinear locus in U.

By Lemma 6.13 and duality for the finite, flat morphism $\pi \colon U \to \operatorname{Hilb}^n(\mathbf{C}^2)$, we have $\pi_*(L\omega_U) \cong (\pi_* L^{-1})^* \omega_H = (\pi_* L^{-1})^*$, where $\omega_H = \mathcal{O}$ is the canonical sheaf on the Hilbert scheme. So we have to show that $\pi_* L^{-1} \cong B^*$ as a B-module, since $\pi_* \mathcal{O}_U = B$.

Now let's examine L as a sub-bundle of $\pi^* B$ on the curvilinear locus in U. To avoid confusion, since we already have regular functions x, y on U, we write x', y' for the variables of B, so the fiber of $\pi^* B$ at (I, P) is $\mathbf{C}[x', y']/I(x', y')$. This given, the fiber of L at (I, P) is the socle of the summand $\mathcal{O}_{S,P}$ in $\mathbf{C}[x', y']/I(x', y')$, since the generator of I_{n-1} mod I belongs to this socle, which is 1-dimensional. Equivalently, the fiber of L is the ideal $(0 : \mathfrak{m}_P(x', y')) = (0 : (x' - x, y' - y))$ in $\pi^* B(I)$. Dualizing this, we see that $L^{-1} = \pi^* B^* / (x' - x, y' - y)\pi^* B^*$, or $\pi^* B^* \otimes_{\mathbf{C}[x', y']} \mathcal{O}_U$, where $\mathbf{C}[x', y']$ acts on $\mathcal{O}_U$ through the homomorphism $x' \mapsto x, y' \mapsto y$.

Now $\pi_* \pi^* B^* = B \otimes B^*$, so $\pi_* L^{-1} = (B \otimes B^*)/(x' - x, y' - y)(B \otimes B^*)$, where x, y act through B and x', y' act through B^*. But this is just another way of writing $\pi_* L^{-1} = B \otimes_B B^* = B^*$. □

Lemma 6.15. *If X_n is Cohen-Macaulay, then it is Gorenstein, and its canonical sheaf ω is the line bundle $\mathcal{O}(-1)$, where $\mathcal{O}(1) = \bigwedge^n B$.*

Proof. By Proposition 4.1, $X_n = \operatorname{Spec} P$ as a scheme affine over $\operatorname{Hilb}^n(\mathbf{C}^2)$, where $P = \sigma_* \mathcal{O}_{X_n}$ is the image of the sheaf homorphism $\phi \colon B^{\otimes n} \to (B^{\otimes n})^* \otimes \mathcal{O}(1)$ in (4.8). If X_n is Cohen-Macaulay this is a vector-bundle homorphism. By construction, this description of P means the bilinear pairing of vector bundles $P \otimes P \to \mathcal{O}(1)$, given by multiplication followed by alternation, is non-degenerate, so $P^* \cong P \otimes \mathcal{O}(-1)$. By duality for the flat, finite morphism $\sigma \colon X_n \to \operatorname{Hilb}^n(\mathbf{C}^2)$, the canonical sheaf ω_{X_n} is the sheaf associated to the P module sheaf $P^* \otimes \omega_H$, which is the same as $P \otimes \mathcal{O}(-1)$, by Lemma 6.13. This shows $\omega_{X_n} = \mathcal{O}(-1)$, and since this is a line bundle, X_n is Gorenstein (by definition). □

We now have all the technical ingredients we need to prove Theorem 4. From this point on we *assume Conjecture 6.2 holds*, and we assume X_{n-1} is Cohen-Macaulay by induction. We shall also assume $n \geq 4$. There is no harm in this since it is trivial to verify the $n!$ conjecture for $n \leq 3$.

To carry the induction forward we introduce the fiber product Y_n indicated by the diagram:

$$\begin{array}{ccc} Y_n & \longrightarrow & H^{n-1,n} \\ \big\downarrow & & \big\downarrow \\ X_{n-1} & \longrightarrow & \operatorname{Hilb}^{n-1}(\mathbf{C}^2). \end{array} \tag{6.9}$$

Since the bottom arrow is flat (by induction), so is the top one. Since Y_n is flat over $H^{n-1,n}$ and generically reduced, it is reduced. A point of Y_n is a pair of ideals $(I_{n-1}, I_n) \in H^{n-1,n}$, together with the spectrum $(x_1, x_n), \ldots, (x_{n-1}, x_{n-1})$ of I_{n-1} in some order. The coordinates of the remaining point (x_n, y_n) in the spectrum of I_n are regular functions on $H^{n-1,n}$ and hence on Y_n, so we obtain a morphism

$$f \colon Y_n \to X_n \tag{6.10}$$

sending $(I_{n-1}, I_n, \mathbf{x}, \mathbf{y})$ to $(I_n, \mathbf{x}, \mathbf{y})$. Note that f is projective, since the map $H^{n-1,n} \to \operatorname{Hilb}^n(\mathbf{C}^2)$ is. Over the locus where the points (x_i, y_i) are all distinct, $X_{n-1} \to (\mathbf{C}^2)^{n-1}$ and $H^{n-1,n} \to \mathbf{C}^2 \times \operatorname{Hilb}^{n-1}(\mathbf{C}^2)$ restrict to isomorphisms, hence so does $Y_n \to X_n \to (\mathbf{C}^2)^n$. The locus where some two points coincide is a proper closed subvariety of the irreducible variety $H^{n-1,n}$. Since Y_n is flat over $H^{n-1,n}$ it cannot have a component contained in the coincidence locus, so the non-coincidence locus is dense in Y_n. This shows Y_n is irreducible and the morphism f is birational.

On Y_n we have, by pullback from $H^{n-1,n}$, the two tautological bundles B_{n-1} and B_n. Let us set $\mathcal{O}(k,l) = (\bigwedge^{n-1} B_{n-1})^k (\bigwedge^n B_n)^l$. By Lemma 6.14 the canonical sheaf on $H^{n-1,n}$ is $\mathcal{O}(1,-1)$ in this notation. By Lemmas 6.13 and 6.15 the relative canonical sheaf of Y_n over $H^{n-1,n}$, which is pulled back from that of X_{n-1} over $\mathrm{Hilb}^{n-1}(\mathbf{C}^2)$, is $\mathcal{O}(-1,0)$. Hence the canonical sheaf ω_{Y_n} is $\mathcal{O}(0,-1)$. In particular, it is a pullback from X_n.

Now we are going to prove that for the derived functor of the pushforward we have $Rf_*\mathcal{O}_{Y_n} = \mathcal{O}_{X_n}$. Since $\omega_{Y_n} = f^*\mathcal{O}_{X_n}(-1)$ this also proves $Rf_*\omega_{Y_n} = \mathcal{O}_{X_n}(-1)$. By duality for the projective morphism f_* we conclude that the sheaf $\mathcal{O}_{X_n}(-1)$ is the dualizing complex on X_n, so X_n is Cohen-Macaulay. (This also shows X_n is Gorenstein with $\omega_{X_n} = \mathcal{O}(-1)$, so we could have made Lemma 6.15 part of the induction.)

Since f is proper and birational, and X_n is normal by Lemma 6.9, we have $f_*\mathcal{O}_{Y_n} = \mathcal{O}_{X_n}$. We have to prove that $R^if_*\mathcal{O}_{Y_n} = 0$ for all $i > 0$. Now the fibers of f are also fibers of the map $H^{n-1,n} \to U$, and thus the fiber dimensions d are bounded by $\binom{d+2}{2} \leq n$, by Lemma 6.11. In particular, since we are assuming $n \geq 4$, this implies $d < n-2$ (exercise for the reader). It follows that $R^if_*\mathcal{O} = 0$ for $i \geq n-2$.

For $i < n-2$ we use the following lemma.

Lemma 6.16. *Let $f\colon Y \to X$ be a morphism and let $x_1, \ldots, x_k$ be global regular functions on X (and so also on Y). Suppose that $\mathbf{x}$ is an $\mathcal{O}$-regular sequence at every point of $V(\mathbf{x})$, both in X and in Y. Let $U = X \setminus V(\mathbf{x})$, $W = f^{-1}(U)$, and $f' = f|_W$. Then $Rf'_*\mathcal{O}_Y = \mathcal{O}_X$ implies $R^if_*\mathcal{O}_Y = 0$ for $0 < i < k-1$.*

Proof. The hypothesis and conclusion are both local with respect to X, so we can assume X is affine. Then we are to show $H^i(Y,\mathcal{O}) = 0$ for $0 < i < k-1$. Let $V = V(\mathbf{x})$ (in both X and Y, by abuse of notation). The regular sequence condition implies $H^i_V(\mathcal{O}) = 0$ for $i < k$, on both X and Y. The hypothesis $Rf'_*\mathcal{O}_Y = \mathcal{O}_X$ implies that $H^i(W,\mathcal{O}_Y) \cong H^i(U,\mathcal{O}_X)$. Then from the local cohomology exact sequences

$$\cdots \to H^i_V(\mathcal{O}_Y) \to H^i(Y,\mathcal{O}) \to H^i(W,\mathcal{O}_Y) \to H^{i+1}_V(\mathcal{O}_Y) \to \cdots$$

and

$$\cdots \to H^i_V(\mathcal{O}_X) \to H^i(X,\mathcal{O}) \to H^i(U,\mathcal{O}_X) \to H^{i+1}_V(\mathcal{O}_X) \to \cdots$$

we obtain $H^i(Y,\mathcal{O}) \cong H^i(W,\mathcal{O}_Y) \cong H^i(U,\mathcal{O}_X) \cong H^i(X,\mathcal{O}) = 0$, for $0 < i < k-1$. □

Conjecture 6.1 implies that $(x_1 - x_2, \ldots, x_{n-1} - x_n)$ is a regular sequence on $R[tJ]$, and hence, by Proposition 6.2, on X_n. To apply the Lemma using this sequence on Y_n and X_n, it remains to prove that the sequence is regular on Y_n, and that $Rf_*\mathcal{O}_Y = \mathcal{O}_X$ outside the locus where all the x_i's coincide.

Note that Y_n can be described directly in terms of X_n as the subscheme of $\mathrm{Hilb}^{n-1}(\mathbf{C}^2)\times X_n$ whose fiber over a point $(I, P_1, \ldots, P_n)$ of X_n consists of all the ideals $I_{n-1} \subseteq I$ for which I_{n-1}/I is a length-1 ideal of the local ring $\mathcal{O}_{S,P_n}$. About a point where the P_i are not all equal, it follows from Lemma 6.3 that there is a neighborhood on which Y_n is locally isomorphic to $X_{r_1} \times \cdots X_{r_{k-1}} \times Y_{r_k}$, where of the distinct points $Q_1, \ldots, Q_k$, Q_k is the one equal to P_n. It also follows that on such a neighborhood the map $f\colon Y_n \to X_n$ is locally given by the identity on the factors X_{r_i}, times the map $f\colon Y_{r_k} \to X_{r_k}$. Hence we have $Rf_*\mathcal{O}_Y = \mathcal{O}_X$ by induction on the locus where the points P_i are not all equal, and, *a fortiori*, on the locus where the coordinates x_i are not all equal.

To conclude, I claim that $x_1, \ldots, x_n$ defines a complete intersection in Y_n, which is Cohen-Macaulay by induction, and hence $\mathbf{x}$ is a regular sequence at each point of $V(\mathbf{x})$. By shifting the origin of coordinates in $\mathbf{C}^2$, this implies that $(x_1 - x_2, \ldots, x_{n-1} - x_n)$ is a regular sequence at every point where the x_i's are all equal. Thus we have to show that $V(\mathbf{x})$ has dimension n. Recall that we already have the analogous result for X_n, Lemma 5.4. By the local product structure described in the preceding paragraph we can assume the result by induction on the open set where the y_i's are not all equal. This reduces us to showing that the dimension of $H_0^{n-1,n} = V(\mathbf{x}, \mathbf{y})$ is $n-1$, since the locus where all the y_i's are equal and all the x_i's are zero is $\mathbf{C}^1 \times H_0^{n-1,n}$.

Now we apply Lemma 6.5. By upper-semicontinuity, the maximal fiber dimension of $H_0^{n-1,n} \to H_0^n$ over the cell C'_μ occurs at I_μ. There, by the remarks preceding Lemma 6.11, the fiber dimension is one less than the number of corners of the diagram of μ. Thus to show that the preimage of C'_μ has dimension at most $n-1$, we just have to check that for any partition of n, we have (number of parts) + (number of corners) $\leq n+1$. But this is clear, since the number of cells in the first column of the diagram is the number of parts, and at most one of them can be a corner.

7. Diagonal harmonics

A polynomial function f on a vector space V is said to be *harmonic* with respect to a group G of linear endomorphisms of V if f is annihilated by all G-invariant partial differential operators without constant term. For $V = (\mathbf{Q}^2)^n = \mathbf{Q}^n \oplus \mathbf{Q}^n$, and G the symmetric group S_n acting "diagonally" by simultaneous coordinate permutations in each summand, we refer to the space D_n of harmonic polynomials as the *diagonal harmonics*.

By Weyl's theorem on the ring of invariants $\mathbf{Q}[\mathbf{x}, \mathbf{y}]^{S_n}$, we may equivalently define D_n as the solution space of the system of differential equations

$$p_{h,k}(\partial \mathbf{x}, \partial \mathbf{y}) f = \sum_i \partial x_i^h \partial y_i^k f = 0, \quad \text{for } 1 \leq h+k \leq n.$$

In particular the diagonal harmonics are solutions of the Laplace equation

$$\sum_i (\partial x_i^2 + \partial y_i^2) f = 0,$$

so they are harmonic polynomials in the classical sense. It is easy to see that the polynomials Δ_μ of Section 3 are diagonal harmonics, and hence the spaces D_μ are subspaces of D_n.

Let $I_n \subseteq \mathbf{Q}[\mathbf{x}, \mathbf{y}]$ be the ideal generated by the polarized power sums $p_{h,k}$ for $h+k > 0$. We may describe D_n in derivative-free terms as follows.

Proposition 7.1. [**19**] *The quotient ring* $\mathbf{Q}[\mathbf{x},\mathbf{y}]/I_n$ *is isomorphic as a doubly graded* S_n *module to* D_n.

For geometric purposes we will work instead with the ring

$$R_n = \mathbf{C}[\mathbf{x},\mathbf{y}]/I_n,$$

I_n being again generated by the polarized power sums, which of course has the same Frobenius series as $\mathbf{Q}[\mathbf{x},\mathbf{y}]/I_n$ or D_n.

Computations have suggested a series of surprising combinatorial conjectures concerning the Hilbert and Frobenius series of the rings R_n. As these are treated at length in [**19**] we here mention only three which are simple to state.

Conjecture 7.1. *The dimension of* R_n *as a vector space, or* $\mathcal{H}_{R_n}(1,1)$, *is equal to* $(n+1)^{n-1}$. *Moreover* $q^{\binom{n}{2}}\mathcal{H}_{R_n}(q,q^{-1}) = (1+q+q^2+\cdots+q^n)^{n-1}$.

Conjecture 7.2. *The specialization* $\mathcal{H}_{R_n}(q,1)$ *enumerates spanning trees* T *on the vertex set* $\{0,1,\ldots,n\}$, *each counted with weight* $q^{i(T)}$, *where* $i(T)$ *is the number of inversions in* T. *An* inversion *is a pair* $i<j$ *for which vertex* j *lies on the unique path in* T *from vertex* 0 *to vertex* i.

Conjecture 7.3. *As an* S_n *module,* R_n *is isomorphic to the sign character tensored with the permutation representation of* S_n *on the finite Abelian group* $(\mathbf{Z}/(n+1)\mathbf{Z})^n/H$, *where* S_n *acts by permuting the factors, and* $H = (\mathbf{Z}/(n+1)\mathbf{Z})\cdot(1,1,\ldots,1)$ *is the subgroup of* S_n*-invariant elements.*

The above conjectures are corollaries to a pair of more general conjectures giving the Frobenius series specializations $\mathcal{F}_{R_n}(X;q,1)$ and $q^{\binom{n}{2}}\mathcal{F}_{R_n}(X;q,1/q)$. In [**12**] we showed that these specializations are in turn corollaries to the following master formula.

Conjecture 7.4. *The Frobenius series of* R_n *is given by*

$$\mathcal{F}_{R_n}(X;q,t) = \sum_{|\mu|=n} \frac{(1-q)(1-t)B_\mu(q,t)\Pi_\mu(q,t)\tilde{H}_\mu(X;q,t)}{\prod_{s\in D(\mu)}(1-q^{-a(s)}t^{1+l(s)})(1-q^{1+a(s)}t^{-l(s)})}, \tag{7.1}$$

where μ *ranges over partitions of* n, *the arms and legs* $a(s)$, $l(s)$ *are as in* (2.16), B_μ *is given by* (2.12), *and*

$$\Pi_\mu = \Omega[1-B_\mu] = \prod_{\substack{(h,k)\in D(\mu)\\(h,k)\neq(0,0)}} (1-q^k t^h).$$

In the remainder of this section we show how formula (7.1) comes about, and prove that it holds if the $n!$ conjecture and a suitable cohomology vanishing hypothesis on X_n are true. The development parallels that in [**20**], to which we refer for some geometric results. There we studied the specialization of (7.1) to the Hilbert series for the S_n-alternating component, which can be expressed without recourse to Maconald polynomials as

$$C_n(q,t) = \sum_{|\mu|=n} \frac{(1-q)(1-t)B_\mu(q,t)\Pi_\mu(q,t)t^{n(\mu)}q^{n(\mu')}}{\prod_{s\in D(\mu)}(1-q^{-a(s)}t^{1+l(s)})(1-q^{1+a(s)}t^{-l(s)})}.$$

This turns out to be a two-parameter analog of the *Catalan number* $C_n = \frac{1}{n+1}\binom{2n}{n}$. We proved that $C_n(q,t)$ is a polynomial in q and t, and that under certain cohomology vanishing hypotheses it is the Hilbert series of the S_n-alternating diagonal harmonics. Because we examined only the alternating component we could work on $\mathrm{Hilb}^n(\mathbf{C}^2)$ without introducing the iso-spectral variety X_n. In essence, what we will now do is to lift these results to X_n.

Proposition 7.2. $\operatorname{Spec} R_n$ *is the scheme theoretic fiber* $\rho^{-1}(0)$ *over the origin, under the canonical map*

$$\rho\colon (\mathbf{C}^2)^n \to S^n\mathbf{C}^2.$$

Proof. The coordinate ring of $S^n\mathbf{C}^2$ is $\mathbf{C}[\mathbf{x},\mathbf{y}]^{S_n}$ and the ideal of the origin is the homogeneous maximal ideal $\mathfrak{m} = (p_{h,k} : h+k>0)$. By definition, the ideal of $\rho^{-1}(0)$ is generated by the image of $\mathfrak{m}$ in $\mathbf{C}[\mathbf{x},\mathbf{y}]$, or I_n. □

In what follows, we *assume the* $n!$ *conjecture holds for all* μ, so X_n is flat over $\mathrm{Hilb}^n(\mathbf{C}^2)$. Consider the fiber square

$$\begin{array}{ccc} X_n & \xrightarrow{\ \psi\ } & (\mathbf{C}^2)^n \\ {\scriptstyle\sigma}\big\downarrow & & \big\downarrow{\scriptstyle\rho} \\ \mathrm{Hilb}^n(\mathbf{C}^2) & \xrightarrow{\ \tau\ } & S^n\mathbf{C}^2. \end{array}$$

Let us define $X_n^0 = (\rho\psi)^{-1}(0) = (\tau\sigma)^{-1}(0)$ to be the scheme-theoretic fiber of X_n over $0 \in S^n\mathbf{C}^2$. This is a *non-reduced* subscheme of X_n. By Proposition 7.2, ψ induces a morphism

$$X_n^0 \xrightarrow{\psi} \rho^{-1}(0) = \operatorname{Spec} R_n,$$

corresponding to a ring homomorphism

$$\psi^\sharp : R_n \to H^0(X_n^0, \mathcal{O}).$$

As a scheme finite over $\mathrm{Hilb}^n(\mathbf{C}^2)$, we have $X_n = \operatorname{Spec} \sigma_*\mathcal{O}_{X_n}$, and since we are assuming the $n!$ conjecture, $\sigma_*\mathcal{O}_{X_n}$ is locally free of rank $n!$, that is, it is the sheaf of sections of a vector bundle P, the image of the homomorphism ϕ in (4.8). Then $X_n^0 = \sigma^{-1}(H_0^n) = \operatorname{Spec} P|_{H_0^n}$ as a scheme over $H_0^n = \tau^{-1}(0)$. Hence we can identify the global sections $H^0(X_n^0,\mathcal{O})$ with $H^0(H_0^n, P)$.

Proposition 7.3. [**20**] *The scheme theoretic zero fiber* $H_0^n = \tau^{-1}(0)$ *in the Hilbert scheme is reduced, Cohen-Macaulay, and has a* $\mathbf{T}$*-equivariant resolution by locally free sheaves on* $\mathrm{Hilb}^n(\mathbf{C}^2)$

$$0 \to B \otimes \bigwedge\nolimits^{n+1} V \to \cdots \to B \otimes \bigwedge\nolimits^{1} V \to B \to \mathcal{O}_{H_0^n} \to 0, \tag{7.2}$$

where $V = B' \oplus \mathcal{O}_t \oplus \mathcal{O}_q$*,* B' *is a summand of the tautological bundle* $B = B' \oplus \mathcal{O}$*, and* $\mathcal{O}_t$*,* $\mathcal{O}_q$ *denote the trival bundle* $\mathcal{O}$ *tensored by the 1-dimensional representation of* $\mathbf{T}$ *with character* t *or* q*, respectively.*

To proceed further we will need to assume the validity of the following conjecture.

Conjecture 7.5. *For all $i > 0$ and $k \geq 0$ we have $H^i(X_n, B^{\otimes k}) = 0$, and for $i = 0$, the canonical map*

$$\mathbf{C}[x_1', y_1', \dots, x_k', y_k', \mathbf{x}, \mathbf{y}] \to H^0(X_n, B^{\otimes k}) \tag{7.3}$$

is surjective.

To clarify, recall that $B = \mathbf{C}[x', y']/I$, where $\mathbf{C}[x', y']$ really means the trivial bundle $\mathcal{O} \otimes_{\mathbf{C}} \mathbf{C}[x', y']$, and we use primes to avoid confusion with the variables $\mathbf{x}, \mathbf{y}$. The map in (7.3) is induced by the maps $\mathbf{C}[x', y'] \to B$, with the identifications $\mathbf{C}[x', y']^{\otimes k} = \mathbf{C}[x_1', y_1', \dots, x_k', y_k']$ and $H^0(X_n, \mathbf{C}[x', y']^{\otimes k}) = \mathbf{C}[\mathbf{x}', \mathbf{y}'] \otimes H^0(X_n, \mathcal{O}) = \mathbf{C}[\mathbf{x}', \mathbf{y}', \mathbf{x}, \mathbf{y}]$. Note that $H^0(X_n, \mathcal{O}) = \mathbf{C}[\mathbf{x}, \mathbf{y}]$ because the map $\psi \colon X_n \to (\mathbf{C}^2)^n$ is proper and birational, and $(\mathbf{C}^2)^n$ is obviously normal. Note also that the exterior power $\bigwedge^k B$ is a summand of $B^{\otimes k}$, so the conjecture extends to tensors of exterior powers as well. We do not use the full strength of Conjecture 7.5 below, only the vanishing property for bundles $B \otimes \bigwedge^k B$ and the surjectivity property for B and $B \otimes B$.

Proposition 7.4. *Assume that Conjecture 7.5 holds. Then the canonical homomorphism*

$$\psi^\sharp : R_n \to H^0(X_n^0, \mathcal{O}) = H^0(H_0^n, P)$$

is an isomorphism.

Proof. Since we are assuming X_n is flat over $\operatorname{Hilb}^n(\mathbf{C}^2)$, the pullback functor σ^* on sheaves is exact. Applying σ^* to the resolution (7.2) we get a resolution on X_n

$$0 \to B \otimes \bigwedge^{n+1} V \to \cdots \to B \otimes \bigwedge^1 V \to B \to \mathcal{O}_{X_n^0} \to 0. \tag{7.4}$$

By our vanishing hypothesis, (7.4) is an acyclic resolution of $\mathcal{O}_{X_n^0}$, and therefore, applying H^0, we get an exact sequence

$$0 \to H^0(X_n, B \otimes \bigwedge^{n+1} V) \to \cdots \to H^0(X_n, B \otimes \bigwedge^1 V) \to H^0(X_n, B) \to H^0(X_n^0, \mathcal{O}) \to 0. \tag{7.5}$$

There is a *trace map* of $\mathcal{O}$-module sheaves

$$\operatorname{tr} \colon B \to \mathcal{O}$$

which sends a section f of B to the function whose value at a point Q is the trace of multiplication by f on the fiber $B(Q)$, divided by n. On X_n, the joint spectrum of the multiplication operators X, Y is $(x_1, y_1), \dots, (x_n, y_n)$, and therefore

$$\operatorname{tr} f(x', y') = \frac{1}{n} \sum_{i=1}^n f(x_i, y_i).$$

By the construction of (7.2) in [**20**], the map $B \to \mathcal{O}_{X_n^0}$ factors as

$$B \underset{\operatorname{tr}}{\to} \mathcal{O} \to \mathcal{O}_{X_n^0}.$$

Hence $H^0(X_n, B) \to H^0(X_n^0, \mathcal{O})$ factors through $H^0(X_n, \mathcal{O}) = \mathbf{C}[\mathbf{x}, \mathbf{y}]$, which shows that $\psi^\sharp$ is surjective.

To prove that ψ is injective, we must show that if $f(x',y',\mathbf{x},\mathbf{y})$ represents a global section in the kernel of $H^0(X_n,B)\to H^0(X_n^0,\mathcal{O})$, then $\operatorname{tr} f\in I_n$. We are using the surjectivity property in Conjecture 7.5 to assume that such a representative polynomial f exists. By (7.5), the kernel in question is the sum of the images of three maps

$$H^0(X_n,B\otimes B')\to H^0(X_n,B); \tag{7.6}$$

$$H^0(X_n,B\otimes \mathcal{O}_t)\to H^0(X_n,B); \tag{7.7}$$

$$H^0(X_n,B\otimes \mathcal{O}_q)\to H^0(X_n,B). \tag{7.8}$$

By the construction of (7.2), the second and third maps are multiplication by x' and y', respectively. For any $f(x',y',\mathbf{x},\mathbf{y})$, the trace map satisfies the identity

$$\operatorname{tr} f - f(0,0,\mathbf{x},\mathbf{y})\in I_n. \tag{7.9}$$

To prove this it is sufficient to take $f=(x')^h(y')^k$, since these monomials generate $\mathbf{C}[x',y',\mathbf{x},\mathbf{y}]$ as a $\mathbf{C}[\mathbf{x},\mathbf{y}]$ module, and the operation we are performing on f is $\mathbf{C}[\mathbf{x},\mathbf{y}]$-linear. We obtain

$$\operatorname{tr}(x')^h(y')^k - 0^h0^k = \begin{cases} 0, & h+k=0\\ p_{h,k}(\mathbf{x},\mathbf{y}), & h+k>0.\end{cases}$$

In particular if f belongs to the ideal (x',y') then $f(0,0,\mathbf{x},\mathbf{y})=0$ and $\operatorname{tr} f\in I_n$.

This leaves us only to consider the first map (7.6). The summand B' is defined to be the kernel of the trace map. Hence if $f(x',y',x'',y'',\mathbf{x},\mathbf{y})$ represents a section in $H^0(X_n,B\otimes B')$, then

$$\sum_i f(x',y',x_i,y_i,\mathbf{x},\mathbf{y})$$

is the zero section in $H^0(X_n,B)$. Now for each j there is a homomorphism of sheaves of $\mathcal{O}_{X_n}$ algebras $B\to\mathcal{O}_{X_n}$ mapping (x',y') to (x_i,y_i). This is so because $X_n\subseteq U^{\times n}$, and the homomorphism $B\to\mathcal{O}_{X_n}$ corresponds to the projection $X_n\to U$ on the j-th factor. Applying these homorphisms to the sum above, we see that

$$\sum_i f(x_j,y_j,x_i,y_i,\mathbf{x},\mathbf{y})=0 \quad \text{in } H^0(X_n,\mathcal{O})=\mathbf{C}[\mathbf{x},\mathbf{y}],$$

for all j. By (7.9) this implies that $f(x_j,y_j,0,0,\mathbf{x},\mathbf{y})\in I_n$ for each j. Summing over j and using (7.9) again we find that $f(0,0,0,0,\mathbf{x},\mathbf{y})\in I_n$.

The map $H^0(B\otimes B')\to H^0(X,B)$ is multiplication in B, which sends $f(x',y',x'',y'',\mathbf{x},\mathbf{y})$ to $f(x',y',x',y',\mathbf{x},\mathbf{y})$. Modulo I_n, the trace map $H^0(X_n,B)\to H^0(X_n,\mathcal{O})$ is the same as evaluation at $(x',y')=(0,0)$, again by (7.9). Hence the image of $f(x',y',x'',y'',\mathbf{x},\mathbf{y})$ in $H^0(X_n,\mathcal{O})$ is given modulo I_n by $f(0,0,0,0,\mathbf{x},\mathbf{y})$, and since the latter belongs to I_n the proof is complete. □

Theorem 5. *Assuming the $n!$ conjecture and Conjecture 7.5 hold, the Frobenius series of R_n is given by the master formula* (7.1) *in Conjecture 7.4.*

Proof. In [**20**] we derived an Atiyah-Bott type Lefschetz formula for $\mathbf{T}$-equivariant vector bundles on H_0^n, using the resolution (7.2) and explicit local parameters for $\mathrm{Hilb}^n(\mathbf{C}^2)$ at the $\mathbf{T}$-fixed points I_μ. This formula takes the form

$$\sum_i (-1)^i \mathcal{F}_{H^i(H_0^n,V)}(X;q,t) = \sum_{|\mu|=n} \frac{(1-q)(1-t)B_\mu(q,t)\Pi_\mu(q,t)\mathcal{F}_{V(I_\mu)}(X;q,t)}{\prod_{s\in D(\mu)}(1-q^{-a(s)}t^{1+l(s)})(1-q^{1+a(s)}t^{-l(s)})}. \tag{7.10}$$

Actually, this formula was derived for Hilbert series, but when V is a bundle of S_n modules it generalizes immediately to Frobenius series. The q,t-Catalan numbers studied in [**20**] correspond to the line bundle $V = \mathcal{O}(1)$.

If the $n!$ conjecture and Conjecture (7.5) hold, then by Proposition 7.4, the Frobenius series of R_n is equal to $\mathcal{F}_{H^0(H_0^n,P)}$, where $P = \sigma_*\mathcal{O}_{X_n}$. Moveover, using the resolution (7.4), we see that Conjecture 7.5 implies $H^i(X_n^0,\mathcal{O}) = 0$ for $i>0$, or equivalently, since σ is finite, $H^i(H_0^n,P) = 0$. Therefore the Euler characteristic on the left-hand side of (7.10) reduces to $\mathcal{F}_{R_n}(X;q,t)$.

By Theorem 3, the $n!$ conjecture implies that $\mathcal{F}_{P(I_\mu)}(X;q,t) = \tilde{H}_\mu(X;q,t)$, and the result follows. □

We conclude with some remarks on the vanishing hypothesis, Conjecture 7.5 . Strong vanishing theorems such as this are a relatively rare phenomenon. The conjecture is the analog, for the tautological bundle B on the iso-spectral Hilbert scheme, of a theorem which does hold for the tautological (quotient) bundle on a Grassmann variety.

In the case of the Hilbert scheme there is some favorable computational evidence. Namely, assuming the $n!$ conjecture—which has been verified for $n \leq 8$—one can use (7.10) to compute the Frobenius series Euler characteristic of any explicit enough bundle V. If V has non-vanishing higher cohomology, we should expect to see some negative terms. For the bundles referred to in the conjecture, and reasonable values of n and k, we have done a number of these computations and the results invariably have positive coefficients. Note also that if the $n!$ conjecture were to fail, we should not even expect to obtain a polynomial in (7.10). For the specialization to Hilbert series, the formula can be evaluated for values of n much larger than those for which we can check the $n!$ conjecture. A. Garsia and I have done some of these computations for n as large as 20, always obtaining polynomials with positive coefficients. I regard this as strong evidence for both the $n!$ conjecture and Conjecture 7.5.

8. The commuting variety

The material in this section is based on my conversations with I. Grojnowski, and represents joint work in progress. At present our results are not definitive, but we have made some observations and conjectures which I will discuss briefly.

Definition. The *commuting variety* C_n is the variety of pairs of $n\times n$ matrices (X,Y) such that $XY = YX$.

Little is known about C_n, except that it is irreducible of dimension n^2+n [**28,31**]. It is not even known whether the equations $XY = YX$ generate its ideal, although this is conjectured to be true. There is also a conjecture, generally attributed to Hochster, that C_n is Cohen-Macaulay.

We will be interested in the open set C_n^0 of pairs for which the vectors $X^hY^ke_1$ span $\mathbf{C}^n$, where e_1 is the first unit coordinate vector. This is an open set, since its complement is defined by the vanishing of the $n\times n$ minors of the matrix whose columns are $X^hY^ke_1$.

Given an ideal $I\subseteq \mathbf{C}[x,y]$ belonging to $\mathrm{Hilb}^n(\mathbf{C}^2)$, let us fix a basis $\{1,v_2,\dots,v_n\}$ of $\mathbf{C}[x,y]/I$. With respect to this basis, the operators X and Y of multiplication by x and y are represented by commuting matrices, and the pair (X,Y) belongs to C_n^0 because we took our first basis vector to be 1. Conversely, given $(X,Y)\in C_n^0$, we have a surjective map $\theta\colon \mathbf{C}[x,y]\to\mathbf{C}^n$ sending $p(x,y)$ to $p(X,Y)e_1$, whose kernel is an ideal $I\in\mathrm{Hilb}^n(\mathbf{C}^2)$. Then θ induces an isomorphism $\mathbf{C}[x,y]/I\to\mathbf{C}^n$, under which the unit coordinate basis $e_1,\dots,e_n$ of $\mathbf{C}^n$ corresponds to a basis $1,v_2,\dots,v_n$ of $\mathbf{C}[x,y]/I$. It is easy to see that these two constructions are mutually inverse and so define a smooth fibration

$$C_n^0\to\mathrm{Hilb}^n(\mathbf{C}^2)$$

with fiber G, where $G\subseteq GL_n$ is the stabilizer of e_1 (so G parametrizes ordered bases of $\mathbf{C}^n$ whose first vector is given). In particular this shows that C_n^0 is non-singular.

Definition. The *iso-spectral commuting variety* IC_n is the variety of tuples $(X,Y,\mathbf{a},\mathbf{b})\in C_n\times\mathbf{C}^{2n}$ such that $(a_1,b_1),\dots,(a_n,b_n)$ is the joint spectrum of X and Y in some order. In other words, we have the identity

$$\det(I+rX+sY)=\prod_{i=1}^{n}(1+ra_i+sb_i), \tag{8.1}$$

where r,s are indeterminates.

Note that if X and Y commute there is a $g\in GL_n$ such that $g^{-1}Xg$ and $g^{-1}Yg$ are both upper triangular, by Lie's theorem. In particular they have a joint spectrum as defined above, given by the diagonal entries of the triangular form. Note also that there is an action of S_n on IC_n, permuting the pairs (a_i,b_i). Under this action we have $IC_n/S_n=C_n$, since the invariants $p_{h,k}(\mathbf{a},\mathbf{b})$ are equal to $\operatorname{tr}X^hY^k$ and so reduce to functions on C_n.

Let IC_n^0 denote the open subset of IC_n lying over C_n^0. From the definition of the iso-spectral Hilbert scheme X_n it follows immediately that we have (set-theoretically) a fiber square

$$\begin{array}{ccc} IC_n^0 & \longrightarrow & X_n \\ \big\downarrow & & \big\downarrow \\ C_n^0 & \longrightarrow & \mathrm{Hilb}^n(\mathbf{C}^2). \end{array}$$

Since the bottom arrow is a smooth morphism, so is the top arrow in the scheme-theoretic fiber square. Hence the scheme-theoretic fiber product is reduced and therefore equal to the set-theoretic fiber product. This proves

Proposition 8.1. *The open set IC_n^0 in IC_n is Cohen-Macaulay (and hence Gorenstein) if and only if X_n is.*

Conjecture 8.1. *The iso-spectral commuting variety IC_n is Gorenstein.*

Note that this implies the conjecture that the commuting variety $C_n = IC_n/S_n$ is Cohen-Macaulay, as well as the $n!$ conjecture. We should point out here that the ideal of IC_n is certainly *not* generated by the ideal of C_n (conjecturally $XY = YX$) together with equations (8.1). This fails even for $n = 2$.

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