LLT POLYNOMIALS IN THE SCHIFFMANN ALGEBRA

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Abstract. We identify certain combinatorially defined rational functions which, under the shuffle to Schiffmann algebra isomorphism, map to LLT polynomials in any of the distinguished copies $\Lambda(X^{m,n}) \subset \mathcal{E}$ of the algebra of symmetric functions embedded in the elliptic Hall algebra $\mathcal{E}$ of Burban and Schiffmann. As a corollary, we deduce an explicit raising operator formula for the ∇ operator applied to any LLT polynomial. In particular, we obtain a formula for $\nabla^m s_\lambda$ which serves as a starting point for our proof of the Loehr-Warrington conjecture in a companion paper to this one.

1. Introduction

In this paper we introduce *Catalanimals*—symmetric rational functions in variables $\mathbf{z} = z_1, \dots, z_l$ defined by

$$H(R_q, R_t, R_{qt}, \lambda) = \sum_{w \in S_l} w\left(\frac{\mathbf{z}^\lambda \prod_{\alpha \in R_{qt}} (1 - q\, t\, \mathbf{z}^\alpha)}{\prod_{\alpha \in R_+} (1 - \mathbf{z}^{-\alpha}) \prod_{\alpha \in R_q} (1 - q\, \mathbf{z}^\alpha) \prod_{\alpha \in R_t} (1 - t\, \mathbf{z}^\alpha)} \right), \tag{1}$$

depending on a weight $\lambda \in \mathbb{Z}^l$ and subsets R_q, R_t, R_{qt} of the set of positive roots $R_+ = \{\epsilon_i - \epsilon_j \mid 1 \leq i < j \leq l\}$ for GL_l. Using Catalanimals, we give explicit formulas for elements of the elliptic Hall algebra $\mathcal{E}$ of Burban and Schiffmann [5] corresponding to arbitrary LLT polynomials $\mathcal{G}_{\boldsymbol{\nu}}(X; q)$. This generalizes a formula of Negut [16] for elements corresponding to ribbon shaped skew Schur functions. The key to our more general result is the use of Catalanimals to combinatorialize Negut's shuffle algebra tools. As a corollary, we also obtain a raising operator formula for $\nabla \mathcal{G}_{\boldsymbol{\nu}}$.

The Schiffmann algebra $\mathcal{E}$ is generated by subalgebras $\Lambda(X^{m,n})$ isomorphic to the ring of symmetric functions over $\Bbbk = \mathbb{Q}(q,t)$, one for each coprime pair $(m,n) \in \mathbb{Z}^2$, along with an additional central subalgebra. The 'right half-plane' subalgebra $\mathcal{E}^+ \subseteq \mathcal{E}$ generated by $\Lambda(X^{m,n})$ for $m > 0$ is known to be isomorphic to a graded algebra $\mathcal{S}_{\widehat{\Gamma}} \subseteq \bigoplus_l \Bbbk(z_1, \dots, z_l)^{S_l}$, called the shuffle algebra, whose degree l component consists of certain symmetric rational functions in l variables. We denote this isomorphism by $\psi_{\widehat{\Gamma}} \colon \mathcal{S}_{\widehat{\Gamma}} \to \mathcal{E}^+$.

Our main result, Theorem 8.3.1 (see also Remark 8.3.2), is the construction of Catalanimals $H^{m,n}_{\boldsymbol{\nu}^m}(\mathbf{z})$ such that

$$\psi_{\widehat{\Gamma}}(H^{m,n}_{\boldsymbol{\nu}^m}(\mathbf{z})) = c^{m,n}_{\boldsymbol{\nu}^m} \mathcal{G}_{\boldsymbol{\nu}}[-M X^{m,n}], \tag{2}$$

where $c^{m,n}_{\boldsymbol{\nu}^m} \in \pm q^{\mathbb{Z}} t^{\mathbb{Z}}$, $M = (1-q)(1-t)$ and the square brackets denote plethystic substitution (see §2.1). Here $\mathcal{G}_{\boldsymbol{\nu}} = \mathcal{G}_{\boldsymbol{\nu}}(X; q)$ is the 'attacking inversions' LLT polynomial (Definition 2.2.1)

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indexed by a tuple of skew shapes $\boldsymbol{\nu} = (\nu_{(1)}, \ldots, \nu_{(k)})$; it is a q-analog of the product of skew Schur functions $s_{\nu_{(1)}} \cdots s_{\nu_{(k)}}$.

The precise definition of $H^{m,n}_{\boldsymbol{\nu}^m}$ is given in §§7.1 and 8.2, but we give a flavor of our results here. For $m = n = 1$, the Catalanimal $H^{1,1}_{\boldsymbol{\nu}}$ has root sets $R_+ \supseteq R_q \supseteq R_t \supseteq R_{qt}$, determined as follows using the same attacking inversion combinatorics as in the definition of the LLT polynomial $\mathcal{G}_{\boldsymbol{\nu}}$:

- $R_+ \setminus R_q \leftrightarrow$ pairs of boxes in the same diagonal,
- $R_q \setminus R_t \leftrightarrow$ the attacking pairs,
- $R_t \setminus R_{qt} \leftrightarrow$ pairs going between adjacent diagonals,

where the boxes of $\boldsymbol{\nu}$ are numbered $1, \ldots, l$ in reading order (see Example 7.1.1). The weight λ is obtained by filling each diagonal D of $\boldsymbol{\nu}$ with the value

$$1 + \chi(D \text{ contains a row start}) - \chi(D \text{ contains a row end}),$$

where $\chi(P) = 1$ if P is true or 0 if P is false, and then reading this filling in the reading order—see Figure 1.

Shuffle algebra representatives for elements of $\Lambda(X^{m,1}) \subseteq \mathcal{E}^+$ carry information about the symmetric function operator ∇ from the theory of Macdonald polynomials. Specifically, if a symmetric function $f(X)$ is related to a Catalanimal $H \in \mathcal{S}_{\widehat{\Gamma}}$ by $\psi_{\widehat{\Gamma}}(H) = f[-MX^{m,1}]$, then $\nabla^m f$ and H are related by

$$(\omega \nabla^m f)(z_1, \ldots, z_l) = H_{\mathrm{pol}}, \tag{3}$$

where H_{pol} is obtained by expanding H as an infinite series of GL_l characters and truncating to polynomial GL_l characters (see §3.5).

Combining (2) and (3), we obtain an explicit raising operator formula for ∇^m on any LLT polynomial

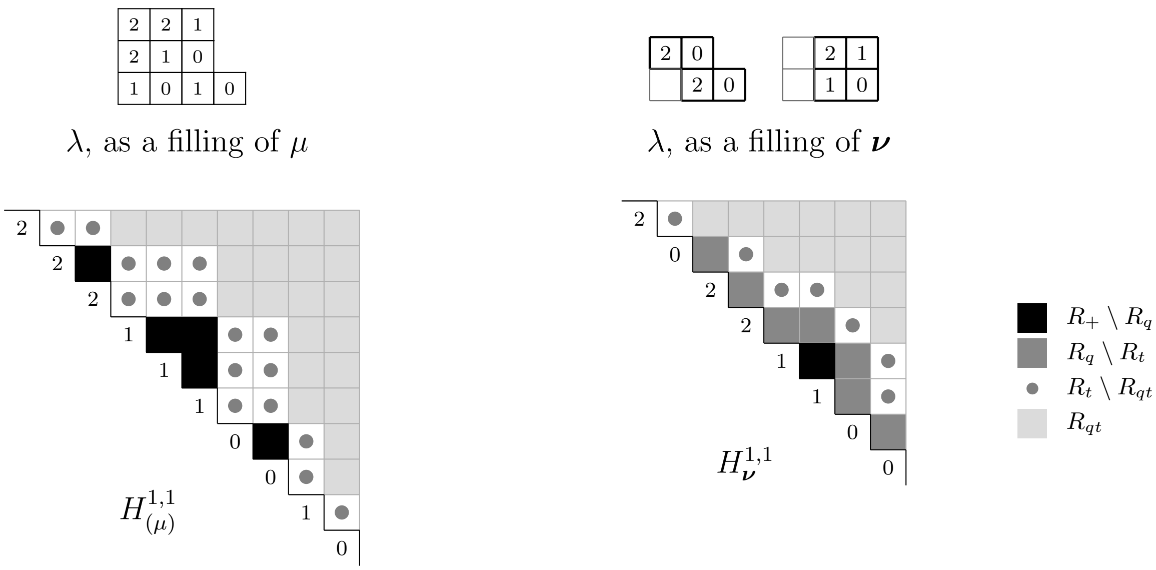
$$(\omega \nabla^m \mathcal{G}_{\boldsymbol{\nu}})(z_1, \ldots, z_l) = (c^{m,1}_{\boldsymbol{\nu}^m})^{-1} (H^{m,1}_{\boldsymbol{\nu}})_{\mathrm{pol}}. \tag{4}$$

In a companion paper [2], we use the case of formula (4) where the LLT polynomial is a Schur function to prove the Loehr-Warrington conjecture [13], a combinatorial formula for $\nabla^m s_\lambda$ in terms of LLT polynomials. By the Schur positivity of LLT polynomials [9], this implies that $\nabla^m s_\lambda$ is Schur positive up to a global sign.

After giving background on symmetric functions and LLT polynomials in §2, we develop the Schiffmann and shuffle algebra tools needed to prove identity (2) in §§3–6. Then, in §§7–8, we give the full description of the Catalanimals $H^{m,n}_{\boldsymbol{\nu}^m}$ and apply these tools to complete the proof.

2. Background on symmetric functions and LLT polynomials

2.1. **Symmetric functions and partition diagrams.** Let $\Lambda = \Lambda_{\Bbbk}(X)$ be the algebra of symmetric functions in an infinite alphabet of variables $X = x_1, x_2, \ldots$, with coefficients in the field $\Bbbk = \mathbb{Q}(q,t)$. We follow the notation of Macdonald [14] for the graded bases of Λ. We write $\omega\colon \Lambda \to \Lambda$ for the involutory $\Bbbk$-algebra automorphism determined by $\omega s_\lambda = s_{\lambda^*}$ for Schur functions s_λ, where λ^* denotes the conjugate partition of λ.



$$\omega\nabla s_{433} = (q\,t)^9 (H^{1,1}_{(\mu)})_{\mathrm{pol}} \qquad \omega\nabla \mathcal{G}_{\boldsymbol{\nu}} = -(q\,t)^4 q^7 (H^{1,1}_{\boldsymbol{\nu}})_{\mathrm{pol}}$$

FIGURE 1. (i) The Catalanimal $H^{1,1}_{(\mu)}$ for $\mu = (433)$. (ii) The Catalanimal $H^{1,1}_{\boldsymbol{\nu}}$ for $\boldsymbol{\nu} = ((32)/(10), (33)/(11))$. These are illustrated by drawing the root sets in an $\ell \times \ell$ grid labeled by matrix-style coordinates, with the sets $R_+ \setminus R_q$, $R_q \setminus R_t$, $R_t \setminus R_{qt}$, R_{qt} specified according to the legend on the right; the weight λ is written on the diagonal with λ_1 in the upper left.

Given $f \in \Lambda$ and an expression A involving indeterminates, such as a polynomial, rational function, or formal series, the plethystic evaluation $f[A]$ is defined by writing f as a polynomial in the power-sums p_k and evaluating with $p_k \mapsto p_k[A]$, where $p_k[A]$ is the result of substituting a^k for every indeterminate a occurring in A. The variables q, t from our ground field $\Bbbk$ count as indeterminates. We will often use the fact that $f(x_1, x_2, \ldots) = f[X]$ with $X = x_1 + x_2 + \cdots$.

The algebra Λ is a Hopf algebra with coproduct given by

$$\Delta f = f[X + Y] \in \Lambda \otimes \Lambda = \Lambda(X) \otimes \Lambda(Y). \tag{5}$$

Here $f \in \Lambda$ and we use separate alphabets X, Y to distinguish the tensor factors.

We fix notation for the series

$$\Omega = 1 + \sum_{k>0} h_k = \exp \sum_{k>0} \frac{p_k}{k}, \quad \text{or} \quad \Omega[a_1 + a_2 + \cdots - b_1 - b_2 - \cdots] = \frac{\prod_i (1 - b_i)}{\prod_i (1 - a_i)} \tag{6}$$

and the quantities

$$M = (1 - q)(1 - t) \qquad \text{and} \qquad \widehat{M} = \big(1 - \frac{1}{q\,t}\big) M. \tag{7}$$

An example of plethystic substitution using these quantities, which will arise again later, is

$$\Omega[-w/y\,\widehat{M}] = \frac{(1-q\,t\,w/y)(1-q^{-1}w/y)(1-t^{-1}w/y)}{(1-(q\,t)^{-1}w/y)(1-q\,w/y)(1-t\,w/y)}\,. \tag{8}$$

The (French style) diagram of a partition λ is the set of lattice points $\{(i,j) \mid 1 \le j \le \ell(\lambda),\ 1 \le i \le \lambda_j\}$, where $\ell(\lambda)$ is the length of λ. We often identify λ and its diagram with the set of lattice squares, or *boxes*, with northeast corner at a point $(i,j) \in \lambda$. For $\mu \subseteq \lambda$, the *skew shape* λ/μ is the set of boxes of λ not contained in μ.

For a box $a = (i,j) \in \mathbb{Z}^2$ (usually in some given skew shape), we let $\mathrm{south}(a) = (i, j-1) \in \mathbb{Z}^2$ denote the box immediately south of a, and define $\mathrm{north}(a) = (i, j+1)$, $\mathrm{west}(a) = (i-1, j)$, and $\mathrm{east}(a) = (i+1, j)$ similarly.

2.2. **LLT polynomials.** We recall the attacking inversions description of LLT polynomials from [11].

Let $\boldsymbol{\nu} = (\nu_{(1)}, \ldots, \nu_{(k)})$ be a tuple of skew shapes. We consider the set of boxes in $\boldsymbol{\nu}$ to be the disjoint union of the sets of boxes in the $\nu_{(i)}$. The *content* of a box $a = (i,j)$ in row j, column i of a skew diagram is $c(a) = i - j$. Fix $\epsilon > 0$ small enough that $k\,\epsilon < 1$. The *adjusted content* of a box $a \in \nu_{(i)}$ is $\tilde{c}(a) = c(a) + i\,\epsilon$. A *diagonal* of $\boldsymbol{\nu}$ is the set of boxes of a fixed adjusted content, or, in other words, the set of boxes of fixed content in one of the shapes $\nu_{(i)}$. We write $\mathrm{diag}(a)$ for the diagonal containing a box a of $\boldsymbol{\nu}$.

The *reading order* on $\boldsymbol{\nu}$ is the total ordering $<$ on the boxes of $\boldsymbol{\nu}$ such that $a < b \Rightarrow \tilde{c}(a) \le \tilde{c}(b)$ and boxes on each diagonal increase from southwest to northeast (see Example 7.1.1). We say that an ordered pair of boxes (a,b) in $\boldsymbol{\nu}$ is an *attacking pair* if $a < b$ in reading order and $0 < \tilde{c}(b) - \tilde{c}(a) < 1$.

A *semistandard tableau* on the tuple $\boldsymbol{\nu}$ is a map $T\colon \boldsymbol{\nu} \to \mathbb{Z}_+$ which restricts to a semistandard Young tableau on each component $\nu_{(i)}$. We write $\mathrm{SSYT}(\boldsymbol{\nu})$ for the set of these. An *attacking inversion* in T is an attacking pair (a,b) such that $T(a) > T(b)$. Let $\mathrm{inv}(T)$ denote the number of attacking inversions in T.

Definition 2.2.1. The *LLT polynomial* indexed by a tuple of skew diagrams $\boldsymbol{\nu}$ is the generating function, which is known to be symmetric [11, 12],

$$\mathcal{G}_{\boldsymbol{\nu}}(X;q) = \sum_{T \in \mathrm{SSYT}(\boldsymbol{\nu})} q^{\mathrm{inv}(T)} \mathbf{x}^T, \tag{9}$$

where $\mathbf{x}^T = \prod_{a \in \boldsymbol{\nu}} x_{T(a)}$.

It follows directly from the definition that

$$\mathcal{G}_{\boldsymbol{\nu}}(x_1, x_2, \ldots, y_1, y_2, \ldots; q) = \sum q^{A(\boldsymbol{\nu}'', \boldsymbol{\nu}')} \mathcal{G}_{\boldsymbol{\nu}'}(X;q) \mathcal{G}_{\boldsymbol{\nu}''}(Y;q), \tag{10}$$

where the sum is over all partitions of $\boldsymbol{\nu}$ into a lower order ideal $\boldsymbol{\nu}'$ and upper order ideal $\boldsymbol{\nu}''$, and $A(\boldsymbol{\nu}'', \boldsymbol{\nu}')$ is the number of attacking pairs (a,b) with $a \in \boldsymbol{\nu}''$, $b \in \boldsymbol{\nu}'$. In other words, LLT polynomials satisfy the coproduct formula

$$\mathcal{G}_{\boldsymbol{\nu}}[X+Y] = \sum q^{A(\boldsymbol{\nu}'', \boldsymbol{\nu}')} \mathcal{G}_{\boldsymbol{\nu}'}[X] \mathcal{G}_{\boldsymbol{\nu}''}[Y]. \tag{11}$$

When we write LLT polynomials in plethystic notation, our convention is to suppress the q.

We will also need a more general combinatorial formalism for LLT polynomials involving a 'signed' alphabet $\mathcal{A} = \mathcal{A}_+ \coprod \mathcal{A}_-$ with a *positive* letter $v \in \mathcal{A}_+$ and a *negative* letter $\overline{v} \in \mathcal{A}_-$ for each $v \in \mathbb{Z}_+$, with total ordering $1 < 2 < \cdots < \overline{1} < \overline{2} < \cdots$.

A *super tableau* on a tuple of skew shapes $\boldsymbol{\nu}$ is a map $T \colon \boldsymbol{\nu} \to \mathcal{A}$, weakly increasing along rows and columns, with positive letters increasing strictly on columns and negative letters increasing strictly on rows.

An attacking inversion in a super tableau is an attacking pair (a, b) such that either $T(a) > T(b)$ in the ordering on $\mathcal{A}$, or $T(a) = T(b) = \overline{v}$ with $\overline{v}$ negative. As before, $\operatorname{inv}(T)$ denotes the number of attacking inversions.

Lemma 2.2.2 ([10, (81–82) and Proposition 4.2])**.** *We have the identity*

$$\omega_Y \mathcal{G}_{\boldsymbol{\nu}}[X+Y] = \sum_T q^{\operatorname{inv}(T)} \mathbf{x}^{T_+} \mathbf{y}^{T_-}, \tag{12}$$

where the sum is over all super tableaux T on $\boldsymbol{\nu}$, and

$$\mathbf{x}^{T_+} \mathbf{y}^{T_-} = \prod_{a \in \boldsymbol{\nu}} \begin{cases} x_i, & T(a) = i \in \mathcal{A}_+, \\ y_i, & T(a) = \overline{i} \in \mathcal{A}_-. \end{cases} \tag{13}$$

3. Catalanimals in the shuffle and Schiffmann algebras

We briefly introduce and fix notation for the shuffle algebra of Feigin et al. [6], Feigin and Tsymbauliak [7], and Negut [16], and the elliptic Hall algebra of Burban and Schiffmann [5], which we refer to as the *Schiffmann algebra*.

We then give a preliminary description of how Catalanimals connect with the shuffle and Schiffmann algebras (§3.4) and how they relate to the ∇ operator (§3.5).

3.1. **The shuffle algebra.** Let $\Gamma = \Gamma(w, y)$ be a non-zero rational function over $\Bbbk$. The *large concrete shuffle algebra* is the graded associative algebra with underlying space

$$\mathcal{R}_\Gamma = \bigoplus_l \mathcal{R}_\Gamma^l = \bigoplus_l \Bbbk(z_1, \ldots, z_l)^{S_l}, \tag{14}$$

equipped with the 'shuffle' product whose graded component $\mathcal{R}_\Gamma^k \times \mathcal{R}_\Gamma^{l-k} \to \mathcal{R}_\Gamma^l$ is defined by

$$f \cdot g = \sum_{w \in S_l/(S_k \times S_{l-k})} w\Big(f(z_1, \ldots, z_k) g(z_{k+1}, \ldots, z_l) \prod_{i=1}^{k} \prod_{j=k+1}^{l} \Gamma(z_i, z_j)\Big). \tag{15}$$

Define symmetrization operators $\sigma_\Gamma^l \colon \Bbbk(z_1, \ldots, z_l) \to \mathcal{R}_\Gamma^l$ by

$$\sigma_\Gamma^l(f) = \sum_{w \in S_l} w\Big(f(z_1, \ldots, z_l) \prod_{i<j} \Gamma(z_i, z_j)\Big). \tag{16}$$

The σ_Γ^l are the components of a surjective graded algebra homomorphism $\sigma_\Gamma \colon U \to \mathcal{R}_\Gamma$, where U is the algebra with underlying space

$$U = \bigoplus_l U^l = \bigoplus_l \Bbbk(z_1, \ldots, z_l) \tag{17}$$

and the 'concatenation' product whose component $U^k \times U^{l-k} \to U^l$ is defined by

$$f \cdot g = f(z_1, \dots, z_k) g(z_{k+1}, \dots, z_l)\,. \tag{18}$$

Let $I_\Gamma^l = \ker(\sigma_\Gamma^l)$, so we have

$$\ker(\sigma_\Gamma) = I_\Gamma = \bigoplus_l I_\Gamma^l \subseteq U \tag{19}$$

and an induced isomorphism $\sigma_\Gamma \colon U/I_\Gamma \xrightarrow{\cong} \mathcal{R}_\Gamma$. We call U/I_Γ the *large abstract shuffle algebra.*

The subalgebra of U consisting of Laurent polynomials,

$$T = \bigoplus_l T^l = \bigoplus_l \Bbbk[z_1^{\pm 1}, \dots, z_l^{\pm 1}] \subseteq U, \tag{20}$$

is generated by the basis elements z_1^a of $T^1 = \Bbbk[z_1^{\pm 1}]$ and is isomorphic to the tensor algebra on these generators.

The *abstract shuffle algebra*, or just *shuffle algebra* for short, is the image

$$S_\Gamma = T/(I_\Gamma \cap T) = (T + I_\Gamma)/I_\Gamma \subseteq U/I_\Gamma \tag{21}$$

of T in the large abstract shuffle algebra U/I_Γ. The isomorphism $U/I_\Gamma \cong \mathcal{R}_\Gamma$ induced by σ_Γ restricts to an isomorphism

$$\sigma_\Gamma \colon S_\Gamma \xrightarrow{\cong} \mathcal{S}_\Gamma \overset{\text{def}}{=} \sigma_\Gamma(T) \subseteq \mathcal{R}_\Gamma. \tag{22}$$

from the shuffle algebra S_Γ to the subalgebra $\mathcal{S}_\Gamma$ of $\mathcal{R}_\Gamma$ generated by the elements $z_1^a \in \mathcal{R}_\Gamma^1$. We call $\mathcal{S}_\Gamma$ the *concrete shuffle algebra.*

The diagram below summarizes the relationships between these algebras.

$$\begin{tikzcd}
T \arrow[r, hook] \arrow[d, two heads] & U \arrow[r, equal] \arrow[d, two heads] & \bigoplus_l \Bbbk(z_1, \dots, z_l) \\
S_\Gamma \arrow[r, hook] \arrow[d, "\sigma_\Gamma", "\wr"'] & U/I_\Gamma \arrow[d, "\sigma_\Gamma", "\wr"'] & \\
\mathcal{S}_\Gamma \arrow[r, hook] & \mathcal{R}_\Gamma \arrow[r, equal] & \bigoplus_l \Bbbk(z_1, \dots, z_l)^{S_l}
\end{tikzcd} \tag{23}$$

Remark 3.1.1. If $\Gamma'(w, y) = h(w, y)\Gamma(w, y)$, where $h(w, y) = h(y, w) \neq 0$, then $I_{\Gamma'} = I_\Gamma$ and $S_{\Gamma'} = S_\Gamma$. In this way, different choices of Γ give rise to different concrete realizations $\mathcal{S}_\Gamma$ of the same shuffle algebra S_Γ. The algebra S_Γ is thus more canonical than $\mathcal{S}_\Gamma$ and has a simpler product (concatenation of Laurent polynomials), but has non-trivial relations. The algebra $\mathcal{S}_\Gamma$ has the advantage that the symmetric rational function $f(z_1, \dots, z_l)$ representing an element of $\mathcal{S}_\Gamma^l$ is unique.

Remark 3.1.2. We can think of the abstract shuffle algebra S_Γ either as a quotient of T or as the subalgebra $(T + I_\Gamma)/I_\Gamma$ of U/I_Γ. From the latter point of view, any rational function $\phi \in T^l + I_\Gamma^l$, that is, ϕ congruent modulo I_Γ^l to a Laurent polynomial, represents an element of S_Γ. An explicit Laurent polynomial η such that $\eta \equiv \phi \pmod{I_\Gamma^l}$ may be hard to compute, is not unique, and need not have any simple form, even if ϕ does. However, we can often avoid the need to construct η, since its image $\sigma_\Gamma(\eta) = \sigma_\Gamma(\phi)$ in the concrete shuffle algebra $\mathcal{S}_\Gamma$ can be computed directly from ϕ.

3.2. **The shuffle to Schiffmann algebra isomorphism.** We use the same notation as in [3, 4] for the Schiffmann algebra $\mathcal{E}$ of [5]. In our notation, $\mathcal{E}$ is generated by subalgebras $\Lambda(X^{m,n})$ isomorphic to the algebra Λ of symmetric functions over $\Bbbk = \mathbb{Q}(q,t)$, one for each pair of coprime integers m, n, and a central Laurent polynomial subalgebra $\Bbbk[c_1^{\pm 1}, c_2^{\pm 1}]$, subject to some defining relations. A translation between this notation and that of [5, 17, 18] can be found in [4, §3.2], and a presentation of the defining relations in [3, §3].

The 'right half-plane' subalgebra $\mathcal{E}^+ \subseteq \mathcal{E}$ is generated by $\Lambda(X^{m,n})$ for $m > 0$, or equivalently (as a consequence of the relations) by the elements $p_1(X^{1,a})$. Schiffmann and Vasserot [18, Theorem 10.1] showed that $\mathcal{E}^+$ is isomorphic to the shuffle algebra S_Γ for a suitable choice of Γ. We use the following version of their theorem, modified the same way as in [4, Proposition 3.5.1].

Theorem 3.2.1 ([18])**.** *Let $\Gamma(w,y)$ be a rational function such that*

$$\Gamma(w,y)/\Gamma(y,w) = \Omega[-w/y\,\widehat{M}], \tag{24}$$

where $\Omega[-w/y\,\widehat{M}]$ is given by (8)*. Then there is an algebra isomorphism $\psi\colon S_\Gamma \to \mathcal{E}^+$ given on the generators by $\psi(z_1^a) = p_1[-MX^{1,a}]$.*

If $\Gamma(w,y)$ and $\Gamma'(w,y)$ both satisfy (24), then they differ by a symmetric factor $h(w,y) = \Gamma'(w,y)/\Gamma(w,y)$, as in Remark 3.1.1. Accordingly, from this point on, for any $\Gamma(w,y)$ satisfying (24), we fix

$$S = S_\Gamma, \quad S^l = S^l_\Gamma, \quad I = I_\Gamma, \quad I^l = I^l_\Gamma\,. \tag{25}$$

Although the abstract shuffle algebra S does not depend on the choice of Γ in Theorem 3.2.1, the concrete shuffle algebra $\mathcal{S}_\Gamma$ does. The following two choices of Γ which satisfy (24) turn out to be convenient.

$$\widehat{\Gamma}(w,y) = \frac{1 - q\,t\,w/y}{(1-y/w)(1-q\,w/y)(1-t\,w/y)}, \tag{26}$$

$$\check{\Gamma}(w,y) = (1-w/y)(1-q\,y/w)(1-t\,y/w)(1-q\,t\,w/y). \tag{27}$$

For either of these we define ψ_Γ to be the isomorphism

$$\psi_\Gamma = \psi \circ \sigma_\Gamma^{-1}\colon \mathcal{S}_\Gamma \xrightarrow{\simeq} \mathcal{E}^+, \tag{28}$$

so we have a commutative diagram, with all arrows isomorphisms,

$$\begin{tikzcd}
\mathcal{S}_{\widehat{\Gamma}} \arrow[dr, "\psi_{\widehat{\Gamma}}"] & \\
S \arrow[u, "\sigma_{\widehat{\Gamma}}"'] \arrow[r, "\psi"] \arrow[d, "\sigma_{\check{\Gamma}}"] & \mathcal{E}^+\,. \\
\mathcal{S}_{\check{\Gamma}} \arrow[ur, "\psi_{\check{\Gamma}}"'] &
\end{tikzcd} \tag{29}$$

3.3. **Grading.** The algebra $\mathcal{E}$ has a $\mathbb{Z}^2$ grading in which $\Bbbk[c_1^{\pm 1}, c_2^{\pm 1}]$ has degree $(0,0)$ and $f(X^{m,n})$ has degree (dm, dn) for $f \in \Lambda$ of degree d. We denote by $\mathcal{E}^{(a,b)}$ the (a,b)-graded component. The subalgebra $\mathcal{E}^+$ is an $\mathbb{N}\times\mathbb{Z}$ graded subalgebra of $\mathcal{E}$. Set $(\mathcal{E}^+)^{(l,\bullet)} = \bigoplus_{d\in\mathbb{Z}} (\mathcal{E}^+)^{(l,d)}$.

The algebra T is $\mathbb{N}\times\mathbb{Z}$ graded, where the component of degree (l,d) consists of Laurent polynomials $\phi \in T^l$ homogeneous of degree d.

If $\Gamma(w,y)$ is a function of w/y satisfying (24), and in particular for $\Gamma = \widehat{\Gamma}$ or $\Gamma = \check{\Gamma}$, the symmetrization operators σ_Γ^l are degree preserving, so $T \cap I = T \cap I_{\widehat{\Gamma}} = T \cap I_{\check{\Gamma}}$ is an $\mathbb{N}\times\mathbb{Z}$ graded ideal. The shuffle algebra S therefore inherits an $\mathbb{N}\times\mathbb{Z}$ grading from T, and $\sigma_{\widehat{\Gamma}}$, $\sigma_{\check{\Gamma}}$ induce $\mathbb{N}\times\mathbb{Z}$ gradings on $\mathcal{S}_{\widehat{\Gamma}}$ and $\mathcal{S}_{\check{\Gamma}}$ such that a symmetric rational function $h(z_1,\dots,z_l)$ in $\mathcal{S}^l_{\widehat{\Gamma}}$ or $\mathcal{S}^l_{\check{\Gamma}}$ belongs to $\mathcal{S}^{(l,d)}_{\widehat{\Gamma}}$ or $\mathcal{S}^{(l,d)}_{\check{\Gamma}}$ if and only if it is homogeneous of degree d in the variables z_i.

The following is clear from the definitions.

Proposition 3.3.1. *The isomorphisms $\psi\colon S \to \mathcal{E}^+$, $\psi_{\widehat{\Gamma}}\colon \mathcal{S}_{\widehat{\Gamma}} \to \mathcal{E}^+$, and $\psi_{\check{\Gamma}}\colon \mathcal{S}_{\check{\Gamma}} \to \mathcal{E}^+$ preserve the $\mathbb{N}\times\mathbb{Z}$ grading.*

3.4. **Catalanimals and their cubs.** Here and throughout, $R = \{\alpha_{ij} \mid 1 \le i,j \le l,\ i \ne j\}$ denotes the set of roots for GL_l, where $\alpha_{ij} = \epsilon_i - \epsilon_j \in \mathbb{Z}^l$, and $R_+ = \{\alpha_{ij} \in R \mid i < j\}$ the set of positive roots. The number l will usually be understood from the context; otherwise we specify it by writing $R(\mathrm{GL}_l)$ or $R_+(\mathrm{GL}_l)$.

Definition 3.4.1. Given a weight $\lambda \in \mathbb{Z}^l$ and subsets $R_q, R_t, R_{qt} \subseteq R_+$, we define the corresponding *Catalanimal* of length l to be the symmetric rational function

$$H(R_q,R_t,R_{qt},\lambda) \overset{\text{def}}{=} \sum_{w\in S_l} w\left(\frac{\mathbf{z}^\lambda \prod_{\alpha\in R_{qt}}(1-q\,t\,\mathbf{z}^\alpha)}{\prod_{\alpha\in R_+}(1-\mathbf{z}^{-\alpha})\prod_{\alpha\in R_q}(1-q\,\mathbf{z}^\alpha)\prod_{\alpha\in R_t}(1-t\,\mathbf{z}^\alpha)}\right), \tag{30}$$

in variables $\mathbf{z} = z_1,\dots,z_l$, where $\mathbf{z}^\lambda$ stands for $z_1^{\lambda_1}\cdots z_l^{\lambda_l}$.

We also define two related functions

$$\phi(R_q,R_t,R_{qt},\lambda) \overset{\text{def}}{=} \frac{\mathbf{z}^\lambda \prod_{\alpha\in R_+\setminus R_q}(1-q\,\mathbf{z}^\alpha)\prod_{\alpha\in R_+\setminus R_t}(1-t\,\mathbf{z}^\alpha)}{\prod_{\alpha\in R_+\setminus R_{qt}}(1-q\,t\,\mathbf{z}^\alpha)}, \tag{31}$$

(32)
$$g(R_q,R_t,R_{qt},\lambda) \overset{\text{def}}{=} \sum_{w\in S_l} w\Big(\mathbf{z}^\lambda \prod_{\alpha\in R_+}(1-\mathbf{z}^\alpha)\prod_{\alpha\in R\setminus R_q}(1-q\,\mathbf{z}^\alpha)\prod_{\alpha\in R\setminus R_t}(1-t\,\mathbf{z}^\alpha)\prod_{\alpha\in R_{qt}}(1-q\,t\,\mathbf{z}^\alpha)\Big),$$

so that

$$\sigma_{\widehat{\Gamma}}\big(\phi(R_q,R_t,R_{qt},\lambda)\big) = H(R_q,R_t,R_{qt},\lambda), \tag{33}$$

$$\sigma_{\check{\Gamma}}\big(\phi(R_q,R_t,R_{qt},\lambda)\big) = g(R_q,R_t,R_{qt},\lambda). \tag{34}$$

Note that the following conditions on a Catalanimal $H = H(R_q,R_t,R_{qt},\lambda)$ of length l and its associated functions $\phi = \phi(R_q,R_t,R_{qt},\lambda)$ and $g = g(R_q,R_t,R_{qt},\lambda)$ are equivalent:

(i) H belongs to the concrete shuffle algebra $\mathcal{S}_{\widehat{\Gamma}}$;
(ii) g belongs to the concrete shuffle algebra $\mathcal{S}_{\check{\Gamma}}$;
(iii) $\phi \in T^l + I^l$, hence ϕ represents an element of the shuffle algebra S, as in Remark 3.1.2.

When these conditions hold, there is a corresponding element of the Schiffmann algebra

$$\zeta = \psi(\phi) = \psi_{\widehat{\Gamma}}(H) = \psi_{\check{\Gamma}}(g) \in \mathcal{E}^{+}\,. \tag{35}$$

Our work here focuses on identifying certain Catalanimals that satisfy the above conditions and have the further property that $\zeta \in \Lambda(X^{m,n})$ for some (m,n) with $m > 0$.

Definition 3.4.2. Let $H = H(R_q, R_t, R_{qt}, \lambda)$ be a Catalanimal. If $\psi_{\widehat{\Gamma}}(H) \in \Lambda(X^{m,n})$, we call the symmetric function $f(X)$ such that $\psi_{\widehat{\Gamma}}(H) = f[-MX^{m,n}]$ its *cub*, and write $\operatorname{cub}(H) = f$.

To summarize the discussion above in the setting of Definition 3.4.2, given a Catalanimal $H = H(R_q, R_t, R_{qt}, \lambda)$ of length l such that $\psi_{\widehat{\Gamma}}(H) = f[-MX^{m,n}]$, the four objects H, $\phi(R_q, R_t, R_{qt}, \lambda)$, $g(R_q, R_t, R_{qt}, \lambda)$, and $f[-MX^{m,n}]$ are related as shown below. We will frequently go back and forth between these viewpoints.

$$\begin{tikzcd}
 & H \in \mathcal{S}^{l}_{\widehat{\Gamma}} \arrow[dr, "\psi_{\widehat{\Gamma}}"] & \\
\phi \in T^l + I^l \arrow[ur, "\sigma^{l}_{\widehat{\Gamma}}"] \arrow[r, two heads] \arrow[dr, "\sigma^{l}_{\check{\Gamma}}"'] & S^l = (T^l + I^l)/I^l \arrow[u, "\sigma^{l}_{\widehat{\Gamma}}"'] \arrow[d, "\sigma^{l}_{\check{\Gamma}}"] \arrow[r, "\psi"] & (\mathcal{E}^{+})^{(l,\bullet)} \ni f[-MX^{m,n}] \\
 & g \in \mathcal{S}^{l}_{\check{\Gamma}} \arrow[ur, "\psi_{\check{\Gamma}}"'] &
\end{tikzcd}$$

3.5. **Catalanimals and the operator ∇.** As touched on in the introduction, one nice consequence of having a Catalanimal representative for an element $f[-MX^{m,1}]$ in the Schiffmann algebra is a raising operator formula for $\nabla^m f(X)$, where ∇ is the linear operator introduced in [1], which acts diagonally on the basis of modified Macdonald polynomials $\tilde{H}_\mu(X; q, t)$ [8] by $\nabla \tilde{H}_\mu = t^{n(\mu)} q^{n(\mu^*)} \tilde{H}_\mu$, with $n(\mu) = \sum_i (i-1)\mu_i$.

The following lemma relates the operator ∇ to the action of $\mathcal{E}$ on Λ constructed by Schiffmann and Vasserot [18]. Here we use the version of this action given by [4, Proposition 3.3.1].

Lemma 3.5.1. *For any symmetric function f, the element $f[-MX^{m,1}] \in \mathcal{E}$ acting on $1 \in \Lambda(X)$ is given by*

$$f[-MX^{m,1}] \cdot 1 = \nabla^m f(X). \tag{36}$$

Proof. By [4, Lemma 3.4.1], $f(X^{m,1})$ acts as $\nabla^m f[-X/M]^\bullet \, \nabla^{-m}$, where $f[-X/M]^\bullet$ is the operator of multiplication by $f[-X/M]$. Since $\nabla(1) = 1$, the result follows. □

We denote the Weyl symmetrization operator for GL_l by

$$\boldsymbol{\sigma}(f(z_1, \ldots, z_l)) = \sum_{w \in S_l} w \left(\frac{f(\mathbf{z})}{\prod_{\alpha \in R_+} (1 - \mathbf{z}^{-\alpha})} \right). \tag{37}$$

If $\lambda \in \mathbb{Z}^l$ is a dominant weight, then $\boldsymbol{\sigma}(\mathbf{z}^\lambda) = \chi_\lambda$ is the corresponding irreducible GL_l character. For any weight μ, we either have $\boldsymbol{\sigma}(\mathbf{z}^\mu) = \pm\chi_\lambda$ for a suitable λ, or $\boldsymbol{\sigma}(\mathbf{z}^\mu) = 0$.

Let $\eta(z_1,\ldots,z_l)\in \Bbbk[z_1^{\pm},\ldots,z_l^{\pm 1}]$ be a Laurent polynomial. A q,t *raising operator series* is an expression of the form

$$h(\mathbf{z}) = \boldsymbol{\sigma}\left(\frac{\eta(\mathbf{z})}{\prod_{\alpha\in R_+}(1-q\,\mathbf{z}^{\alpha})\prod_{\alpha\in R_+}(1-t\,\mathbf{z}^{\alpha})}\right), \tag{38}$$

interpreted as an infinite formal linear combination of irreducible GL_l characters χ_λ with coefficients in $\Bbbk$ by expanding the factors $(1-q\,\mathbf{z}^{\alpha})^{-1} = 1+q\,\mathbf{z}^{\alpha}+\cdots$ and $(1-t\,\mathbf{z}^{\alpha})^{-1} = 1+t\,\mathbf{z}^{\alpha}+\cdots$ as geometric series before applying $\boldsymbol{\sigma}$. This makes sense since for each λ, the set of weights μ such that $\boldsymbol{\sigma}(\mathbf{z}^{\mu})=\pm\chi_\lambda$ is finite.

By writing $\eta(\mathbf{z})$ in the form $c(q,t)\eta'(\mathbf{z})$, where $\eta'\in\mathbb{Z}[q,t][z_1^{\pm1},\ldots,z_l^{\pm1}]$, we see that $h(\mathbf{z})$ can be expressed as a scalar in $\Bbbk$ times a power series in q, t over the ring of virtual GL_l characters $\mathbb{Z}[z_1^{\pm1},\ldots,z_l^{\pm1}]^{S_l}$, and also that this power series expansion is the same as the raising operator series expansion. It follows that $h(\mathbf{z})$ considered as a symmetric rational function over $\Bbbk$ determines its raising operator series expansion (not every symmetric rational function is a raising operator series, however).

In most of this paper, we regard Catalanimals merely as symmetric rational functions. However, rewriting (30) as

$$H(R_q,R_t,R_{qt},\lambda) = \boldsymbol{\sigma}\left(\frac{\mathbf{z}^{\lambda}\prod_{\alpha\in R_{qt}}(1-q\,t\,\mathbf{z}^{\alpha})}{\prod_{\alpha\in R_q}(1-q\,\mathbf{z}^{\alpha})\prod_{\alpha\in R_t}(1-t\,\mathbf{z}^{\alpha})}\right), \tag{39}$$

we see that every Catalanimal can also be viewed as a q,t raising operator series.

The *polynomial characters* of GL_l are the irreducible characters χ_λ for which $\lambda\in\mathbb{N}^l$, that is, λ is a partition. We define the *polynomial part* $h(\mathbf{z})_{\mathrm{pol}}$ of a raising operator series $h(\mathbf{z})$ to be its truncation to polynomial characters. If $h(\mathbf{z})$ is homogeneous of degree d, then all irreducible characters χ_λ in it have $\lambda_1+\cdots+\lambda_l=d$, so $h(\mathbf{z})_{\mathrm{pol}}$ is a finite linear combination of polynomial GL_l characters—that is, a symmetric polynomial in l variables over $\Bbbk$.

Proposition 3.5.2. *Let $H = H(R_q,R_t,R_{qt},\lambda)$, be a Catalanimal of length l such that $\psi_{\widehat{\Gamma}}(H)\in\Lambda(X^{m,1})$, and let $f(X)$ be its cub. Then*

$$(\omega\nabla^m f)(z_1,\ldots,z_l) = H_{\mathrm{pol}}. \tag{40}$$

Moreover, this determines $\nabla^m f$, since the Schur expansion of the symmetric function $\omega\nabla^m f(X)$ contains only terms s_λ with $\ell(\lambda)\le l$.

Proof. By Lemma 3.5.1, we can replace $\nabla^m f$ with $f[-MX^{m,1}]\cdot 1$. The result now follows from [4, Proposition 3.5.2] by taking ϕ in [4, (48)] to be a Laurent polynomial congruent modulo I^l to $\phi(R_q,R_t,R_{qt},\lambda)$ and noting that the right hand side of [4, (48)] then becomes H_{pol}. □

3.6. **Shuffle algebra toolkit.** Negut [16] provides a useful toolkit for working with the shuffle algebra, which we will use extensively in §§4–6, below. For the convenience of readers who wish to compare our versions of results cited from [16] with the originals, we briefly discuss how Negut's notation and conventions are related to ours.

For $m>0$, the elements of $\mathcal{E}$ denoted $u_{km,kn}$ in [5] and [16] are $\omega p_k(X^{n,m})$ in our notation. Because we have switched the indices m,n, the positive half subalgebra denoted $\mathcal{E}^+$ in [16] is actually the upper half-plane subalgebra generated by $\Lambda(X^{n,m})$ for $m > 0$ in our notation. However, there is an automorphism of $\mathcal{E}$ which carries $f(X^{n,m})$ to $f(X^{m,-n})$ for $m > 0$, and we use this to identify our right half-plane subalgebra $\mathcal{E}^+$ with the subalgebra generated by the $u_{km,kn}$ for $m > 0$ in [5], so that $u_{km,kn}$ corresponds to $\omega p_k(X^{m,-n})$.

The shuffle algebra $\mathcal{A}^+$ in [16] is related to our shuffle algebra as follows. The parameters q_1, q_2 in [16] are $t, q \in \Bbbk$, and the algebra $\mathcal{A}^+$ with z_i replaced by z_i^{-1} coincides with our concrete shuffle algebra $\mathcal{S}_\Gamma$ for

$$\Gamma(w, y) = \frac{(1 - w/y)(1 - q\,t\,w/y)}{(1 - q\,w/y)(1 - t\,w/y)}. \tag{41}$$

This function $\Gamma(w, y)$ is $\omega(y/w)$ in the notation of [16, (2.3)].

The isomorphism Υ in Negut [16, Theorem 3.1] sends $u_{1,-a}$ to z_1^{-a}. Hence, Υ^{-1} corresponds in our notation to an isomorphism $S \cong \mathcal{S}_\Gamma \xrightarrow{\simeq} \mathcal{E}^+$ sending z_1^a to $p_1(X^{1,a}) \in \mathcal{E}^+$. Since we defined $\psi\colon S \to \mathcal{E}^+$ in Theorem 3.2.1 by $\psi(z_1^a) = p_1[-MX^{1,a}] = -Mp_1(X^{1,a})$, we see that ψ differs by a factor $(-M)^l$ on S^l from the isomorphism corresponding to Υ^{-1}. By [16, Theorem 1.1], the elements $P_{k,d} \in \mathcal{A}^+$ defined by [16, (1.2)] are given by $P_{k,d} = \Upsilon(u_{k,d})$. Using the identification of $u_{km,-kn}$ with $\omega p_k(X^{m,n})$, we have the following diagram of isomorphisms and corresponding elements.

$$\begin{array}{ccccc} \mathcal{A}^+ & \xrightarrow[z_i \mapsto z_i^{-1}]{\simeq} & S & \xrightarrow[\psi]{\simeq} & \mathcal{E}^+ \\ \uplus & & & & \uplus \\ P_{km,-kn} & & \longmapsto & & (-M)^{km}\omega p_k(X^{m,n}) \end{array} \tag{42}$$

4. Cuddly Catalanimals

In this section we identify combinatorial conditions which guarantee that a Catalanimal belongs to $\mathcal{S}_{\widehat{\Gamma}}$ and that its image under $\psi_{\widehat{\Gamma}}$ belongs to one of the distinguished subalgebras $\Lambda(X^{m,n})$ of the Schiffmann algebra.

4.1. Tame Catalanimals.

Negut [16, Theorem 2.2] gives a criterion based on the wheel condition of Feigin et al. [6] for a symmetric rational function to belong to the concrete shuffle algebra. For $\mathcal{S}_{\check{\Gamma}}$, Negut's criterion takes the form given by the theorem below. Note that, because $\check{\Gamma}(w, y)$ in (27) is a Laurent polynomial, the elements of $\mathcal{S}_{\check{\Gamma}}$ are symmetric Laurent polynomials and not just rational functions.

A symmetric Laurent polynomial $g(\mathbf{z}) \in \Bbbk[z_1^{\pm 1}, \ldots, z_l^{\pm 1}]^{S_l}$ satisfies the *wheel condition* if it vanishes whenever any three of the variables z_i, z_j, z_k are in the ratio $(z_i : z_j : z_k) = (1 : q : q\,t)$ or $(1 : t : q\,t)$. If $l < 3$, the wheel condition holds vacuously.

Theorem 4.1.1 ([16])**.** *A symmetric Laurent polynomial $g(z_1, \ldots, z_l)$ belongs to $\mathcal{S}^l_{\check{\Gamma}}$ if and only if it satisfies the wheel condition and vanishes whenever $z_i = z_j$ for $i \neq j$.*

To connect this with [16, Theorem 2.2] we remark that, up to an irrelevant factor, our $g(\mathbf{z})$ is the numerator in [16, (2.4)] with the variables inverted. Because $g(\mathbf{z})$ is symmetric, the condition that it vanishes whenever $z_i = z_j$ is equivalent to $\prod_{i<j}(z_i - z_j)^2$ dividing $g(\mathbf{z})$. The wheel condition then applies to the factor that remains after dividing by $\prod_{i<j}(z_i - z_j)^2$.

If $A, B \subseteq R_+$ are subsets of the positive roots for GL_l, we set $[A, B] = R_+ \cap (A + B)$. The reason for this notation is that if $\mathfrak{g}_C = \sum_{\alpha \in C} \mathfrak{g}_\alpha$ denotes a sum of root spaces, then $\mathfrak{g}_{[A,B]} = [\mathfrak{g}_A, \mathfrak{g}_B]$ in the Lie algebra $\mathfrak{gl}_l$.

Definition 4.1.2. A Catalanimal $H(R_q, R_t, R_{qt}, \lambda)$ is *tame* if the root sets R_q, R_t, R_{qt} satisfy

$$[R_q, R_t] \subseteq R_{qt}. \tag{43}$$

Strictly speaking, tameness is a condition on the root sets rather than the rational function $H(R_q, R_t, R_{qt}, \lambda)$, but it has the following consequence for the function.

Proposition 4.1.3. *If $H = H(R_q, R_t, R_{qt}, \lambda)$ is a tame Catalanimal, then $H \in \mathcal{S}_{\widehat{\Gamma}}$; hence it corresponds to an element $\psi_{\widehat{\Gamma}}(H) \in \mathcal{E}^+$ in the Schiffmann algebra.*

Proof. It's equivalent to show that $g(\mathbf{z}) = g(R_q, R_t, R_{qt}, \lambda)$ belongs to $\mathcal{S}_{\check{\Gamma}}$. Using Theorem 4.1.1, we need to check that $g(\mathbf{z})$ vanishes when any two distinct variables z_i, z_j are set equal, or when any three are in the ratio $(1 : q : q\,t)$ or $(1 : t : q\,t)$. We will verify that the $w = 1$ term in (32) vanishes under any of these conditions; this suffices since the conditions are symmetric in the variables z_i.

The factor $\prod_{\alpha \in R_+} (1 - \mathbf{z}^\alpha)$ gives the required vanishing when $z_i = z_j$.

Suppose $(z_i : z_j : z_k) = (1 : q : q\,t)$. If $\alpha_{ij} \notin R_q$, the factor $(1 - q\,\mathbf{z}^{\alpha_{ij}})$ in the product over $R \setminus R_q$ in (32) vanishes, while if $\alpha_{jk} \notin R_t$, the factor $(1 - t\,\mathbf{z}^{\alpha_{jk}})$ in the product over $R \setminus R_t$ vanishes. If neither of these factors appears, we have $\alpha_{ij} \in R_q$ and $\alpha_{jk} \in R_t$, hence $\alpha_{ik} \in R_{qt}$ by hypothesis. Thus, the product over R_{qt} contains the factor $(1 - q\,t\,\mathbf{z}^{\alpha_{ik}})$ which vanishes. The same reasoning with the roles of q and t exchanged applies if $(z_i : z_j : z_k) = (1 : t : q\,t)$. □

4.2. Cuddly Catalanimals.

Definition 4.2.1. We use the abbreviations $I^c = [l] \setminus I$ for $I \subseteq [l] = \{1, \ldots, l\}$ and $A^{I,J} = \{\alpha_{ij} \in A \mid i \in I,\, j \in J\}$ for $A \subseteq R(GL_l)$ and $I, J \subseteq [l]$. We also write $\sum A = \sum_{\alpha \in A} \alpha$ for the sum of the roots in A. For any weight $\nu = (\nu_1, \ldots, \nu_l)$, we define $|\nu| = \nu_1 + \cdots + \nu_l$, and let $\nu_I = (\nu_{i_1}, \ldots, \nu_{i_k})$ denote the subsequence with index set $I = \{i_1 < \cdots < i_k\} \subseteq [l]$.

Let $(m, n) \in \mathbb{Z}_+ \times \mathbb{Z}$ be a pair of coprime integers. A Catalanimal $H(R_q, R_t, R_{qt}, \lambda)$ of length l is *(m,n)-cuddly* if

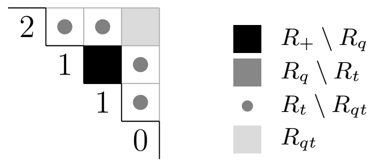
(a) it is tame, that is, $[R_q, R_t] \subseteq R_{qt}$;
(b) $|\lambda| = ln/m$ (in particular, m must divide l); and
(c) it satisfies the *cuddliness bounds*

$$\bigl|\, \lambda[I]_I \,\bigr| \leq |I|\, \frac{n}{m} \quad \text{for all } I \subseteq \{1, \ldots, l\}, \tag{44}$$

where

$$\lambda[I] = \lambda + \sum R_+^{I,I^c} - \sum R_q^{I,I^c} - \sum R_t^{I,I^c} + \sum R_{qt}^{I,I^c}. \tag{45}$$

Example 4.2.2. The Catalanimal below is $(1,1)$-cuddly. It is drawn with the same conventions as Figure 1.



Condition (a) holds since $[R_q, R_t] = R_{qt}$, and $|\lambda| = l = 4$ verifies (b). For (c), we must check $|\lambda[I]_I| \le |I|$ for all subsets $I \subset \{1,2,3,4\}$; this computation is illustrated below for the subsets of size 2, with bold letters indicating the subsequence $\lambda[I]_I$ of $\lambda[I]$.

I	12	13	14	23	24	34	
$\lambda[I]$	**11**11	**1**2**0**1	**0**22**0**	2**00**2	2**2**0**0**	21**10**	
$\lvert\lambda[I]_I\rvert$	2	1	0	0	2	1	$\le \lvert I\rvert$

Below we will show that if H is an (m,n)-cuddly Catalanimal then $\psi_{\widehat{\Gamma}}(H) \in \Lambda(X^{m,n})$. For this we need the following criterion for $\psi_{\check{\Gamma}}$ to map an element of $\mathcal{S}_{\check{\Gamma}}$ into the subalgebra of $\mathcal{E}^+$ generated by the $\Lambda(X^{m,n})$ for $n/m \le p$.

Theorem 4.2.3 ([16]). *Let $\mathcal{E}^+_{\le p}$ be the subalgebra of $\mathcal{E}^+$ generated by the $\Lambda(X^{m,n})$ for $n/m \le p$. Given $g = g(z_1, \dots, z_l) \in \mathcal{S}^l_{\check{\Gamma}}$, the image $\psi_{\check{\Gamma}}(g) \in (\mathcal{E}^+)^{(l,\bullet)}$ belongs to $\mathcal{E}^+_{\le p}$ if and only if g is supported on monomials in the S_l orbits of dominant weights μ satisfying $\mu_1 + \cdots + \mu_k \le 2k(l-k) + kp$ for all $k = 1, \dots, l$.*

We briefly explain how this follows from [16]. As noted in §3.6, there is an isomorphism $\mathcal{A}^+ \cong S$ between Negut's shuffle algebra and ours that inverts the variables z_i. There is also an anti-isomorphism that does not invert the variables, under which the subalgebra generated by the elements $P_{k,d} \in \mathcal{A}^+$ for $d/k \le p$ corresponds to $\psi^{-1}(\mathcal{E}^+_{\le p}) \subseteq S$. The proof of [16, Theorem 1.1] shows that this subalgebra is the same as the one denoted $\mathcal{A}^p \subseteq \mathcal{A}^+$ in [16, Proposition 2.3]. The criterion on the support of g in Theorem 4.2.3 is a reformulation in terms of $\mathcal{S}_{\check{\Gamma}}$ of the condition [16, (2.7)] that defines $\mathcal{A}^p$.

Proposition 4.2.4. *Let $H = H(R_q, R_t, R_{qt}, \lambda)$ be a Catalanimal of length l and let $g(\mathbf{z}) = g(R_q, R_t, R_{qt}, \lambda)$ be the corresponding symmetric Laurent polynomial defined in* (32).

(i) *If H satisfies the cuddliness bounds* (44) *for (m,n), then $g(\mathbf{z})$ satisfies the condition in Theorem 4.2.3 for $p = n/m$.*
(ii) *If H is (m,n)-cuddly, then $H \in \mathcal{S}_{\widehat{\Gamma}}$ with $\psi_{\widehat{\Gamma}}(H) \in \Lambda(X^{m,n})$, so H has a cub (Definition 3.4.2).*

Proof. For (i), we must show that for every monomial $\mathbf{z}^\nu$ occurring in $g(\mathbf{z})$ and every $I \subseteq [l]$ of size $|I| = k$, we have $|\nu_I| \le 2k(l-k) + kn/m$. We will show that in fact this holds for each term in (32). By symmetry, it is enough to check it for the $w = 1$ term,

$$\mathbf{z}^\lambda \prod_{\alpha \in R_+} (1 - \mathbf{z}^\alpha) \prod_{\alpha \in R \setminus R_q} (1 - q\, \mathbf{z}^\alpha) \prod_{\alpha \in R \setminus R_t} (1 - t\, \mathbf{z}^\alpha) \prod_{\alpha \in R_{qt}} (1 - q\, t\, \mathbf{z}^\alpha). \tag{46}$$

To get a term $\mathbf{z}^\nu$ maximizing $|\nu_I|$ in this product, we must choose the $\mathbf{z}^\alpha$ term from the factors with $\alpha \in R^{I,I^c}$ and the constant term from the factors with $\alpha \in R^{I^c,I}$. For $\alpha \in R^{I,I} \cup R^{I^c,I^c}$, we can choose either term from the factor in question, since $|\alpha_I| = 0$. Since $|R^{I,I^c}| = k(l-k)$ for any I of size k, $|(R \setminus R_q)^{I,I^c}| = k(l-k) - |R_q^{I,I^c}|$ and $|(R \setminus R_t)^{I,I^c}| = k(l-k) - |R_t^{I,I^c}|$. Hence, ν chosen as indicated satisfies

$$|\nu_I| = |\lambda[I]_I| + 2k(l-k) \tag{47}$$

and the cuddliness bound (44) implies $|\nu_I| \leq 2k(l-k) + kn/m$.

For (ii), we know from Proposition 4.1.3 that $H \in \mathcal{S}_{\widehat{\Gamma}}$ and from (i) that $\psi_{\widehat{\Gamma}}(H) \in \mathcal{E}^+_{\leq n/m}$. The definition of $\mathcal{E}^+_{\leq n/m}$ and the $\mathbb{N} \times \mathbb{Z}$ grading of $\mathcal{E}^+$ imply that $(\mathcal{E}^+_{\leq n/m})^{(l,ln/m)} \subseteq \Lambda(X^{m,n})$. Since H is homogeneous of degree $|\lambda| = ln/m$, it follows from Proposition 3.3.1 that $\psi_{\widehat{\Gamma}}(H) \in \Lambda(X^{m,n})$. □

5. The coproduct

The full Schiffmann algebra $\mathcal{E}$ is constructed in [5] as the Drinfeld double of a subalgebra of the Hall algebra of coherent sheaves on an elliptic curve. This subalgebra corresponds to the algebra denoted $\mathcal{A}^{\geq}$ in Negut [16, Proposition 4.1] and, under the identifications in §3.6, to the subalgebra $\mathcal{E}^{\geq}$ of $\mathcal{E}$ generated by $\mathcal{E}^+$ and $\Lambda(X^{0,-1})$ in our notation. The relations in $\mathcal{E}^{\geq}$ yield a tensor product decomposition $\mathcal{E}^{\geq} = \Lambda(X^{0,-1}) \otimes \mathcal{E}^+$ (as a vector space).

By [5, Proposition 4.5], the Hall algebra realization gives rise to a geometrically defined coproduct Δ on $\mathcal{E}^{\geq}$ taking values in a suitably completed tensor product $\mathcal{E}^{\geq} \mathbin{\widehat{\otimes}} \mathcal{E}^{\geq}$; the corresponding coproduct on the shuffle algebra is described in [16]. Here we will use properties of Δ to obtain a combinatorial coproduct formula for the cub of a cuddly Catalanimal.

5.1. Leading term. When evaluated on $\Lambda(X^{m,n})$, the coproduct Δ on $\mathcal{E}^{\geq}$ has a leading term that coincides with the standard coproduct on $\Lambda(X^{m,n})$. To make this precise we define $\mathcal{E}^+_{<p}$, $\mathcal{E}^+_{>p}$ to be the subalgebras of $\mathcal{E}^+$ generated by the $\Lambda(X^{m,n})$ for $n/m < p$ or $n/m > p$, respectively (similar to the definition of $\mathcal{E}^+_{\leq p}$ in Theorem 4.2.3). We also define $\mathcal{E}^{\geq}_{<p}$ to be the subalgebra of $\mathcal{E}^{\geq}$ generated by $\mathcal{E}^+_{<p}$ and $\Lambda(X^{0,-1})$; it decomposes as $\mathcal{E}^{\geq}_{<p} = \Lambda(X^{0,-1}) \otimes \mathcal{E}^+_{<p}$.

The following proposition is a consequence of either [5, p. 1212, line 6] or [16, Lemma 5.3], with a geometric proof in [5] and a shuffle algebra proof in [16].

Proposition 5.1.1 ([5, 16])**.** *For any $f \in \Lambda$ and coprime integers m, n with $m > 0$, the coproduct in $\mathcal{E}^{\geq}$ evaluated on $f(X^{m,n})$ has the form*

$$\Delta(f(X^{m,n})) = f[X^{m,n}_{(1)} + X^{m,n}_{(2)}] + (\text{terms in } \mathcal{E}^{\geq}_{<n/m} \mathbin{\widehat{\otimes}} \mathcal{E}^+_{>n/m}), \tag{48}$$

where the first term is in $\Lambda(X^{m,n}) \otimes \Lambda(X^{m,n})$, and the subscripts $X^{m,n}_{(1)}$, $X^{m,n}_{(2)}$ distinguish the tensor factors.

5.2. Coproduct on the shuffle algebra. Recall that for any $\Gamma = \Gamma(w, y)$ satisfying (24), we have an isomorphism $\psi_\Gamma \colon \mathcal{S}_\Gamma \xrightarrow{\cong} \mathcal{E}^+$. Let $\mathcal{S}^{\geq}_\Gamma = \Lambda(X^{0,-1}) \otimes \mathcal{S}_\Gamma$ be the extended algebra isomorphic to $\mathcal{E}^{\geq}$ via an isomorphism $\psi^{\geq}_\Gamma \colon \mathcal{S}^{\geq}_\Gamma \xrightarrow{\cong} \mathcal{E}^{\geq}$ that is ψ_Γ on $\mathcal{S}_\Gamma$ and the identity on $\Lambda(X^{0,-1})$. We then have a coproduct Δ^Γ on $\Lambda(X^{0,-1}) \otimes \mathcal{S}_\Gamma$ corresponding under $\psi^{\geq}_\Gamma$ to the coproduct on $\mathcal{E}^{\geq}$.

Negut [16] gives the following formula (written in our notation) for the component $\Delta^\Gamma_{k,l-k}$ of Δ^Γ with values in $(\Lambda(X^{0,-1}) \otimes \mathcal{S}^k_\Gamma) \mathbin{\widehat{\otimes}} \mathcal{S}^{l-k}_\Gamma$, when evaluated on $\mathcal{S}_\Gamma$. The symbol $\widehat{\otimes}$ here indicates that the values are infinite sums of elements of different degrees in $(\Lambda(X^{0,-1}) \otimes \mathcal{S}_\Gamma^k) \otimes \mathcal{S}_\Gamma^{l-k}$, as we explain below after stating the result.

We remark that, although Negut uses a different Γ than we do, his proof is not specific to the choice.

Proposition 5.2.1 ([16, Proposition 4.1]). *Assume that $\Gamma(w,y)$ satisfies* (24) *and is a function of w/y. For any $H(z_1,\ldots,z_l) \in \mathcal{S}_\Gamma^l$, we have*

$$\Delta_{k,l-k}^{\Gamma}(H(z_1,\ldots,z_l)) = \omega\Omega[-Y\widehat{M}X^{0,-1}]\frac{H(w_1,\ldots,w_k,y_1,\ldots,y_{l-k})}{\prod_{i=1}^{k}\prod_{j=1}^{l-k}\Gamma(y_j,w_i)}, \tag{49}$$

where $Y = y_1 + \cdots + y_{l-k}$, and we distinguish the factors in $(\Lambda(X^{0,-1}) \otimes \mathcal{S}_\Gamma^k) \mathbin{\widehat{\otimes}} \mathcal{S}_\Gamma^{l-k}$ by writing elements of $\mathcal{S}_\Gamma$ as functions of $w_1,\ldots,w_k$ in the first tensor factor, or $y_1,\ldots,y_{l-k}$ in the second. Elements of $\Lambda(X^{0,-1})$ are understood to belong to the first tensor factor.

More precisely, let $(\Lambda(X^{0,-1}) \otimes \mathcal{S}_\Gamma^k)_d = (\psi_\Gamma^{\geq})^{-1}((\mathcal{E}^{\geq})^{(k,d)})$ be the subspace consisting of functions $h(X^{0,-1})f(\mathbf{w})$ homogeneous of degree d, where $X^{0,-1}$ has degree -1, that is, $h(X^{0,-1})$ has degree $-m$ if $h(X)$ is homogeneous of degree m, and let $(\mathcal{S}_\Gamma^{l-k})_d = \psi_\Gamma^{-1}((\mathcal{E}^{+})^{(l-k,d)})$ be the subspace consisting of functions $g(\mathbf{y})$ homogeneous of degree d. The coproduct $\Delta_{k,l-k}^{\Gamma}(H(\mathbf{z}))$ on the left hand side of (49) is an infinite sum with components in the spaces $(\Lambda(X^{0,-1}) \otimes \mathcal{S}_\Gamma^k)_{d_1} \otimes (\mathcal{S}_\Gamma^{l-k})_{d_2}$ for some set of degrees with d_1 bounded above and d_2 bounded below.

The assumption on $\Gamma(w,y)$ ensures that the right hand side can be expanded as a formal Laurent series in the w_i^{-1} and y_j, multiplied by rational functions of w_i/w_j and y_i/y_j (which are thus homogeneous of degree zero in both $\mathbf{w}$ and $\mathbf{y}$). The meaning of (49) is that the component of $\Delta_{k,l-k}^{\Gamma}(H(\mathbf{z}))$ in $(\Lambda(X^{0,-1}) \otimes \mathcal{S}_\Gamma^k)_{d_1} \otimes (\mathcal{S}_\Gamma^{l-k})_{d_2}$ is given by the homogeneous component of these degrees in the series expansion on the right hand side.

Suppose now that $\psi_\Gamma(H(\mathbf{z})) = f(X^{m,n})$, where $f \in \Lambda$ is homogeneous of degree d. Then $f(X^{m,n}) \in (\mathcal{E}^{+})^{(dm,dn)}$, and $H(\mathbf{z})$ is a function of $l = dm$ variables, homogeneous of degree dn. Hence, $\Delta_{k,l-k}^{\Gamma}(H(\mathbf{z}))$ has components in $(\Lambda(X^{0,-1}) \otimes \mathcal{S}_\Gamma^k)_{d_1} \otimes (\mathcal{S}_\Gamma^{l-k})_{d_2}$ for $d_1 + d_2 = dn$. Components contributing to the terms of (48) in $\mathcal{E}^{\geq}_{<n/m} \mathbin{\widehat{\otimes}} \mathcal{E}^{+}_{>n/m}$ must have $d_1 < kn/m$, $d_2 > (l-k)n/m$, while those contributing to the first term have $d_1 = kn/m$, $d_2 = (l-k)n/m$, necessarily with k a multiple of m.

Since the first term in (48) is in $\mathcal{E}^{+} \otimes \mathcal{E}^{+}$, all terms of (49) contributing to it involve only the constant term in the factor $\omega\Omega[-Y\widehat{M}X^{0,-1}]$. These observations yield the following corollary to Propositions 5.1.1 and 5.2.1.

Corollary 5.2.2. *Assume that Γ satisfies the hypothesis of Proposition 5.2.1. Suppose that $H(\mathbf{z}) = H(z_1,\ldots,z_l) \in \mathcal{S}_\Gamma^l$ has $\psi_\Gamma(H(\mathbf{z})) = f(X^{m,n})$, where $f \in \Lambda$ is homogeneous of degree d and (therefore) $l = dm$. Given $0 \leq e \leq d$, let $k = em$ and let*

$$h(\mathbf{w},\mathbf{y}) = h(w_1,\ldots,w_k,y_1,\ldots,y_{l-k}) = \left(\frac{H(w_1,\ldots,w_k,y_1,\ldots,y_{l-k})}{\prod_{i=1}^{k}\prod_{j=1}^{l-k}\Gamma(y_j,w_i)}\right)_{\max} \tag{50}$$

be the homogeneous component of maximum possible degree en in $\mathbf{w}$ and minimum possible degree $(d-e)n$ in $\mathbf{y}$. Regard $h(\mathbf{w},\mathbf{y})$ as an element of $\mathcal{S}_\Gamma^k \otimes \mathcal{S}_\Gamma^{l-k}$, with variables $\mathbf{w}$ in the *first tensor factor and* $\mathbf{y}$ *in the second. Then we have*

$$(\psi_\Gamma \otimes \psi_\Gamma)(h(\mathbf{w},\mathbf{y})) = f[X^{m,n}_{(1)} + X^{m,n}_{(2)}]_{e,d-e} \in \Lambda(X^{m,n}) \otimes \Lambda(X^{m,n}), \tag{51}$$

where the subscripts $X^{m,n}_{(1)}$, $X^{m,n}_{(2)}$ *distinguish the tensor factors, and* $f[X+Y]_{e,d-e}$ *designates the homogeneous component of* $f[X+Y]$ *of degree* e *in* X *and* $d-e$ *in* Y.

5.3. **Coproduct formula for cuddly Catalanimals.** By Proposition 4.2.4, every (m,n)-cuddly Catalanimal has a cub (Definition 3.4.2). Using Corollary 5.2.2, we will now obtain a combinatorial expression for the coproduct of the cub.

We again use the notation $A^{I,J} = \{\alpha_{ij} \in A \mid i \in I,\, j \in J\}$ from Definition 4.2.1. Given $A \subseteq R(\mathrm{GL}_l)$ and $I \subseteq [l]$ of size $|I| = k$, we also define $A|_I$ to be the set of roots $\{\alpha_{ij} \mid \alpha_{\pi(i)\,\pi(j)} \in A^{I,I}\} \subseteq R(\mathrm{GL}_k)$, where $\pi\colon [k] \to I$ is the unique increasing bijection.

Theorem 5.3.1. *Let* $H = H(R_q, R_t, R_{qt}, \lambda)$ *be an* (m,n)*-cuddly Catalanimal of length* $l = dm$*, so its cub has degree* d*. If* $I \subseteq [l]$ *attains the cuddliness bound* $|\lambda[I]_I| = kn/m$*, where* $k = |I|$ *is necessarily a multiple of* m*, then the restricted Catalanimals*

$$H'_I = H(R_q|_I, R_t|_I, R_{qt}|_I, \lambda[I]_I), \qquad H''_I = H(R_q|_{I^c}, R_t|_{I^c}, R_{qt}|_{I^c}, \lambda[I]_{I^c}) \tag{52}$$

are (m,n)*-cuddly, and the coproduct in* Λ *of* $f(X) = \mathrm{cub}(H)$ *is given by*

$$f[X+Y] = \sum_I (-1)^{|R_+^{I,I^c}|}(-q)^{-|R_q^{I,I^c}|}(-t)^{-|R_t^{I,I^c}|}(-q\,t)^{|R_{qt}^{I,I^c}|}\,\mathrm{cub}(H'_I)(X)\cdot \mathrm{cub}(H''_I)(Y), \tag{53}$$

where the sum is over subsets I *that attain the cuddliness bound.*

Proof. First, one verifies directly from the definitions the identities

$$|(\lambda[I]_I)[J']_{J'}| = |\lambda[J]_J| \qquad \text{for } J \subseteq I, \tag{54}$$

$$|(\lambda[I]_{I^c})[J']_{J'}| = |\lambda[I\cup J]_{I\cup J}| - |\lambda[I]_I| \qquad \text{for } J \subseteq I^c, \tag{55}$$

where, if $I \subseteq [l]$ has size $|I| = k$, we take $J' \subseteq [k]$ in (54) and $J' \subseteq [l-k]$ in (55) to be the subsets such that $\pi(J') = J$, where $\pi\colon [k] \to I$ (resp. $\pi\colon [l-k] \to I^c$) is again the unique increasing bijection. It then follows from the (m,n)-cuddliness of H that H'_I and H''_I are (m,n)-cuddly if I attains the cuddliness bound.

We now apply Corollary 5.2.2 with the given Catalanimal H as $H(\mathbf{z})$, and $\Gamma = \widehat{\Gamma}(w,y)$, which is the relevant choice for Catalanimals and their cubs. Expanding $\widehat{\Gamma}(y,w)^{-1}$ as a Laurent series in y/w gives

$$\frac{1}{\widehat{\Gamma}(y,w)} = \frac{(1-w/y)(1-q\,y/w)(1-t\,y/w)}{1-q\,t\,y/w} = -\frac{w}{y}\,(1+O(y/w)). \tag{56}$$

Upon replacing the factor $\prod_{i=1}^k \prod_{j=1}^{l-k} \widehat{\Gamma}(y_j,w_i)^{-1}$ with its leading term, (50) simplifies to

$$h(\mathbf{w},\mathbf{y}) = \Big(\prod_{i=1}^{k}\prod_{j=1}^{l-k}(-w_i/y_j)\Big)\, H(\mathbf{w},\mathbf{y})_{\max}. \tag{57}$$

Since $h(\mathbf{w},\mathbf{y})$ has degree kn/m in $\mathbf{w}$ and $(l-k)n/m$ in $\mathbf{y}$, the notation $H(\mathbf{w},\mathbf{y})_{\max}$ here means the term in $H(\mathbf{w},\mathbf{y})$ of degree $kn/m - k(l-k)$ in $\mathbf{w}$ and $(l-k)n/m + k(l-k)$ in $\mathbf{y}$.

We now turn to evaluating the leading term $H(\mathbf{w},\mathbf{y})_{\max}$ of $H(\mathbf{w},\mathbf{y})$ expanded as a formal Laurent series in the w_i^{-1} and y_j. Fix a subset $I \subseteq [l]$ of size $|I| = k$, and consider the terms for which $v(I) = [k]$ in the formula that defines $H(\mathbf{w},\mathbf{y})$, namely,

$$\sum_{v\in S_l} v\left(\frac{\mathbf{z}^{\lambda}\prod_{\alpha\in R_{qt}}(1-q\,t\,\mathbf{z}^{\alpha})}{\prod_{\alpha\in R_+}(1-\mathbf{z}^{-\alpha})\prod_{\alpha\in R_q}(1-q\,\mathbf{z}^{\alpha})\prod_{\alpha\in R_t}(1-t\,\mathbf{z}^{\alpha})}\right)\Bigg|_{\mathbf{z}=(w_1,\ldots,w_k,y_1,\ldots,y_{l-k})}. \tag{58}$$

The terms in question are given by evaluating the expression inside the parentheses with the z_i for $i \in I$ specialized to a permutation of $w_1,\ldots,w_k$, and the z_i for $i \in I^c = [l]\setminus I$ to a permutation of $y_1,\ldots,y_{l-k}$.

When evaluated in this way, each factor $(1-q\,t\,\mathbf{z}^{\alpha})$ with $\alpha \in R_{qt}^{I,I^c}$ has leading term $-q\,t\,\mathbf{z}^{\alpha}$ since in this case $\mathbf{z}^{\alpha}$ evaluates to some w_i/y_j. If $\alpha \in R_{qt}^{I,I} \cup R_{qt}^{I^c,I^c}$, the entire factor becomes homogeneous of degree zero in either $\mathbf{w}$ or $\mathbf{y}$. Otherwise, if $\alpha \in R_{qt}^{I^c,I}$, the leading term is 1. Similarly, expanding $(1-q\,\mathbf{z}^{\alpha})^{-1}$ as a Laurent series in y_j/w_i if $\alpha \notin R_q^{I,I} \cup R_q^{I^c,I^c}$, its leading term is $-q^{-1}\mathbf{z}^{-\alpha}$ if $\alpha \in R_q^{I,I^c}$, or 1 otherwise. The same holds with t in place of q for factors $(1-t\,\mathbf{z}^{\alpha})^{-1}$. Factors $(1-\mathbf{z}^{-\alpha})^{-1}$ have leading term 1 if $\alpha \in R_+^{I,I^c}$, or $-\mathbf{z}^{\alpha}$ if $\alpha \in R_+^{I^c,I}$.

All of these factors become homogeneous of degree zero if $\alpha \in R^{I,I} \cup R^{I^c,I^c}$.

Putting all this together, and abbreviating the notation $A^{I,I}$, A^{I^c,I^c} to A^I, A^{I^c}, we find that the contribution to $H(\mathbf{w},\mathbf{y})_{\max}$ from the terms in (58) for a fixed I is given by

$$\begin{aligned}(-1)^{|R_+^{I^c,I}|}&(-q)^{-|R_q^{I,I^c}|}(-t)^{-|R_t^{I,I^c}|}(-q\,t)^{|R_{qt}^{I,I^c}|}\\ &\times\sum_{v(I)=[k]} v\left(\frac{\mathbf{z}^{\lambda+\sum R_+^{I^c,I}-\sum R_q^{I,I^c}-\sum R_t^{I,I^c}+\sum R_{qt}^{I,I^c}}\prod_{\alpha\in R_{qt}^{I}\cup R_{qt}^{I^c}}(1-q\,t\,\mathbf{z}^{\alpha})}{\prod_{\alpha\in R_+^{I}\cup R_+^{I^c}}(1-\mathbf{z}^{-\alpha})\prod_{\alpha\in R_q^{I}\cup R_q^{I^c}}(1-q\,\mathbf{z}^{\alpha})\prod_{\alpha\in R_t^{I}\cup R_t^{I^c}}(1-t\,\mathbf{z}^{\alpha})}\right)\Bigg|_{\mathbf{z}=(\mathbf{w},\mathbf{y})}\end{aligned} \tag{59}$$

if the degree of this expression (which is homogeneous) is $kn/m - k(l-k)$ in $\mathbf{w}$ and $(l-k)n/m + k(l-k)$ in $\mathbf{y}$. Otherwise, its contribution to $H(\mathbf{w},\mathbf{y})_{\max}$ is zero. Observing that

$$\begin{aligned}(-1)^{|R_+^{I,I^c}|+|R_+^{I^c,I}|}\,v\big(\mathbf{z}^{\sum R_+^{I,I^c}-\sum R_+^{I^c,I}}\big)\Big|_{\mathbf{z}=(\mathbf{w},\mathbf{y})} &= (-1)^{|R^{I,I^c}|}\,v(\mathbf{z}^{\sum R^{I,I^c}})\Big|_{\mathbf{z}=(\mathbf{w},\mathbf{y})}\\ &= \prod_{i=1}^{k}\prod_{j=1}^{l-k}(-w_i/y_j),\end{aligned} \tag{60}$$

we see that the effect of the extra factor in (57) is to replace $(-1)^{|R_+^{I^c,I}|}$ with $(-1)^{|R_+^{I,I^c}|}$ and $\mathbf{z}^{\sum R_+^{I^c,I}}$ with $\mathbf{z}^{\sum R_+^{I,I^c}}$ in (59), and change the $\mathbf{w}$ and $\mathbf{y}$ degrees back to kn/m, $(l-k)n/m$. The exponent of $\mathbf{z}$ then becomes $\lambda[I]$, and (59) reduces to

$$(-1)^{|R_+^{I,I^c}|}(-q)^{-|R_q^{I,I^c}|}(-t)^{-|R_t^{I,I^c}|}(-q\,t)^{|R_{qt}^{I,I^c}|}H_I'(\mathbf{w})H_I''(\mathbf{y}), \tag{61}$$

which is homogeneous of degree $|\lambda[I]_I|$ in $\mathbf{w}$ and $|\lambda[I]_{I^c}|$ in $\mathbf{y}$. These are the maximum (resp. minimum) permissible degrees precisely when I attains the cuddliness bound $|\lambda[I]_I| = kn/m$.

Corollary 5.2.2 now implies that the image under $\psi_{\widehat{\Gamma}}\otimes\psi_{\widehat{\Gamma}}$ of the expression in (61), summed over all k and I attaining the cuddliness bounds, yields $f[X_{(1)}^{m,n}+X_{(2)}^{m,n}]$. Identity (53) is this same result expressed in terms of the cubs. $\square$

6. Principal specialization and evaluating cubs

The general strategy to determine the cub f of a cuddly Catalanimal, justified by the lemma below, has three steps: show that it exists, determine the inner terms of its coproduct, and evaluate the specialization $(\omega f)[1-q]$. We completed the first two steps in §§4–5. In this section we first give the lemma, then establish the required specialization for the last step, and finally package the resulting criterion for determining cubs as Corollary 6.1.3.

This strategy is similar to that used by Negut [16] to establish the special case discussed in Remark 8.3.4.

Lemma 6.1.1. *If $f \in \Lambda(X)$ is homogeneous of degree d, then f is determined by the terms of the coproduct $f[X_1 + X_2]$ of degree k, $d-k$ in X_1, X_2 for $0 < k < d$, together with the specialization $(\omega f)[1-q]$.*

Proof. The terms of degree k, $d-k$ for $0 < k < d$ in $f[X_1+X_2]$ determine f up to adding a primitive (an element x of a Hopf algebra is primitive if $\Delta(x) = x \otimes 1 + 1 \otimes x$). By [15, Prop. 4.17], the power-sums $p_j(X)$ span the vector space of primitives in $\Lambda(X)$. Hence, the result follows if the specialization of $p_d(X)$ does not vanish. But, indeed, $(\omega p_d)[1-q] = (-1)^{d-1}(1-q^d) \neq 0$. □

Negut [16, Propositions 6.4, 6.5] gives the value of elements in the concrete shuffle algebra $\mathcal{S}_\Gamma$ when specialized at $\mathbf{z} = (1, t, \ldots, t^{l-1})$. The following theorem is more or less a corollary to Negut's formulas, but we have added what is needed to express the result in terms of functions ϕ representing elements in the abstract shuffle algebra S. Note, in particular, that since $\phi(z_1, \ldots, z_l)$ is not symmetric, it matters that the powers of t in (62), below, are in increasing order.

Theorem 6.1.2. *Let $\phi = \phi(z_1, \ldots, z_l) \in T^l + I^l$ be a rational function such that $\psi(\phi) = f[-MX^{m,n}]$, where $\psi\colon S = (T+I)/I \to \mathcal{E}^+$ is the shuffle to Schiffmann isomorphism in Theorem 3.2.1, and $f(X)$ is homogeneous of degree d, so $l = dm$. Assume that the denominator of ϕ does not vanish when evaluated at any permutation of $(1, t, \ldots, t^{l-1})$. Then*

$$\phi(1, t, \ldots, t^{l-1}) = \frac{t^a (\omega f)[1-q]}{(1-q)^l}, \tag{62}$$

where $a = \frac{1}{2} d(dmn - m - n + 1)$.

Proof. First we show that $\phi(1, t, \ldots, t^{l-1})$ depends only on $\psi(\phi)$, or equivalently on

$$g(\mathbf{z}) = \psi_{\check{\Gamma}}(\phi) = \sum_{w \in S_l} w\Big(\phi(\mathbf{z}) \prod_{\alpha \in R_+} \big((1-\mathbf{z}^\alpha)(1-q\,\mathbf{z}^{-\alpha})(1-t\,\mathbf{z}^{-\alpha})(1-q\,t\,\mathbf{z}^\alpha)\big)\Big). \tag{63}$$

If $w \neq 1$, there is some index i such that $j = w^{-1}(i+1) < w^{-1}(i) = k$. Then the factor $w(1 - t\, z_k/z_j) = (1 - t\, z_i/z_{i+1})$ in $w(\prod_{\alpha \in R_+}(1 - t\,\mathbf{z}^{-\alpha}))$ vanishes at $\mathbf{z} = (1, t, \ldots, t^{l-1})$. By our assumption on the denominator of ϕ, the entire w term in (63) vanishes for $w \neq 1$, leaving

$$g(1, t, \ldots, t^{l-1}) = \phi(1, t, \ldots, t^{l-1}) \prod_{i<j} \big((1-t^{i-j})(1-q\,t^{j-i})(1-t^{j-i+1})(1-q\,t^{i-j+1})\big). \tag{64}$$

The product factor is fixed and non-zero, so $g(1, t, \ldots, t^{l-1})$ determines $\phi(1, t, \ldots, t^{l-1})$.

Next observe that for fixed m, n,

$$a(d) = \frac{1}{2}d(dmn - m - n + 1) \tag{65}$$

satisfies

$$a(d_1 + d_2) = a(d_1) + a(d_2) + d_1 d_2 mn. \tag{66}$$

Given d_1, d_2 such that $d_1 + d_2 = d$, suppose that (62) holds for two functions $\phi_1(z_1, \ldots, z_k)$ and $\phi_2(z_1, \ldots, z_{l-k})$ with corresponding $f_1(X)$, $f_2(X)$ homogeneous of degrees d_1, d_2. Then $k = d_1 m$, $l - k = d_2 m$, $l = dm$, the functions ϕ_1, ϕ_2 are homogeneous of degrees $d_1 n$, $d_2 n$, and we have

$$\phi_1(1, t, \ldots, t^{k-1})\phi_2(t^k, \ldots, t^{l-1}) = t^{d_1 d_2 mn}\phi_1(1, t, \ldots, t^{k-1})\phi_2(1, t, \ldots, t^{l-k-1}). \tag{67}$$

The product in S being concatenation, the left hand side of (67) is $(\phi_1 \cdot \phi_2)(1, t, \ldots, t^{l-1})$. Using (66), it follows that (62) holds for $\phi = \phi_1 \cdot \phi_2$ and $f = f_1 f_2$. Since (62) is also clearly linear in ϕ and f, it's enough to prove it when $f = p_d$.

Negut [16, §6.3] defines a linear map $\varphi \colon \mathcal{A}^+ \to \Bbbk$ which is characterized by the property in [16, Proposition 6.4] and its values $\varphi(z_1^d) = (t^{1/2} - t^{-1/2})^{-1}$ on $\mathcal{A}^+_{1,d}$. Using this one can verify that φ corresponds in our notation to the evaluation map $S \to \Bbbk$ that sends $\phi \in S^l$ to $\phi(t^{(1-l)/2}, \ldots, t^{(l-1)/2})/(t^{1/2} - t^{-1/2})^l$. Then, using [16, Proposition 6.5] and (42), one can calculate that for $f = p_d$, we have $\phi(1, t, \ldots, t^{l-1}) = (-1)^{d-1} t^a (1 - q^d)/(1 - q)^l$. This agrees with the desired value and completes the proof. ☐

We can now give a criterion to determine the cub of a cuddly Catalanimal.

Corollary 6.1.3. *Let $H = H(R_q, R_t, R_{qt}, \lambda)$ be an (m, n)-cuddly Catalanimal of length $l = dm$, and let $f \in \Lambda$ be homogeneous of degree d. To show that* $\operatorname{cub}(H) = f$ *it suffices to verify the following.*

(1) For $0 < k < d$, the component $f[X + Y]_{k, d-k}$ of degrees k, $d - k$ in X, Y is given by

$$\sum_I (-1)^{|R_+^{I,I^c}|}(-q)^{-|R_q^{I,I^c}|}(-t)^{-|R_t^{I,I^c}|}(-q\,t)^{|R_{qt}^{I,I^c}|}\operatorname{cub}(H'_I)(X) \cdot \operatorname{cub}(H''_I)(Y) \tag{68}$$

where H'_I and H''_I are as in (52)*, and the sum is over index sets I of size km that attain the cuddliness bound $|\lambda[I]_I| = kn$.*

(2) The function $\phi(\mathbf{z}) = \phi(R_q, R_t, R_{qt}, \lambda)$ defined in (31) *satisfies $\phi(1, t, \ldots, t^{l-1}) = t^a(\omega f)[1 - q]/(1 - q)^l$, where $a = \frac{1}{2}d(dmn - m - n + 1)$.*

Proof. This follows directly from Lemma 6.1.1, Theorem 5.3.1, and Theorem 6.1.2. The only thing to check is the condition on the denominator in Theorem 6.1.2, which clearly holds since the denominator in this case is a product of factors of the form $(1 - q\,t\,z_i/z_j)$. ☐

7. $(1,0)$-cuddly LLT Catalanimals

For any tuple of skew shapes $\boldsymbol{\nu}$, we introduce a $(1, 0)$-cuddly Catalanimal $H_{\boldsymbol{\nu}}$ and prove that its cub is essentially the LLT polynomial $\mathcal{G}_{\boldsymbol{\nu}}(X; q)$. It will be convenient to establish this case first before turning to the general (m, n) case in the next section.

7.1. **Definition of the LLT Catalanimals.** We briefly recall the combinatorial concepts used to define LLT polynomials in §2.2. The adjusted content of a box $a = (u, v) \in \nu_{(i)}$ in a tuple of skew shapes $\boldsymbol{\nu} = (\nu_{(1)}, \dots, \nu_{(k)})$ is $\tilde{c}(a) = u - v + i\epsilon$, and the reading order is the total ordering on the boxes of $\boldsymbol{\nu}$ such that $\tilde{c}$ is increasing and boxes on each diagonal increase from southwest to northeast. An ordered pair of boxes (a, b) in $\boldsymbol{\nu}$ is an attacking pair if $a < b$ in reading order and $0 < \tilde{c}(b) - \tilde{c}(a) < 1$.

We let $\boldsymbol{\nu}(1), \dots, \boldsymbol{\nu}(l)$ denote the boxes of $\boldsymbol{\nu}$ in increasing reading order and set

$$\boldsymbol{\nu}(I) = \{\boldsymbol{\nu}(i) \mid i \in I\} \tag{69}$$

for any subset $I \subseteq [l]$.

Example 7.1.1. For the tuple of skew shapes $\boldsymbol{\nu} = ((32)/(10), (33)/(11))$, the numbering of boxes in increasing reading order is

$$\left(\begin{array}{ccc} \boxed{1} & \boxed{2} & \\ \square & \boxed{4} & \boxed{7} \end{array}, \; \begin{array}{ccc} \square & \boxed{3} & \boxed{6} \\ \square & \boxed{5} & \boxed{8} \end{array} \right)$$

The following definition is the special case for $(m, n) = (1, 0)$ of a more general construction in Section 8.

Definition 7.1.2. The ((1,0) case) *LLT Catalanimal* associated to a tuple of skew shapes $\boldsymbol{\nu}$ with a total of l boxes is the length l Catalanimal $H_{\boldsymbol{\nu}} = H(R_q, R_t, R_{qt}, \lambda)$, as in Definition 3.4.1, given by the following data:

$$R_q = \big\{\alpha_{ij} \in R_+ \mid \tilde{c}(\boldsymbol{\nu}(i)) < \tilde{c}(\boldsymbol{\nu}(j))\big\}, \tag{70}$$

$$R_t = \big\{\alpha_{ij} \in R_+ \mid \tilde{c}(\boldsymbol{\nu}(i)) + 1 \leq \tilde{c}(\boldsymbol{\nu}(j))\big\}, \tag{71}$$

$$R_{qt} = \big\{\alpha_{ij} \in R_+ \mid \tilde{c}(\boldsymbol{\nu}(i)) + 1 < \tilde{c}(\boldsymbol{\nu}(j))\big\}, \tag{72}$$

$$\begin{aligned} \lambda_i = {} & \chi(\mathrm{diag}(\boldsymbol{\nu}(i)) \text{ contains the first box in a row}) \\ & - \chi(\mathrm{diag}(\boldsymbol{\nu}(i)) \text{ contains the last box in a row}), \end{aligned} \tag{73}$$

where $\mathrm{diag}(a)$ is the diagonal of $\boldsymbol{\nu}$ containing a box a, as in §2.2, and $\chi(P) = 1$ if P is true, $\chi(P) = 0$ otherwise.

Remark 7.1.3. (i) The root sets in (70)–(73) satisfy $R_+ \supseteq R_q \supseteq R_t \supseteq R_{qt}$. It is convenient to think of these as providing a partition of the positive roots into the following four subsets defined by the combinatorial features of the LLT polynomial $\mathcal{G}_{\boldsymbol{\nu}}$:

$$\begin{aligned} R_+ \setminus R_q &= \{\alpha_{ij} \in R_+ \mid \boldsymbol{\nu}(i),\, \boldsymbol{\nu}(j) \text{ are on the same diagonal}\}, \\ R_q \setminus R_t &= \{\alpha_{ij} \in R_+ \mid \boldsymbol{\nu}(i),\, \boldsymbol{\nu}(j) \text{ form an attacking pair}\}, \\ R_t \setminus R_{qt} &= \{\alpha_{ij} \in R_+ \mid \boldsymbol{\nu}(i),\, \boldsymbol{\nu}(j) \text{ are on adjacent diagonals}\}, \\ R_{qt} &= \{\text{all other } \alpha_{ij} \in R_+\}. \end{aligned}$$

We say that diagonals in $\boldsymbol{\nu}$ are *adjacent* if their adjusted contents differ by 1, that is, they are in the same skew shape $\nu_{(r)}$ and their ordinary contents differ by 1.

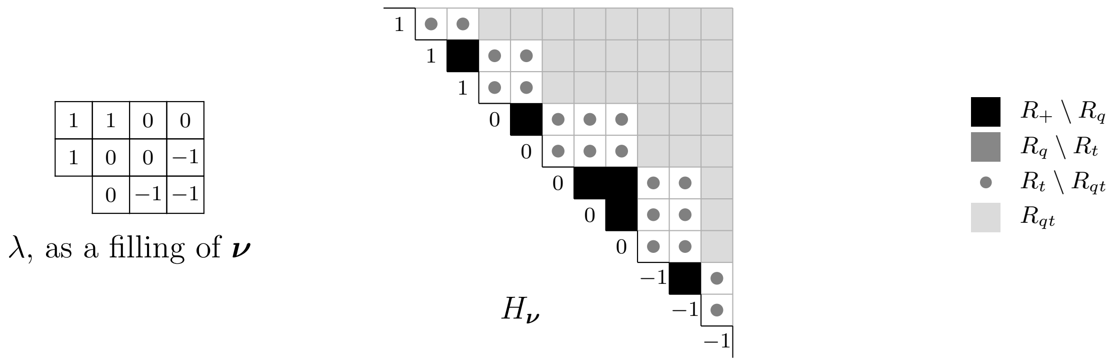


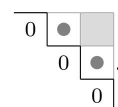
FIGURE 2. The LLT Catalanimal $H_{\boldsymbol{\nu}}$ for $\boldsymbol{\nu}$ the single skew shape $(444)/(1)$, drawn with same conventions as in Figure 1.

(ii) The data R_q, R_t, R_{qt} and λ are constant on diagonals of $\boldsymbol{\nu}$, in the sense that λ_i depends only on $\mathrm{diag}(\boldsymbol{\nu}(i))$, and whether or not α_{ij} belongs to R_q depends only on $\mathrm{diag}(\boldsymbol{\nu}(i))$ and $\mathrm{diag}(\boldsymbol{\nu}(j))$; likewise for R_t and R_{qt}.

(iii) One way to picture the weight λ is as a filling of $\boldsymbol{\nu}$ with λ_i in box $\boldsymbol{\nu}(i)$, so that λ is the list of labels in the filling in reading order. Viewed as a filling, λ is constant on each diagonal $D \subseteq \boldsymbol{\nu}$, with value ± 1 or 0 depending on whether the boxes at the southwest and northeast ends of D are the first or last boxes in their row. This is illustrated in Figures 2 and 3.

Example 7.1.4. (i) If $\boldsymbol{\nu}$ is a single straight shape $((111))$, then $R_q = R_t = R_+ = \{\alpha_{12}, \alpha_{13}, \alpha_{23}\}$, $R_{qt} = \{\alpha_{13}\}$, and $\lambda = (000)$. This gives

$$H_{\boldsymbol{\nu}} = \sum_{w \in S_3} w\Big(\frac{(1 - q\,t\,z_1/z_3)}{\prod_{1\le i<j\le 3}(1 - z_j/z_i)(1 - q\,z_i/z_j)(1 - t\,z_i/z_j)}\Big), \quad \text{drawn as}$$



(ii) If $\boldsymbol{\nu}$ is a single ribbon skew shape (θ), then $R_q = R_t = R_+$ and $R_{qt} = [R_+, R_+] = \{\alpha_{ij} \in R_+ \mid j - i > 1\}$. The weight λ, viewed as a filling of θ as in Remark 7.1.3 (iii), is obtained by writing $1, 0, \dots, 0, -1$ across each row, or 0 if the row has one box. This case relates to prior work of Negut as will be discussed in Remark 8.3.4.

(iii) More generally, if $\boldsymbol{\nu}$ is an arbitrary single skew shape (θ), then $R_q = R_t$ is the set of roots corresponding to positions above the diagonal blocks in a block matrix with block sizes equal to the lengths of the diagonals in θ, and R_{qt} corresponds to the subset of these blocks obtained by removing the blocks immediately above the diagonal, as shown for $\theta = (444)/(1)$ in Figure 2. See also Figure 1 (i) for the case $\theta = (433)$ (but note that the LLT Catalanimal $H^{1,1}_{\boldsymbol{\nu}}$ shown there is obtained from that of $H_{\boldsymbol{\nu}}$ by adding 1^l to the weight, as will be explained in §8).

(iv) The case $\boldsymbol{\nu} = ((444)/(1), (11))$ in Figure 3 illustrates the general construction.

Definition 7.1.5. Define a partial order $\prec$ on boxes in $\boldsymbol{\nu}$ by setting $a \preceq b$ if boxes a, b belong to the same skew shape $\nu_{(i)}$ and a is weakly southwest of b, i.e., $a \le b$ in the usual product order on $\mathbb{N} \times \mathbb{N}$. Let $R_+^{\prec} = \{\alpha_{ij} \in R_+ \mid \boldsymbol{\nu}(i) \prec \boldsymbol{\nu}(j)\}$.

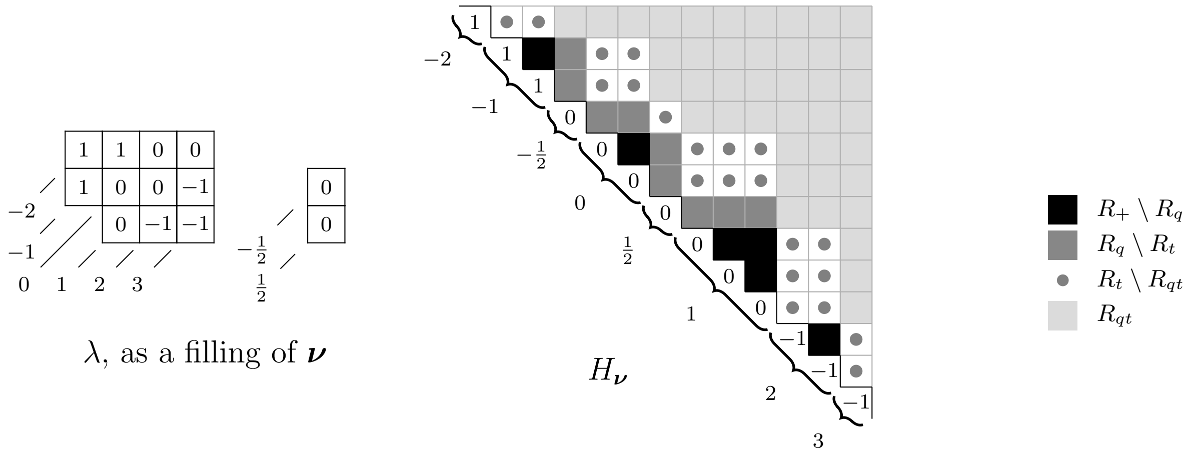


FIGURE 3. The LLT Catalanimal $H_{\boldsymbol{\nu}}$ for $\boldsymbol{\nu} = ((444)/(1),(11))$. We have marked the adjusted contents of λ for $\epsilon = 1/2$, along with the corresponding parabolic blocks in the Catalanimal.

Proposition 7.1.6. *The weight* λ *in* (73) *has the following alternative descriptions:*

$$\lambda_i = \chi(\mathrm{diag}(\boldsymbol{\nu}(i)) \text{ does not contain the last box in a row}) - \chi(\mathrm{diag}(\boldsymbol{\nu}(i)) \text{ does not contain the first box in a row}); \tag{74}$$

$$\lambda = -\sum(R_+ \setminus R_q) + \sum((R_t \setminus R_{qt}) \cap R_+^{\prec}) \tag{75}$$

Proof. The first description is just a reformulation of (73). For the second, let μ denote the right side of (75). We compute the contribution to μ_i from each of the sums. Let $D = \mathrm{diag}(\boldsymbol{\nu}(i))$ and let C, E be the (possibly empty) adjacent diagonals with adjusted contents $\tilde{c}(\boldsymbol{\nu}(i)) - 1$, $\tilde{c}(\boldsymbol{\nu}(i)) + 1$, respectively. Then, setting $\boldsymbol{\nu}_{\succ i} = \{a \in \boldsymbol{\nu} \mid a \succ \boldsymbol{\nu}(i)\}$, $\boldsymbol{\nu}_{\prec i} = \{a \in \boldsymbol{\nu} \mid a \prec \boldsymbol{\nu}(i)\}$, we have

$$\mu_i = -|D \cap \boldsymbol{\nu}_{\succ i}| + |D \cap \boldsymbol{\nu}_{\prec i}| - |C \cap \boldsymbol{\nu}_{\prec i}| + |E \cap \boldsymbol{\nu}_{\succ i}|. \tag{76}$$

The middle two terms sum to $-\chi(D$ does not contain the first box in a row$)$, while the first and last terms sum to $\chi(D$ does not contain the last box in a row$)$, so this matches (74). $\square$

7.2. **Statistics on $\boldsymbol{\nu}$.** In preparation for determining the cubs of LLT Catalanimals, we require the following statistics associated to a tuple of skew shapes $\boldsymbol{\nu}$:

$$\gamma(\boldsymbol{\nu}) \overset{\text{def}}{=} \text{sequence of lengths of diagonals in } \boldsymbol{\nu}\text{, in increasing reading order;} \tag{77}$$

$$n'(\gamma) \overset{\text{def}}{=} \sum_i \binom{\gamma_i}{2} \quad \text{for any } \gamma\text{, but chiefly used for } n'(\gamma(\boldsymbol{\nu}))\text{;} \tag{78}$$

$$p(\boldsymbol{\nu}) \overset{\text{def}}{=} \sum_{\text{diagonals } D \subseteq \boldsymbol{\nu}} \chi(D \text{ does not contain the first box in a row}) \cdot |D|\,; \tag{79}$$

$$A(\boldsymbol{\nu}) \overset{\text{def}}{=} \text{number of attacking pairs in } \boldsymbol{\nu}. \tag{80}$$

We also refer to $p(\boldsymbol{\nu})$ as the *magic number* of $\boldsymbol{\nu}$.

Example 7.2.1. The statistics associated to the LLT Catalanimals of Figures 1, 2, 3 are as follows.

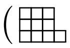
Figure 1 (i), $\boldsymbol{\nu} = ($$)$: $p(\boldsymbol{\nu}) = 4$, $\gamma(\boldsymbol{\nu}) = (1,2,3,2,1,1)$, $n'(\gamma(\boldsymbol{\nu})) = 5$, $A(\boldsymbol{\nu}) = 0$.

Figure 1 (ii), $\boldsymbol{\nu} = ($$,$$)$: $p(\boldsymbol{\nu}) = 3$, $\gamma(\boldsymbol{\nu}) = (1,1,1,1,2,1,1)$, $n'(\gamma(\boldsymbol{\nu})) = 1$, $A(\boldsymbol{\nu}) = 7$.

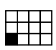
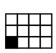
Figure 2, $\boldsymbol{\nu} = ($$)$: $p(\boldsymbol{\nu}) = 5$, $\gamma(\boldsymbol{\nu}) = (1,2,2,3,2,1)$, $n'(\gamma(\boldsymbol{\nu})) = 6$, $A(\boldsymbol{\nu}) = 0$.

Figure 3, $\boldsymbol{\nu} = ($$,$$)$: $p(\boldsymbol{\nu}) = 5$, $\gamma(\boldsymbol{\nu}) = (1,2,1,2,1,3,2,1)$, $n'(\gamma(\boldsymbol{\nu})) = 6$, $A(\boldsymbol{\nu}) = 9$.

Lemma 7.2.2. *For each diagonal D in $\boldsymbol{\nu}$, let D_+ denote the (possibly empty) adjacent diagonal southeast of D, that is, the adjusted contents of boxes $d \in D$ and $e \in D_+$ satisfy $\tilde{c}(e) = \tilde{c}(d) + 1$. Then*

$$n'(\gamma(\boldsymbol{\nu})) + p(\boldsymbol{\nu}) = \sum_D |\{(d,e) \in D \times D_+ \mid d \prec e\}|, \tag{81}$$

summed over all diagonals D in $\boldsymbol{\nu}$.

Proof. Letting $e' = \mathrm{south}(d)$, the sum in (81) is almost the same as the number $n'(\gamma(\boldsymbol{\nu}))$ of pairs of boxes (e', e) such that e' and e are on the same diagonal and $e' \prec e$. The only difference is that (81) also counts pairs (d, e) for which $\mathrm{south}(d)$ is not in $\boldsymbol{\nu}$. But this means that the diagonal D_+ containing e does not contain the first box in a row, and that $d = \mathrm{west}(e_1)$, where e_1 is the first box of D_+ in reading order. Hence, the number of pairs not counted by $n'(\gamma(\boldsymbol{\nu}))$ is $p(\boldsymbol{\nu})$. $\square$

7.3. **Proof of cuddliness and determining the cubs.** Before determining the cubs of LLT Catalanimals, we observe that some formulas involved in Theorem 5.3.1 and Corollary 6.1.3 simplify when $R_q \supseteq R_t \supseteq R_{qt}$, as is the case for LLT Catalanimals. Here and below it will be convenient to use the abbreviations

$$R_{+\setminus q} = R_+ \setminus R_q, \quad R_{q\setminus t} = R_q \setminus R_t, \quad R_{t\setminus qt} = R_t \setminus R_{qt}. \tag{82}$$

The adjusted weight $\lambda[I]$ in (45) simplifies to

$$\lambda[I] = \lambda + \sum R^{I,I^c}_{+\setminus q} - \sum R^{I,I^c}_{t\setminus qt} \tag{83}$$

and the quantity $|\lambda[I]_I|$ in the cuddliness bound (44) to

$$|\lambda[I]_I| = \sum_{i\in I} \lambda_i + |R^{I,I^c}_{+\setminus q}| - |R^{I,I^c}_{t\setminus qt}|. \tag{84}$$

Formula (68) in Corollary 6.1.3 part (1), which is also the right hand side of (53), can be rewritten as

$$\sum_I (-1)^{|R^{I,I^c}_{+\setminus q}|+|R^{I,I^c}_{t\setminus qt}|}(qt)^{-|R^{I,I^c}_{t\setminus qt}|}q^{-|R^{I,I^c}_{q\setminus t}|}\operatorname{cub}(H'_I)(X)\cdot \operatorname{cub}(H''_I)(Y), \tag{85}$$

where the sum is still over index sets $I \subseteq [l]$ of size km attaining the cuddliness bound $|\lambda[I]_I| = kn$, and H'_I, H''_I are as in (52).

To invoke Corollary 6.1.3 part (2) we will need the following lemma, which is the special case for $(m, n) = (1, 0)$ of Lemma 8.3.7, below—see Remark 8.3.8.

Lemma 7.3.1. *The principal specialization of the function $\phi(\mathbf{z}) = \phi(R_q, R_t, R_{qt}, \lambda)$ in (31) associated to the LLT Catalanimal $H_{\boldsymbol\nu} = H(R_q, R_t, R_{qt}, \lambda)$ is given by*

$$\phi(1, t, \ldots, t^{l-1}) = (\omega f)[1 - q]/(1 - q)^l, \tag{86}$$

where $f = (-1)^{p(\boldsymbol\nu)}(qt)^{-p(\boldsymbol\nu)-n'(\gamma(\boldsymbol\nu))}q^{-A(\boldsymbol\nu)}\mathcal{G}_{\boldsymbol\nu}(X; q)$.

Theorem 7.3.2. *For any tuple of skew shapes $\boldsymbol\nu$, the LLT Catalanimal $H_{\boldsymbol\nu} = H(R_q, R_t, R_{qt}, \lambda)$ is $(1, 0)$-cuddly with cub related to the corresponding LLT polynomial by*

$$\operatorname{cub}(H_{\boldsymbol\nu}) = (-1)^p(q\,t)^{-p-n'(\gamma)}q^{-A}\mathcal{G}_{\boldsymbol\nu}(X; q), \tag{87}$$

where $A = A(\boldsymbol\nu)$ is the number of attacking pairs in $\boldsymbol\nu$, $p = p(\boldsymbol\nu)$ is the magic number, and $\gamma = \gamma(\boldsymbol\nu)$ are the diagonal lengths.

Proof. We first verify that $H_{\boldsymbol\nu}$ is $(1, 0)$-cuddly, then use Corollary 6.1.3 to determine its cub.

Checking $(1, 0)$*-cuddliness.* We start by checking the tameness condition $[R_q, R_t] \subseteq R_{qt}$. For $\alpha_{ij} \in R_q$ and $\alpha_{jk} \in R_t$, (70) and (71) give $\tilde c(\boldsymbol\nu(i)) < \tilde c(\boldsymbol\nu(j)) \le \tilde c(\boldsymbol\nu(k)) - 1$, which ensures $\alpha_{ik} \in R_{qt}$ by (72), as desired. A similar argument works with the roles of R_q and R_t interchanged.

We next prove that for all $I \subseteq [l]$, we have

$$|\lambda[I]_I| \le 0, \text{ with equality if and only if } \boldsymbol\nu(I) \text{ is a lower order ideal for } \prec. \tag{88}$$

This establishes the cuddliness bounds (44) for $H_{\boldsymbol\nu}$ and completes the proof that it is $(1, 0)$-cuddly. The second part of the claim will help later on to determine the cub.

In verifying (88) it will be convenient to use the abbreviation

$$r_I = |\lambda[I]_I|. \tag{89}$$

Let $I \subseteq [l]$. First suppose there is a pair $x - 1, x \in [l]$ such that $x \in I$, $x - 1 \notin I$, and $\boldsymbol\nu(x - 1)$ and $\boldsymbol\nu(x)$ belong to the same diagonal. Set $J = I \cup \{x - 1\} \setminus \{x\}$. Then by (84),

$$r_J - r_I = \lambda_{x-1} - \lambda_x + |R^{J,J^c}_{+\setminus q}| - |R^{I,I^c}_{+\setminus q}| - |R^{J,J^c}_{t\setminus qt}| + |R^{I,I^c}_{t\setminus qt}| = 1, \tag{90}$$

where for the second equality, Remark 7.1.3 (i)–(ii) imply that $\lambda_{x-1} = \lambda_x$, $|R^{J,J^c}_{+\backslash q}| - |R^{I,I^c}_{+\backslash q}| = 1$, and $|R^{J,J^c}_{t\backslash qt}| = |R^{I,I^c}_{t\backslash qt}|$. Hence, if such a pair $x-1$, x exists, then $r_J > r_I$. Thus, it suffices to prove (88) for sets I having no such pair, that is, such that

$$\text{for each diagonal } D \subseteq \boldsymbol{\nu},\ D \cap \boldsymbol{\nu}(I) \text{ is a lower order ideal of } (D, \prec). \tag{91}$$

We prove this restricted statement—that the claim in (88) holds for I satisfying (91)—by induction on $|I|$. The base case $I = \varnothing$ is trivial. Now suppose $I \neq \varnothing$. Choose an element $x \in I$ such that $\boldsymbol{\nu}(x)$ is $\prec$-maximal in $\boldsymbol{\nu}(I)$; this means north$(\boldsymbol{\nu}(x))$, east$(\boldsymbol{\nu}(x)) \notin \boldsymbol{\nu}(I)$. Set $K = I \setminus \{x\}$. Let $D = \operatorname{diag}(\boldsymbol{\nu}(x))$ and let C, E be the (possibly empty) adjacent diagonals with adjusted contents $\tilde{c}(\boldsymbol{\nu}(x)) - 1$, $\tilde{c}(\boldsymbol{\nu}(x)) + 1$. We compute

$$\begin{aligned}
r_K - r_I &= |R^{K,K^c}_{+\backslash q}| - |R^{I,I^c}_{+\backslash q}| - |R^{K,K^c}_{t\backslash qt}| + |R^{I,I^c}_{t\backslash qt}| - \lambda_x \\
&= |R^{K,x}_{+\backslash q}| - |R^{x,I^c}_{+\backslash q}| - |R^{K,x}_{t\backslash qt}| + |R^{x,I^c}_{t\backslash qt}| - \lambda_x \\
&= |D \cap \boldsymbol{\nu}(K)| - |D \cap \boldsymbol{\nu}(I^c)| - |C \cap \boldsymbol{\nu}(I)| + |E \cap \boldsymbol{\nu}(I^c)| - \lambda_x \\
&= |D \cap \boldsymbol{\nu}(K)| - |C \cap \boldsymbol{\nu}(I)| + \chi(D \text{ does not contain a row start}) \\
&\qquad - |D \cap \boldsymbol{\nu}(I^c)| + |E \cap \boldsymbol{\nu}(I^c)| - \chi(D \text{ does not contain a row end}),
\end{aligned}$$

where we have used (74) for the last equality. Since $C \cap \boldsymbol{\nu}(I)$ and $D \cap \boldsymbol{\nu}(K)$ are lower order ideals in C and D, and north$(\boldsymbol{\nu}(x)) \notin \boldsymbol{\nu}(I)$,

$$|D \cap \boldsymbol{\nu}(K)| - |C \cap \boldsymbol{\nu}(I)| + \chi(D \text{ does not contain a row start}) \geq 0$$

with equality if and only if $C \cap \boldsymbol{\nu}(I) = \{b \in C \mid b \prec \text{north}(\boldsymbol{\nu}(x))\}$. Similarly,

$$-|D \cap \boldsymbol{\nu}(I^c)| + |E \cap \boldsymbol{\nu}(I^c)| - \chi(D \text{ does not contain a row end}) \geq 0$$

with equality if and only if $E \cap \boldsymbol{\nu}(I^c) = \{b \in E \mid \text{east}(\boldsymbol{\nu}(x)) \preceq b\}$.

Thus, we have shown that

$$r_K = r_I \quad \text{if } \boldsymbol{\nu} \cap \{\text{south}(\boldsymbol{\nu}(x)), \text{west}(\boldsymbol{\nu}(x))\} \subseteq \boldsymbol{\nu}(I), \quad \text{and otherwise } r_K > r_I. \tag{92}$$

If $\boldsymbol{\nu}(I)$ is a lower order ideal, then so is $\boldsymbol{\nu}(K)$. In this case we have $r_K = 0$ by induction and $r_I = r_K = 0$ by (92). If $\boldsymbol{\nu}(I)$ is not a lower order ideal then either $\boldsymbol{\nu} \cap \{\text{west}(\boldsymbol{\nu}(x)), \text{south}(\boldsymbol{\nu}(x))\} \not\subseteq \boldsymbol{\nu}(I)$ or $\boldsymbol{\nu}(K)$ is not a lower order ideal; by induction we have $0 \geq r_K > r_I$ if the former holds and $0 > r_K \geq r_I$ if the latter holds.

Determining the cub. We now prove (87) by verifying the two conditions in Corollary 6.1.3. The condition in part (2) holds by Lemma 7.3.1. For the condition in part (1) we can assume by induction on the number of boxes in $\boldsymbol{\nu}$ that Theorem 7.3.2 applies to the two Catalanimals H'_I and H''_I in (85). Our remaining task is to relate the coproduct formula (85) to the coproduct formula (11) for LLT polynomials. We first address the coefficients in (85). Let I be a subset of $[l]$ appearing in (85); by (88), this is equivalent to $\boldsymbol{\nu}(I)$ being a lower order ideal. Let p_I, γ_I, A_I denote the magic number, diagonal lengths, and number of attacking pairs of $\boldsymbol{\nu}(I)$; let p_{I^c}, γ_{I^c}, A_{I^c} denote the corresponding data for $\boldsymbol{\nu}(I^c)$.

We begin by computing $|R^{I,I^c}_{q\backslash t}|$. Recall from Remark 7.1.3 (i) that $\{(\boldsymbol{\nu}(i), \boldsymbol{\nu}(j)) \mid \alpha_{ij} \in R_{q\backslash t}\}$ is the set of attacking pairs in $\boldsymbol{\nu}$. Hence, $\{(\boldsymbol{\nu}(i), \boldsymbol{\nu}(j)) \mid \alpha_{ij} \in R^{I^c,I}_{q\backslash t}\}$ is the set of attacking pairs going from $\boldsymbol{\nu}(I^c)$ to $\boldsymbol{\nu}(I)$, which has size $A(\boldsymbol{\nu}(I^c), \boldsymbol{\nu}(I))$ in the notation of (11). Thus,

$$|R^{I,I^c}_{q\setminus t}| = |R_{q\setminus t}| - |R^{I^c,I}_{q\setminus t}| - |R^{I,I}_{q\setminus t}| - |R^{I^c,I^c}_{q\setminus t}| = A - A(\boldsymbol{\nu}(I^c), \boldsymbol{\nu}(I)) - A_I - A_{I^c}. \tag{93}$$

Next we compute $|R^{I,I^c}_{t\setminus qt}|$. Recall that $R_{t\setminus qt}$ is the set of roots $\alpha_{ij} \in R_+$ with $\boldsymbol{\nu}(i)$ and $\boldsymbol{\nu}(j)$ in consecutive diagonals. Then since $\boldsymbol{\nu}(I)$ is a lower order ideal, we have $R^{I,I^c}_{t\setminus qt} \subseteq R^{\prec}_+$ and $R^{I^c,I}_{t\setminus qt} \cap R^{\prec}_+ = \varnothing$, with $R^{\prec}_+$ as in Definition 7.1.5, allowing us to write

$$|R^{I,I^c}_{t\setminus qt}| = |R_{t\setminus qt} \cap R^{\prec}_+| - |R^{I,I}_{t\setminus qt} \cap R^{\prec}_+| - |R^{I^c,I^c}_{t\setminus qt} \cap R^{\prec}_+|. \tag{94}$$

Using Lemma 7.2.2, this becomes

$$|R^{I,I^c}_{t\setminus qt}| = p + n'(\gamma) - p_I - n'(\gamma_I) - p_{I^c} - n'(\gamma_{I^c}). \tag{95}$$

For the sign in (85), it remains to compute $|R^{I,I^c}_{+\setminus q}|$. Recall that $R_{+\setminus q}$ is the set of roots $\alpha_{ij} \in R_+$ with $\boldsymbol{\nu}(i)$ and $\boldsymbol{\nu}(j)$ in the same diagonal, hence $|R_{+\setminus q}| = n'(\gamma)$. Since $R_{+\setminus q} \subseteq R^{\prec}_+$, we have $R^{I^c,I}_{+\setminus q} = \varnothing$, and we conclude that

$$|R^{I,I^c}_{+\setminus q}| = |R_{+\setminus q}| - |R^{I,I}_{+\setminus q}| - |R^{I^c,I^c}_{+\setminus q}| = n'(\gamma) - n'(\gamma_I) - n'(\gamma_{I^c}). \tag{96}$$

Combining (93), (95), and (96), we can express the coefficient in the term for I in (85) as

$$\begin{aligned} &(-1)^{|R^{I,I^c}_{+\setminus q}| + |R^{I,I^c}_{t\setminus qt}|} (q\,t)^{-|R^{I,I^c}_{t\setminus qt}|} q^{-|R^{I,I^c}_{q\setminus t}|} = \\ &\quad q^{A(\boldsymbol{\nu}(I^c),\boldsymbol{\nu}(I))} \cdot (-1)^p (q\,t)^{-p-n'(\gamma)} q^{-A} \cdot (-1)^{p_I} (q\,t)^{p_I + n'(\gamma_I)} q^{A_I} \cdot (-1)^{p_{I^c}} (q\,t)^{p_{I^c} + n'(\gamma_{I^c})} q^{A_{I^c}}. \end{aligned} \tag{97}$$

We next consider the restricted Catalanimals $(H_{\boldsymbol{\nu}})'_I = H(R_q|_I, R_t|_I, R_{qt}|_I, \lambda[I]_I)$ and $(H_{\boldsymbol{\nu}})''_I = H(R_q|_{I^c}, R_t|_{I^c}, R_{qt}|_{I^c}, \lambda[I]_{I^c})$ in (85). We claim that $(H_{\boldsymbol{\nu}})'_I = H_{\boldsymbol{\nu}(I)}$ and $(H_{\boldsymbol{\nu}})''_I = H_{\boldsymbol{\nu}(I^c)}$. It is clear from Remark 7.1.3 (i) that $R_q|_I, R_t|_I, R_{qt}|_I$ are the root sets defining the Catalanimal $H_{\boldsymbol{\nu}(I)}$ and similarly for I^c. It remains to consider the weights. By (75) and (83), we have

$$\lambda[I] = -\sum R_{+\setminus q} + \sum (R_{t\setminus qt} \cap R^{\prec}_+) + \sum R^{I,I^c}_{+\setminus q} - \sum R^{I,I^c}_{t\setminus qt}. \tag{98}$$

Using the same reasoning that gave (94) and (96) to compute $\sum R^{I,I^c}_{+\setminus q} - \sum R_{+\setminus q}$ and $\sum (R_{t\setminus qt} \cap R^{\prec}_+) - \sum R^{I,I^c}_{t\setminus qt}$ yields

$$\lambda[I] = -\sum R^{I,I}_{+\setminus q} + \sum (R^{I,I}_{t\setminus qt} \cap R^{\prec}_+) - \sum R^{I^c,I^c}_{+\setminus q} + \sum (R^{I^c,I^c}_{t\setminus qt} \cap R^{\prec}_+). \tag{99}$$

By (75) again, $\lambda[I]_I$ is the weight for $H_{\boldsymbol{\nu}(I)}$ and $\lambda[I]_{I^c}$ the weight for $H_{\boldsymbol{\nu}(I^c)}$. Thus, $(H_{\boldsymbol{\nu}})'_I = H_{\boldsymbol{\nu}(I)}$ and $(H_{\boldsymbol{\nu}})''_I = H_{\boldsymbol{\nu}(I^c)}$, as asserted.

Combining this with (97), the term indexed by I in (85) becomes

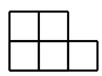
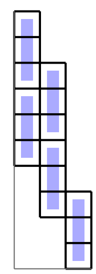
$$\begin{aligned} &(-1)^p (q\,t)^{-p-n'(\gamma)} q^{-A} q^{A(\boldsymbol{\nu}(I^c),\boldsymbol{\nu}(I))} \\ &\quad \times \Big((-1)^{p_I} (q\,t)^{p_I + n'(\gamma_I)} q^{A_I} \operatorname{cub}(H_{\boldsymbol{\nu}(I)})(X) \Big) \Big((-1)^{p_{I^c}} (q\,t)^{p_{I^c} + n'(\gamma_{I^c})} q^{A_{I^c}} \operatorname{cub}(H_{\boldsymbol{\nu}(I^c)})(Y) \Big). \end{aligned}$$

For $0 < |I| < l$, this is equal to $(-1)^p(q\,t)^{-p-n'(\gamma)}q^{-A}q^{A(\boldsymbol{\nu}(I^c),\boldsymbol{\nu}(I))}\mathcal{G}_{\boldsymbol{\nu}(I)}(X;q)\mathcal{G}_{\boldsymbol{\nu}(I^c)}(Y;q)$ by induction. Hence, by (11), the symmetric function $f = (-1)^p(q\,t)^{-p-n'(\gamma)}q^{-A}\mathcal{G}_{\boldsymbol{\nu}}(X;q)$ satisfies the condition in Corollary 6.1.3 part (1). $\square$

8. LLT Catalanimals

We now generalize the results of the previous section by constructing, for any tuple of skew shapes $\boldsymbol{\nu}$, an (m,n)-cuddly Catalanimal whose cub is a scalar multiple of the LLT polynomial $\mathcal{G}_{\boldsymbol{\nu}}$. As a corollary, we obtain a raising operator formula for $\nabla^m\mathcal{G}_{\boldsymbol{\nu}}$.

8.1. **m-stretching.** Let θ be a skew shape and $m \in \mathbb{Z}_+, o \in \mathbb{Z}$. We construct a new skew shape $\theta^{m,o}$ by stretching θ vertically by a factor of m as follows: for each box x of content c in θ, place m boxes of contents $o+mc, o+mc-1, \ldots, o+mc-m+1$ in the same column as x. The set of m boxes arising from x in this way is called a *stretched box*, denoted $\mathrm{stretch}(x) \subseteq \theta^{m,o}$. Thus, the stretched boxes partition the boxes of $\theta^{m,o}$ into $|\theta|$ sets of size m. For example, for $m = 3$ and $o = -4$,

$\theta = (3,2)/(0,0) =$  yields $\theta^{m,o} = \theta^{3,-4} =$ 

where the shaded rectangles indicate the stretched boxes.

Definition 8.1.1. Given a tuple of skew shapes $\boldsymbol{\nu} = (\nu_{(1)}, \ldots, \nu_{(k)})$, $m \in \mathbb{Z}_+$, and $\mathbf{o} = (o_1, \ldots, o_k) \in \mathbb{Z}^k$ satisfying

$$o_1 \leq o_2 \leq \cdots \leq o_k < m + o_1, \tag{100}$$

the associated *m-stretching of $\boldsymbol{\nu}$* is $\boldsymbol{\nu}^m = \boldsymbol{\nu}(m,\mathbf{o}) = (\nu_{(1)}^{m,o_1}, \ldots, \nu_{(k)}^{m,o_k})$. We often use the abbreviation $\boldsymbol{\nu}^m$ even though it actually depends on $\mathbf{o}$ as well.

We use the same notation $\boldsymbol{\nu}^m(i)$, $\boldsymbol{\nu}^m(I)$ as in (69) for the boxes of $\boldsymbol{\nu}^m$ numbered in reading order, and for the set of boxes corresponding to a set of indices $I \subseteq [l]$, where $l = dm = |\boldsymbol{\nu}^m|$ if $d = |\boldsymbol{\nu}|$.

The assumption (100) on the offsets $\mathbf{o}$ allows us to relate attacking pairs in $\boldsymbol{\nu}$ to attacking pairs in $\boldsymbol{\nu}^m$, as follows. Let $\mathbf{A}(\boldsymbol{\nu})$ denote the set of attacking pairs in $\boldsymbol{\nu}$ (with the pair in increasing reading order, as always). For $(\boldsymbol{\nu}(i), \boldsymbol{\nu}(j)) \in \mathbf{A}(\boldsymbol{\nu})$, with $\boldsymbol{\nu}(i) \in \nu_{(r)}$ and $\boldsymbol{\nu}(j) \in \nu_{(s)}$, we set

$$\begin{aligned} a_{ij} &\overset{\text{def}}{=} \left|\left\{(a,b) \in \mathbf{A}(\boldsymbol{\nu}^m) \mid a \in \mathrm{stretch}(\boldsymbol{\nu}(i)), b \in \mathrm{stretch}(\boldsymbol{\nu}(j))\right\}\right| \\ &= \begin{cases} m + o_r - o_s & \text{if } r < s \\ 1 + o_r - o_s & \text{if } r > s. \end{cases} \end{aligned} \tag{101}$$

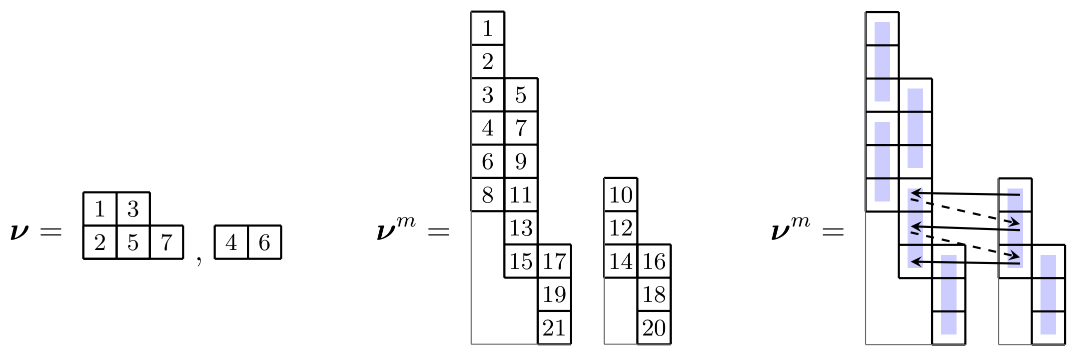
Note that (100) implies $a_{ij} \geq 1$, and furthermore,

$$a_{ij} - 1 = \left|\left\{(a,b) \in \mathbf{A}(\boldsymbol{\nu}^m) \mid a \in \operatorname{stretch}(\boldsymbol{\nu}(j)), b \in \operatorname{stretch}(\boldsymbol{\nu}(i))\right\}\right|, \tag{102}$$

as illustrated in Example 8.1.2. Finally, for any $1 \leq i < j \leq d$,

$$(\boldsymbol{\nu}(i), \boldsymbol{\nu}(j)) \notin \mathbf{A}(\boldsymbol{\nu}) \Rightarrow \left\{(a,b) \in \mathbf{A}(\boldsymbol{\nu}^m) \mid a, b \in \operatorname{stretch}(\boldsymbol{\nu}(i)) \cup \operatorname{stretch}(\boldsymbol{\nu}(j))\right\} = \varnothing. \tag{103}$$

Example 8.1.2. Let $\boldsymbol{\nu} = ((32)/(00), (2)/(0))$, $m = 3$, and $\mathbf{o} = (-4, -2)$. The tuples $\boldsymbol{\nu}$ and $\boldsymbol{\nu}^m$ are shown below with boxes numbered in reading order, along with another drawing of $\boldsymbol{\nu}^m$ to illustrate additional features. There are five attacking pairs in $\boldsymbol{\nu}$, with $a_{24} = a_{34} = a_{56} = 1$, $a_{45} = a_{67} = 3$. The three attacking pairs in $\boldsymbol{\nu}^m$ counted by a_{45} are indicated by the solid arrows and the two attacking pairs counted by $a_{45} - 1$ as in (102) are indicated by the dashed arrows.



8.2. **Definition of LLT Catalanimals.** For $(m, n) \in \mathbb{Z}_+ \times \mathbb{Z}$ (not necessarily coprime), we define a vector of integers $\mathbf{b}(m, n)$ of length m by

$$\mathbf{b}(m,n)_i = \lceil in/m \rceil - \lceil (i-1)n/m \rceil \qquad (i = 1, \ldots, m). \tag{104}$$

More pictorially, if $n \geq 0$, then $\mathbf{b}(m, n)_i$ is the number of south steps on the line $x = i - 1$ in the highest south/east lattice path weakly below the line segment from $(0, n)$ to $(m, 0)$. Note that $\mathbf{b}(dm, dn)$ is the concatenation of d copies of $\mathbf{b}(m, n)$.

Definition 8.2.1. Let $(m, n) \in \mathbb{Z}_+ \times \mathbb{Z}$ be a pair of coprime integers, let $\boldsymbol{\nu}$ be a tuple of skew shapes with $d = |\boldsymbol{\nu}|$ boxes, and let $\boldsymbol{\nu}^m = \boldsymbol{\nu}(m, \mathbf{o})$ be an m-stretching of $\boldsymbol{\nu}$. Let $l = dm = |\boldsymbol{\nu}^m|$ be the number of boxes in $\boldsymbol{\nu}^m$.

We define the *LLT Catalanimal* $H^{m,n}_{\boldsymbol{\nu}^m} = H(R_q, R_t, R_{qt}, \lambda)$ as follows. The root sets R_q, R_t, R_{qt} are defined as in (70)–(72) but with $\boldsymbol{\nu}^m$ in place of $\boldsymbol{\nu}$. The weight λ is defined by

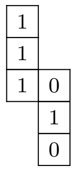
$$\lambda = \hat{\lambda} + \widetilde{\mathbf{b}}, \tag{105}$$

where $\hat{\lambda}$ is the weight given by (73) with $\boldsymbol{\nu}^m$ in place of $\boldsymbol{\nu}$, and $\widetilde{\mathbf{b}} \in \mathbb{Z}^l$ is given by

$$\widetilde{\mathbf{b}}_i = \mathbf{b}(m,n)_{\operatorname{mod}_m(c - o_s)}, \quad \text{where } \boldsymbol{\nu}^m(i) \text{ is a box of content } c \text{ in } \nu^{m,o_s}_{(s)}. \tag{106}$$

Here $\operatorname{mod}_m(c - o_s)$ denotes the integer $j \in [m]$ such that $c - o_s \equiv j \pmod{m}$, and $\mathbf{b}(m, n)$ is defined above.

Viewed as a filling of $\boldsymbol{\nu}^m$, as in Remark 7.1.3 (iii), the weight λ is obtained from $\hat{\lambda}$ by adding the vector $\mathbf{b}(m, n)$ to each stretched box from north to south as in Example 8.2.2 (i).



λ, as a filling of $\boldsymbol{\nu}^m$

$p(\boldsymbol{\nu}^m) = 1$
$\gamma(\boldsymbol{\nu}^m) = (1,1,1,1,1,1,1,1,1)$
$n'(\gamma(\boldsymbol{\nu}^m)) = 0$
$\mathcal{A} = \sum_{(i,j)\in \mathbf{A}(\boldsymbol{\nu})} a_{ij} = 4$

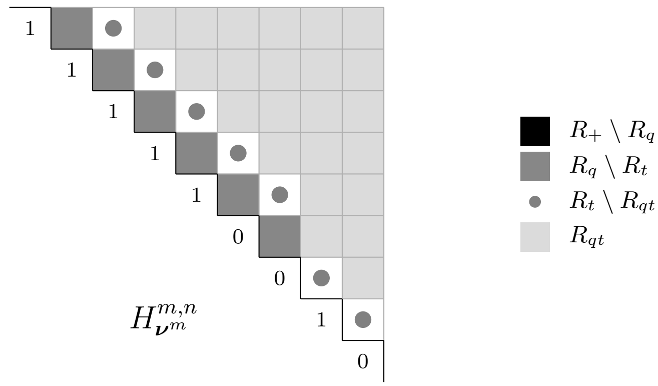
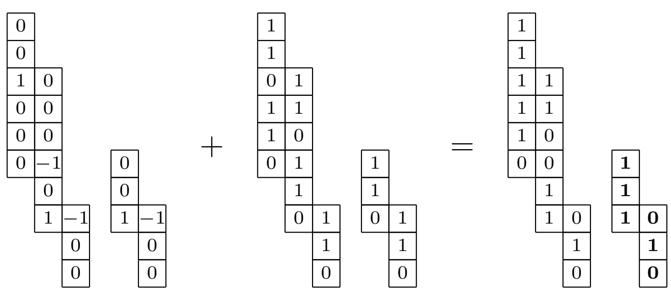


FIGURE 4. The LLT Catalanimal $H^{m,n}_{\boldsymbol{\nu}^m}$ for $m = 3$, $n = 2$, $\boldsymbol{\nu} = ((2),(1))$, $\mathbf{o} = (-2,-2)$, and its associated statistics. By Theorem 8.3.1, $\psi_{\widehat{\Gamma}}(-q^5 t H^{m,n}_{\boldsymbol{\nu}^m}) = \mathcal{G}_{\boldsymbol{\nu}}[-MX^{m,n}]$.

Example 8.2.2. (i) Let $\boldsymbol{\nu} = (\,\text{⊞}\,,\ \text{▭}\,)$, $m = 3$, $\mathbf{o} = (-4,-2)$ be as in Example 8.1.2 and $n = 2$. We have $\mathbf{b}(m,n) = (1,1,0)$. The weights $\hat{\lambda}, \widetilde{\mathbf{b}}, \lambda$ are drawn below as fillings of $\boldsymbol{\nu}(m,\mathbf{o})$. For the weight λ, entries of the second shape are shown in bold to help translate between its filling and vector depictions.



$$\hat{\lambda} \quad + \quad \widetilde{\mathbf{b}} \quad = \quad \lambda$$

$$\lambda = (1\ 1\ 1\ 1\ 1\ 1\ 1\ 0\ 0\ \mathbf{1}\ 0\ \mathbf{1}\ 1\ \mathbf{1}\ 1\ \mathbf{0}\ 0\ \mathbf{1}\ 1\ \mathbf{0}\ 0)$$

(ii) The LLT Catalanimal $H^{3,2}_{\boldsymbol{\nu}(3,\mathbf{o})}$ for $\boldsymbol{\nu} = ((2),(1))$, $\mathbf{o} = (-2,-2)$ is shown in Figure 4.

Remark 8.2.3. Define a binary operation $\uplus$ on sets of roots as follows: for $A \subseteq R_+(\mathrm{GL}_l)$ and $B \subseteq R_+(\mathrm{GL}_{l'})$,

$$A \uplus B \overset{\text{def}}{=} A \sqcup \big\{(i+l, j+l) \mid (i,j) \in B\big\} \sqcup \big\{(i,j) \mid 1 \le i \le l < j \le l+l'\big\} \subseteq R_+(\mathrm{GL}_{l+l'}). \tag{107}$$

The product of two Catalanimals $H = H(R_q, R_t, R_{qt}, \lambda)$ and $H' = H(R'_q, R'_t, R'_{qt}, \lambda')$ in the concrete shuffle algebra $\mathcal{S}_{\widehat{\Gamma}}$ is another Catalanimal,

$$HH' = H(R_q \uplus R'_q, R_t \uplus R'_t, R_{qt} \uplus R'_{qt}, (\lambda;\lambda')), \tag{108}$$

where $(\lambda;\lambda')$ denotes the concatenation of λ and λ'.

The definition of the LLT Catalanimals $H^{m,n}_{\boldsymbol{\nu}(m,\mathbf{o})}$ interacts well with this product in the following sense. If $\boldsymbol{\nu}$ decomposes as $\boldsymbol{\nu} = \boldsymbol{\nu}' \sqcup \boldsymbol{\nu}''$ (meaning that $\boldsymbol{\nu}_{(i)} = \boldsymbol{\nu}'_{(i)} \sqcup \boldsymbol{\nu}''_{(i)}$ for each i), and $\tilde{c}(a)+1 < \tilde{c}(b)$ for all boxes $a \in \boldsymbol{\nu}'$, $b \in \boldsymbol{\nu}''$, then the root sets and weights of $H^{m,n}_{\boldsymbol{\nu}(m,\mathbf{o})}$ are constructed from those of $H^{m,n}_{\boldsymbol{\nu}'(m,\mathbf{o})}, H^{m,n}_{\boldsymbol{\nu}''(m,\mathbf{o})}$ as in (108), giving $H^{m,n}_{\boldsymbol{\nu}(m,\mathbf{o})} = H^{m,n}_{\boldsymbol{\nu}'(m,\mathbf{o})} H^{m,n}_{\boldsymbol{\nu}''(m,\mathbf{o})}$ in $\mathcal{S}_{\widehat{\Gamma}}$.

8.3. **Determining the cubs.** We now come to our main theorem giving a Catalanimal formula for LLT polynomials in any of the subalgebras $\Lambda(X^{m,n})$ of $\mathcal{E}^+$.

Theorem 8.3.1. *For any tuple of skew shapes $\boldsymbol{\nu}$ and any m-stretching $\boldsymbol{\nu}^m = \boldsymbol{\nu}(m,\mathbf{o})$, the LLT Catalanimal $H^{m,n}_{\boldsymbol{\nu}^m} = H(R_q, R_t, R_{qt}, \lambda)$ is (m,n)-cuddly with cub given by*

$$\operatorname{cub}(H^{m,n}_{\boldsymbol{\nu}^m}) = (-1)^p (q\,t)^{-p-n'(\gamma)} q^{-\mathcal{A}} \mathcal{G}_{\boldsymbol{\nu}}(X;q), \tag{109}$$

where $p = p(\boldsymbol{\nu}^m)$, $\gamma = \gamma(\boldsymbol{\nu}^m)$ are the magic number and diagonal lengths of $\boldsymbol{\nu}^m$, and $\mathcal{A} = \sum_{(i,j)\in \mathbf{A}(\boldsymbol{\nu})} a_{ij}$, with $\mathbf{A}(\boldsymbol{\nu})$ and a_{ij} as in (101).

Remark 8.3.2. Equation (109) can also be written as the following more precise form of the formula (2) mentioned in the introduction:

$$\psi_{\widehat{\Gamma}}(H^{m,n}_{\boldsymbol{\nu}^m}(\mathbf{z})) = (-1)^p (q\,t)^{-p-n'(\gamma)} q^{-\mathcal{A}} \mathcal{G}_{\boldsymbol{\nu}}[-MX^{m,n}], \tag{110}$$

where $\psi_{\widehat{\Gamma}}$ is the isomorphism from the shuffle algebra to the Schiffmann algebra defined in §3.2.

The proof will be given below after some further remarks and preliminary lemmas.

Remark 8.3.3. (i) One can check that in fact $p(\boldsymbol{\nu}^m) = p(\boldsymbol{\nu})$ and $n'(\gamma(\boldsymbol{\nu}^m)) = m \cdot n'(\gamma(\boldsymbol{\nu}))$.

(ii) For constant offsets $\mathbf{o} = (c, c, \dots, c)$, $\mathcal{A} = \sum_{(i,j)\in \mathbf{A}(\boldsymbol{\nu})} a_{ij}$ is m times the number of attacking pairs in $\boldsymbol{\nu}$ in which both boxes have the same content, plus the number of attacking pairs in which the boxes have different contents.

(iii) If $\boldsymbol{\nu}$ is a single skew shape (θ), the only effect of the offset is to translate the m-stretched diagram $\boldsymbol{\nu}^m = (\theta^{m,o})$ vertically. In this case there are no attacking pairs, so $\mathcal{A} = 0$, and the Catalanimal $H^{m,n}_{\boldsymbol{\nu}^m}$, whose cub is $(-1)^p (q\,t)^{-p-n'(\gamma)} s_\theta(X)$, does not depend on the offset.

Remark 8.3.4. If $\boldsymbol{\nu} = (\theta)$, where θ is a ribbon skew shape of size d, then $\theta^{m,o}$ is a ribbon of size $l = dm$, and the function $\phi(\mathbf{z}) = \phi(R_q, R_t, R_{qt}, \lambda)$ in (31) such that $H^{m,n}_{\boldsymbol{\nu}^m} = \sigma_{\widehat{\Gamma}}(\phi(\mathbf{z}))$ is given by

$$\phi(\mathbf{z}) = \frac{\mathbf{z}^\lambda}{\prod_{i=1}^{l-1}(1 - q\,t\,z_i/z_{i+1})}, \tag{111}$$

where $\lambda = \mathbf{b}(dm, dn) + \sum_{i\in I} \alpha_{mi,mi+1}$, with $I \subseteq [d-1]$ the set of indices such that boxes $\boldsymbol{\nu}(i)$ and $\boldsymbol{\nu}(i+1)$ of θ are in the same row.

Negut [16, Proposition 6.1] showed that image under $\sigma_{\widehat{\Gamma}}$ of the rational function in (111) lies in the shuffle algebra $\mathcal{S}_{\widehat{\Gamma}}$ for any weight λ. For the specific weight λ occurring here, translating [16, Proposition 6.7] into our notation using (42) gives $\psi(\phi(\mathbf{z})) = (-qt)^{-|I|} s_\theta[-MX^{m,n}]$, which agrees with (109), because for ribbons we have $n'(\gamma) = 0$, and the magic number p

is equal to $|I|$. This result of Negut is thus the special case of Theorem 8.3.1 for $\boldsymbol{\nu}$ a single ribbon skew shape.

Lemma 8.3.5. *If each component of $\boldsymbol{\nu}$ is a disjoint union of ribbon skew shapes, and $\boldsymbol{\nu}$ has no attacking pairs, then $\mathcal{G}_{\boldsymbol{\nu}}(X;q)$ is a product of ribbon skew Schur functions, with no dependence on q. Otherwise, $\omega\mathcal{G}_{\boldsymbol{\nu}}[1-q]=0$.*

Proof. The first statement is clear from the definition of $\mathcal{G}_{\boldsymbol{\nu}}$. For the second, note that $\omega\mathcal{G}_{\boldsymbol{\nu}}[1-q]=(-1)^{|\boldsymbol{\nu}|}\mathcal{G}_{\boldsymbol{\nu}}[q-1]$, and that $\mathcal{G}_{\boldsymbol{\nu}}[q-1]$ is given by evaluating $\omega_Y\mathcal{G}_{\boldsymbol{\nu}}[X+Y]$ at $X=q$ and $Y=y_1$, followed by setting $y_1=-1$. It then follows from Lemma 2.2.2 that $\mathcal{G}_{\boldsymbol{\nu}}[q-1]=\sum_{T\in\mathcal{T}}q^{\mathrm{inv}(T)}q^{\#1\text{'s}}(-1)^{\#\bar{1}\text{'s}}$, where the sum is over the set $\mathcal{T}$ of super tableaux on $\boldsymbol{\nu}$ in the alphabet $1<\bar{1}$.

If $\boldsymbol{\nu}$ is not a disjoint union of ribbon skew shapes, the set $\mathcal{T}$ is empty and we are done.

It remains to show that $\mathcal{G}_{\boldsymbol{\nu}}[q-1]=0$ if $\boldsymbol{\nu}$ has an attacking pair. If so, adapting the argument in the proof of [10, Lemma 5.1], we construct an involution Ψ on $\mathcal{T}$ that cancels the terms $q^{\mathrm{inv}(T)}q^{\#1\text{'s}}(-1)^{\#\bar{1}\text{'s}}$. To define Ψ, let b be the last box in reading order that is part of an attacking pair and let a be the last box in reading order such that (a,b) is an attacking pair; then we let ΨT be the tableau obtained from T by changing the sign of $T(a)$. It is not hard to see that a is necessarily the southeast corner of the ribbon containing it, and therefore that ΨT is indeed a super tableau on $\boldsymbol{\nu}$. Since a and b only depend on $\boldsymbol{\nu}$, it is clear that $\Psi\Psi T=T$. For T with $T(a)=1$, changing this 1 to a $\bar{1}$ adds one to the number of inversions and subtracts one from the number of 1's, hence the contributions of T and ΨT to $\mathcal{G}_{\boldsymbol{\nu}}[q-1]$ cancel. □

Lemma 8.3.6. *Given coprime integers $(m,n)\in\mathbb{Z}_+\times\mathbb{Z}$ and an integer $d>0$, the vector $\mathbf{b}(dm,dn)$ satisfies*

$$\sum_{i=1}^{dm}(i-1)\mathbf{b}(dm,dn)_i = a \overset{\text{def}}{=} \frac{1}{2}d(dmn-m-n+1), \tag{112}$$

where a is the exponent in Theorem 6.1.2 and Corollary 6.1.3.

Proof. Adding rm to n adds a constant vector $(r,r,\dots,r)$ to $\mathbf{b}(dm,dn)$ and thus increases both sides of (112) by $r\binom{dm}{2}$. It therefore suffices to prove (112) for $n\geq 0$.

The left hand side of (112) is then the area under the highest lattice path weakly below the line segment from $(0,dn)$ to $(dm,0)$, or the number of complete lattice squares below the diagonal in a $dn\times dm$ rectangle. Call this number b. Then d^2mn-2b is the number of lattice squares cut by the diagonal.

If $d=1$, the cut squares form a ribbon of size $m+n-1$. In general, they are a union of d copies of this ribbon. Hence, $b=\frac{1}{2}(d^2mn-d(m+n-1))=a$. □

Lemma 8.3.7. *The element $\phi(\mathbf{z})=\phi(R_q,R_t,R_{qt},\lambda)$ defined in (31) corresponding to the LLT Catalanimal $H^{m,n}_{\boldsymbol{\nu}^m}=H(R_q,R_t,R_{qt},\lambda)$ has principal specialization*

$$\phi(1,t,\dots,t^{l-1})=t^a(\omega f)[1-q]/(1-q)^l, \tag{113}$$

where $f=(-1)^p(q\,t)^{-p-n'(\gamma)}q^{-\mathcal{A}}\mathcal{G}_{\boldsymbol{\nu}}(X;q)$ is the right hand side of (109), $l=|\boldsymbol{\nu}^m|$, and $a=\frac{1}{2}d(dmn-m-n+1)$ with $d=|\boldsymbol{\nu}|$.

Proof. If $\boldsymbol{\nu}$ is not a disjoint union of mutually non-attacking ribbon shapes, then any m-stretching of $\boldsymbol{\nu}$ contains two successive boxes in reading order that are either on the same diagonal (if some component is not a ribbon) or form an attacking pair. The Catalanimal $H^{m,n}_{\boldsymbol{\nu}^m}$ then has at least one simple root $\alpha_{i,i+1} \notin R_t$. The factor $\prod_{\alpha\in R_+\setminus R_t}(1 - t\,\mathbf{z}^\alpha)$ in $\phi(\mathbf{z})$ includes $(1 - t\,z_i/z_{i+1})$, so $\phi(1, t, \ldots, t^{l-1}) = 0$. By Lemma 8.3.5, $\omega\mathcal{G}_{\boldsymbol{\nu}}[1 - q] = 0$ as well.

When $\boldsymbol{\nu} = (\theta)$ is a single ribbon shape, $\phi(\mathbf{z})$ is given by (111) and specializes to

$$\phi(1, t, \ldots, t^{l-1}) = t^{a-p}/(1 - q)^{l-1}, \tag{114}$$

using Lemma 8.3.6 and the fact that the magic number p for this $\boldsymbol{\nu}$ is equal to $|I|$, where I is as defined after (111). On the right hand side of (113) we have $t^a(-q\,t)^{-p}(\omega s_\theta)[1-q]/(1-q)^l$. This agrees with (114) because $(\omega s_\theta)[1 - q] = (-q)^p(1 - q)$, noting that p is one less than the number of columns of θ.

Finally, when $\boldsymbol{\nu}$ is a disjoint union of non-attacking ribbons, we use the fact that, as in the proof of Theorem 6.1.2, (66) and (67) imply that $t^{-a}\phi(1, t, \ldots, t^{l-1})$ is multiplicative for elements $\phi \in S$ such that $\psi(\phi) \in \Lambda(X^{m,n})$. By Remark 8.2.3, the Catalanimal $H^{m,n}_{\boldsymbol{\nu}^m}$ in this case is the product in $\mathcal{S}_{\widehat{\Gamma}}$ of the Catalanimals for the individual ribbons θ, so $\phi(\mathbf{z})$ is the product in S of the corresponding functions ϕ_θ. On the right hand side of (113), the function f is the product of the functions $(-qt)^{-p_\theta}s_\theta(X)$, so (113) reduces to the single ribbon case. $\square$

Remark 8.3.8. When $(m, n) = (1, 0)$, the m-stretching is trivial, so p and $n'(\gamma)$ in Lemma 8.3.7 are the same as for $\boldsymbol{\nu}$. The offset $\mathbf{o}$ is necessarily constant, so the number a_{ij} in (101) is equal to 1 for every attacking pair, and $\mathcal{A} = A(\boldsymbol{\nu})$ is the number of attacking pairs. The exponent a is zero. Hence, Lemma 8.3.7 reduces to Lemma 7.3.1 in this case.

Lemma 8.3.9. *Let $(m, n) \in \mathbb{Z}_+ \times \mathbb{Z}$ be a coprime pair.*
(i) For any interval $J = \{a, a + 1, \ldots, b\} \subseteq [dm]$, we have

$$\sum_{j\in J} \mathbf{b}(dm, dn)_j < |J|\,\frac{n}{m} + 1. \tag{115}$$

(ii) For $J = \{a, a + 1, \ldots, dm\}$, we have

$$\sum_{j\in J} \mathbf{b}(dm, dn)_j \le |J|\,\frac{n}{m} \tag{116}$$

with equality if and only if a is one more than a multiple of m.

Proof. Computing directly from the definition (104) of $\mathbf{b}(dm, dn)$, for any interval $J = \{a, a+1, \ldots, b\}$, there holds $|J|\,\frac{n}{m} - \sum_{j\in J}\mathbf{b}(dm, dn)_j = [(a-1)\frac{n}{m}] - [b\frac{n}{m}]$, using the notation $[x] = \lceil x\rceil - x$. Both parts of the lemma follow. $\square$

Proof of Theorem 8.3.1. We verify that $H^{m,n}_{\boldsymbol{\nu}^m}$ is (m, n)-cuddly and use Corollary 6.1.3 to determine its cub.

Let $d = |\boldsymbol{\nu}|$ and $l = dm = |\boldsymbol{\nu}^m|$. For $I \subseteq [l]$, we again use the abbreviation

$$r_I = |\lambda[I]_I| = \sum_{i\in I}\lambda_i + |R^{I,I^c}_{+\setminus q}| - |R^{I,I^c}_{q\setminus t}|, \tag{117}$$

now with the root sets and weight for $H^{m,n}_{\boldsymbol{\nu}^m}$. Replacing λ with the weight $\hat{\lambda}$ for the $(1,0)$-cuddly LLT Catalanimal $H_{\boldsymbol{\nu}^m}$, we also set

$$\hat{r}_I = |\hat{\lambda}[I]_I| = \sum_{i\in I} \hat{\lambda}_i + |R^{I,I^c}_{+\setminus q}| - |R^{I,I^c}_{q\setminus t}|, \tag{118}$$

so that $r_I = \hat{r}_I + \sum_{i\in I} \widetilde{\mathbf{b}}_i$, by (105).

Checking the cuddliness conditions. The tameness condition $[R_q, R_t] \subseteq R_{qt}$ holds by the same argument as in Theorem 7.3.2, applied to $\boldsymbol{\nu}^m$ instead of $\boldsymbol{\nu}$. For the cuddliness bound, it will be convenient to prove the following stronger claim:

(119) $r_I \leq |I|\,\frac{n}{m}$ for all $I \subseteq [l]$, with equality if and only if $\boldsymbol{\nu}^m(I)$ is a lower order ideal in $(\boldsymbol{\nu}^m, \prec)$ and $\boldsymbol{\nu}^m(I)$ is a union of stretched boxes.

If $\boldsymbol{\nu}^m(I)$ is a lower order ideal for $(\boldsymbol{\nu}^m, \prec)$, then $\hat{r}_I = 0$ by (88) in the proof of $(1,0)$-cuddliness for $H_{\boldsymbol{\nu}^m}$. Hence,

$$r_I = \sum_{i\in I} \widetilde{\mathbf{b}}_i = \sum_{x\in\boldsymbol{\nu}} \sum_{\substack{\boldsymbol{\nu}^m(i)\in\\ \mathrm{stretch}(x)\cap\boldsymbol{\nu}^m(I)}} \widetilde{\mathbf{b}}_i\,. \tag{120}$$

Since $\boldsymbol{\nu}^m(I)$ is a lower order ideal, each sum $\sum_{\boldsymbol{\nu}^m(i)\in\mathrm{stretch}(x)\cap\boldsymbol{\nu}^m(I)} \widetilde{\mathbf{b}}_i$ equals $\sum_{i=j}^m \mathbf{b}(m,n)_i$ for some j. Thus, it follows from Lemma 8.3.9 (ii) that $r_I \leq |I|\,\frac{n}{m}$, with equality if and only if $\boldsymbol{\nu}^m(I)$ is a union of stretched boxes.

By the same argument as in the proof of (88), if there is a pair $x-1$, $x \in [l]$ such that $x \in I$, $x-1 \notin I$, and $\boldsymbol{\nu}^m(x-1)$ and $\boldsymbol{\nu}^m(x)$ belong to the same diagonal, then $r_J > r_I$, where $J = I \cup \{x-1\} \setminus \{x\}$. Here we are using the fact that the weight λ is constant on diagonals of $\boldsymbol{\nu}^m$. It therefore suffices to prove (119) for I satisfying

(121) for each diagonal $D \subseteq \boldsymbol{\nu}^m$, $D \cap \boldsymbol{\nu}^m(I)$ is a lower order ideal of $(D, \prec)$.

We prove this restricted statement by induction on $|I|$. We can assume that $\boldsymbol{\nu}^m(I)$ is not a lower order ideal, since we have already dealt with the case when it is. Choose a column C_0 in one of the skew shapes of $\boldsymbol{\nu}^m$ such that C_0 contains at least one box of $\boldsymbol{\nu}^m(I)$, and no box of $\boldsymbol{\nu}^m(I)$ is in the column immediately east of C_0.

If every box $b \in \boldsymbol{\nu}^m(I) \cap C_0$ has $\boldsymbol{\nu}^m \cap \{\mathrm{south}(b), \mathrm{west}(b)\} \subseteq \boldsymbol{\nu}^m(I)$, let $C = \boldsymbol{\nu}^m(I) \cap C_0$; call this Case 1. Otherwise, let $\boldsymbol{\nu}^m(y)$ be the northernmost box of $\boldsymbol{\nu}^m(I) \cap C_0$ such that $\boldsymbol{\nu}^m \cap \{\mathrm{south}(\boldsymbol{\nu}^m(y)), \mathrm{west}(\boldsymbol{\nu}^m(y))\} \not\subseteq \boldsymbol{\nu}^m(I)$, and let C be the set of boxes of $\boldsymbol{\nu}^m(I) \cap C_0$ north of and including $\boldsymbol{\nu}^m(y)$; call this Case 2. Let K be the set of indices such that $\boldsymbol{\nu}^m(K) = \boldsymbol{\nu}^m(I) \setminus C$. If we remove the boxes of C one at a time from north to south, each of these boxes is $\prec$-maximal in the set remaining just before we remove it. Using (92) with $\boldsymbol{\nu}^m$ in place of $\boldsymbol{\nu}$ at each step, we obtain $\hat{r}_I = \hat{r}_{K\cup\{y\}} < \hat{r}_K$ in Case 2, and $\hat{r}_I = \hat{r}_K$ in Case 1. Note that the boxes of C are contiguous in C_0 in both cases. Hence, since the entries $\widetilde{\mathbf{b}}_i$ for $\boldsymbol{\nu}^m(i) \in C_0$ form the sequence $\mathbf{b}(rm, rn)$ for some r, the entries $\widetilde{\mathbf{b}}_i$ for $\boldsymbol{\nu}^m(i) \in C$ form an interval in this sequence. In Case 1, C contains the southernmost box of C_0. Then using $\hat{r}_I = \hat{r}_K$ and Lemma 8.3.9 (ii) we obtain

$$|I|\,\frac{n}{m} - r_I - (|K|\,\frac{n}{m} - r_K) = |C|\,\frac{n}{m} - \hat{r}_I + \hat{r}_K - \sum_{\boldsymbol{\nu}^m(i)\in C} \widetilde{\mathbf{b}}_i = |C|\,\frac{n}{m} - \sum_{\boldsymbol{\nu}^m(i)\in C} \widetilde{\mathbf{b}}_i \geq 0.$$

Since $\boldsymbol{\nu}^m(I)$ is not a lower order ideal, there is some box $b \in \boldsymbol{\nu}^m(I)$ such that $\boldsymbol{\nu}^m \cap \{\mathrm{south}(b), \mathrm{west}(b)\} \not\subseteq \boldsymbol{\nu}^m(I)$. Since we are in Case 1, such a box b does not belong to C, so $b \in \boldsymbol{\nu}^m(K)$. This shows that $\boldsymbol{\nu}^m(K)$ is not a lower order ideal. Thus, by induction, $|K|\,\frac{n}{m} - r_K > 0$, hence $|I|\,\frac{n}{m} - r_I > 0$.

In Case 2, using $\hat{r}_I + 1 \leq \hat{r}_K$ and Lemma 8.3.9 (i) we obtain

$$|I|\,\frac{n}{m} - r_I - (|K|\,\frac{n}{m} - r_K) = |C|\,\frac{n}{m} - \hat{r}_I + \hat{r}_K - \sum_{\boldsymbol{\nu}^m(i)\in C} \widetilde{\mathbf{b}}_i \geq |C|\,\frac{n}{m} - \sum_{\boldsymbol{\nu}^m(i)\in C} \widetilde{\mathbf{b}}_i + 1 > 0.$$

By induction, $|K|\,\frac{n}{m} - r_K \geq 0$, hence $|I|\,\frac{n}{m} - r_I > 0$. This completes the proof of (119).

Determining the cub. We now prove (109) using Corollary 6.1.3, assuming by induction that Theorem 8.3.1 holds for smaller shapes $\boldsymbol{\nu}$.

By (119), the subsets I indexing the summands in (85) are characterized by the property that $\boldsymbol{\nu}^m(I)$ is a lower order ideal and a union of stretched boxes. Given such an I, let $J \subseteq [d]$ be the set of indices such that $\boldsymbol{\nu}^m(I)$ is the m-stretching $\boldsymbol{\nu}(J)^m$ of the lower order ideal $\boldsymbol{\nu}(J)$ in $\boldsymbol{\nu}$, with the same offsets $\mathbf{o}$ as for $\boldsymbol{\nu}^m = \boldsymbol{\nu}(m, \mathbf{o})$. Then we also have $\boldsymbol{\nu}^m(I^c) = \boldsymbol{\nu}(J^c)^m$.

Our first task is to show that H'_I and H''_I in (85) for $H = H^{m,n}_{\boldsymbol{\nu}^m}$ are given by $H'_I = H^{m,n}_{\boldsymbol{\nu}(J)^m}$, $H''_I = H^{m,n}_{\boldsymbol{\nu}(J^c)^m}$. The proof is similar to the proof of the $(m, n) = (1, 0)$ case in Theorem 7.3.2. In particular, the root sets clearly agree. We only discuss the adjustment needed to see that the Catalanimals $(H^{m,n}_{\boldsymbol{\nu}^m})'_I$ and $H^{m,n}_{\boldsymbol{\nu}(J)^m}$ have the same weight. The adjustment for $(H^{m,n}_{\boldsymbol{\nu}^m})''_I$ and $H^{m,n}_{\boldsymbol{\nu}(J^c)^m}$ is similar.

Since $\boldsymbol{\nu}^m(I)$ is a union of stretched boxes, $\lambda[I]_I - \hat{\lambda}[I]_I = \widetilde{\mathbf{b}}_I$ consists of copies of the vector $\mathbf{b}(m, n)$ in the m indices corresponding to the individual boxes in each stretched box. The proof of Theorem 7.3.2 shows that the weight of the $(1, 0)$-cuddly Catalanimal $H_{\boldsymbol{\nu}^m(I)} = H_{\boldsymbol{\nu}(J)^m}$ is $\hat{\lambda}[I]_I$. By construction the weight of $H^{m,n}_{\boldsymbol{\nu}(J)^m}$ is obtained by adding $\widetilde{\mathbf{b}}_I$ to this, so its weight is $\lambda[I]_I$, as desired.

Now we turn to the coefficients in (85). Since $H^{m,n}_{\boldsymbol{\nu}^m}$ and $H_{\boldsymbol{\nu}^m}$ have the same root sets, the computation is almost the same as in the proof of Theorem 7.3.2, except that the statistic in the exponent of q is more complicated. Let $p_I = p(\boldsymbol{\nu}^m(I))$, $\gamma_I = \gamma(\boldsymbol{\nu}^m(I))$, $p_{I^c} = p(\boldsymbol{\nu}^m(I^c))$, $\gamma_{I^c} = \gamma(\boldsymbol{\nu}^m(I^c))$ denote the magic number and diagonal lengths of $\boldsymbol{\nu}^m(I)$ and $\boldsymbol{\nu}^m(I^c)$. Just as in (95, 96), we have

$$|R^{I,I^c}_{t\setminus qt}| = p + n'(\gamma) - p_I - n'(\gamma_I) - p_{I^c} - n'(\gamma_{I^c}), \tag{122}$$

$$|R^{I,I^c}_{+\setminus q}| = n'(\gamma) - n'(\gamma_I) - n'(\gamma_{I^c}). \tag{123}$$

Next, by Remark 7.1.3 (i), $|R^{I,I^c}_{q\setminus t}| = A(\boldsymbol{\nu}^m(I), \boldsymbol{\nu}^m(I^c))$. We need to relate this to attacking pairs in $\boldsymbol{\nu}$. Let $\mathbf{A} = \mathbf{A}(\boldsymbol{\nu})$ be the set of attacking pairs in $\boldsymbol{\nu}$, and let $\mathbf{A}^{X,Y}$ denote the set of attacking pairs from $\boldsymbol{\nu}(X)$ to $\boldsymbol{\nu}(Y)$ for any subsets $X, Y \subseteq [d]$. Using (101)–(103), we have

$$
\begin{aligned}
|R^{I,I^c}_{q\setminus t}| = A(\boldsymbol{\nu}^m(I), \boldsymbol{\nu}^m(I^c)) &= \sum_{(i,j)\in \mathbf{A}^{J,J^c}} a_{ij} + \sum_{(i,j)\in \mathbf{A}^{J^c,J}} (a_{ij}-1) \\
&= \sum_{(i,j)\in \mathbf{A}} a_{ij} - \sum_{(i,j)\in \mathbf{A}^{J,J}} a_{ij} - \sum_{(i,j)\in \mathbf{A}^{J^c,J^c}} a_{ij} - |\mathbf{A}^{J^c,J}| \\
&= \mathcal{A} - \mathcal{A}_J - \mathcal{A}_{J^c} - A(\boldsymbol{\nu}(J^c), \boldsymbol{\nu}(J)),
\end{aligned} \tag{124}
$$

where $\mathcal{A}_J = \sum_{(i,j)\in \mathbf{A}^{J,J}} a_{ij}$, $\mathcal{A}_{J^c} = \sum_{(i,j)\in \mathbf{A}^{J^c,J^c}} a_{ij}$.

Combining (122)–(124), the term indexed by I in (85) becomes

$$
\begin{aligned}
&(-1)^p (q\,t)^{-p-n'(\gamma)} q^{-\mathcal{A}} q^{A(\boldsymbol{\nu}(J^c),\boldsymbol{\nu}(J))} \\
&\times \Big((-1)^{p_I}(q\,t)^{p_I+n'(\gamma_I)} q^{\mathcal{A}_J}\operatorname{cub}(H^{m,n}_{\boldsymbol{\nu}(J)^m})(X)\Big)\Big((-1)^{p_{I^c}}(q\,t)^{p_{I^c}+n'(\gamma_{I^c})} q^{\mathcal{A}_{J^c}}\operatorname{cub}(H^{m,n}_{\boldsymbol{\nu}(J^c)^m})(Y)\Big).
\end{aligned}
$$

The desired formula (109) now follows from (11), Lemma 8.3.7, and Corollary 6.1.3 just as in the proof of Theorem 7.3.2. □

8.4. **Formulas for ∇ on LLT polynomials.** Combining Theorem 8.3.1 and Proposition 3.5.2, we obtain the following formulas. Recall that $\boldsymbol{\sigma}$ denotes the Weyl symmetrization operator in (37).

Corollary 8.4.1. *For any tuple of skew shapes $\boldsymbol{\nu}$ with $|\boldsymbol{\nu}| = l$, we have the following raising operator style formula for ∇ applied to the associated LLT polynomial:*

$$
(\omega\nabla\mathcal{G}_{\boldsymbol{\nu}})(z_1,\dots,z_l) = (-1)^p (q\,t)^{p+n'(\gamma)} q^A\, \boldsymbol{\sigma}\Big(\frac{\mathbf{z}^{\lambda+(1,\dots,1)}\prod_{\alpha\in R_{qt}}(1-q\,t\,\mathbf{z}^\alpha)}{\prod_{\alpha\in R_q}(1-q\,\mathbf{z}^\alpha)\prod_{\alpha\in R_t}(1-t\,\mathbf{z}^\alpha)}\Big)_{\mathrm{pol}},
$$

where R_q, R_t, R_{qt}, λ are as in (70)–(73), $A = A(\boldsymbol{\nu})$ is the number of attacking pairs in $\boldsymbol{\nu}$, and $p = p(\boldsymbol{\nu})$, $\gamma = \gamma(\boldsymbol{\nu})$ are the magic number and diagonal lengths of $\boldsymbol{\nu}$.

Corollary 8.4.2. *More generally, ∇^m on any LLT polynomial $\mathcal{G}_{\boldsymbol{\nu}}$ is given by*

$$
(\omega\nabla^m\mathcal{G}_{\boldsymbol{\nu}})(z_1,\dots,z_l) = (-1)^p (q\,t)^{p+m\,n'(\gamma)} q^{\mathcal{A}}\, \boldsymbol{\sigma}\Big(\frac{\mathbf{z}^{\lambda}\prod_{\alpha\in R_{qt}}(1-q\,t\,\mathbf{z}^\alpha)}{\prod_{\alpha\in R_q}(1-q\,\mathbf{z}^\alpha)\prod_{\alpha\in R_t}(1-t\,\mathbf{z}^\alpha)}\Big)_{\mathrm{pol}},
$$

where $l = m|\boldsymbol{\nu}|$; R_q, R_t, R_{qt}, λ are as in Definition 8.2.1 for $n = 1$ and the m-stretching $\boldsymbol{\nu}(m, \mathbf{o})$ of $\boldsymbol{\nu}$ with constant offsets $\mathbf{o} = (c, c, \dots, c)$; $p = p(\boldsymbol{\nu})$, $\gamma = \gamma(\boldsymbol{\nu})$, and $\mathcal{A}$ is m times the number of attacking pairs in $\boldsymbol{\nu}$ with the same content, plus the number of attacking pairs with different contents.

Remark 8.4.3. (i) If a Catalanimal $H(R_q, R_t, R_{qt}, \lambda)$ is $(1, 0)$-cuddly, then $H(R_q, R_t, R_{qt}, \lambda + (1, \dots, 1))$ is $(1, 1)$-cuddly. This explains why when $m = 1$, the λ in Corollary 8.4.2 becomes $\lambda + (1, \dots, 1)$ in Corollary 8.4.1.

(ii) Corollary 8.4.2 also holds for any m-stretching $\boldsymbol{\nu}(m, \mathbf{o})$ of $\boldsymbol{\nu}$, but now with $\mathcal{A} = \sum_{(i,j)\in\mathbf{A}(\boldsymbol{\nu})} a_{ij}$, where $\mathbf{A}(\boldsymbol{\nu})$ and a_{ij} are as in (101).

Example 8.4.4. (i) Continuing Example 7.1.4 (i), Corollary 8.4.1 gives

$$(\omega\nabla e_3)(z_1,z_2,z_3) = \big(z_1z_2z_3\, H_{((111))}\big)_{\mathrm{pol}} = \boldsymbol{\sigma}\Big(\frac{z_1z_2z_3\,(1-q\,t\,z_1/z_3)}{\prod_{1\le i<j\le 3}((1-q\,z_i/z_j)(1-t\,z_i/z_j))}\Big)_{\mathrm{pol}}$$

$$= s_{111} + (q+t+q^2+q\,t+t^2)s_{21} + (q\,t+q^3+q^2\,t+q\,t^2+t^3)s_3.$$

(ii) When $\boldsymbol{\nu}$ is the single shape $((433))$, $\mathcal{G}_{\boldsymbol{\nu}} = s_{433}$. In this case, the LLT Catalanimal $H^{1,1}_{((433))} = z_1\cdots z_l\, H_{((433))}$ is shown in Figure 1 (i), and its associated statistics in Example 7.2.1. By Corollary 8.4.1, we have $(\omega\nabla s_{433})(z_1,\dots,z_l) = (q\,t)^9\big(z_1\cdots z_l\, H_{((433))}\big)_{\mathrm{pol}}$.

(iii) For $\boldsymbol{\nu} = ((32)/(10),(33)/(11))$, the LLT Catalanimal $H^{1,1}_{\boldsymbol{\nu}} = z_1\cdots z_l\, H_{\boldsymbol{\nu}}$ is shown in Figure 1 (ii), and its associated statistics in Example 7.2.1. By Corollary 8.4.1, we have $(\omega\nabla\mathcal{G}_{\boldsymbol{\nu}})(z_1,\dots,z_l) = -(q\,t)^4q^7\big(z_1\cdots z_l\, H_{\boldsymbol{\nu}}\big)_{\mathrm{pol}}$.

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