Journal of Combinatorial Theory, Series A **82**, 74–111 (1998)
Article No. TA972842

A Random q, t-Hook Walk and a Sum of Pieri Coefficients

A. M. Garsia* and M. Haiman*

Department of Mathematics, University of California, San Diego, La Jolla, California 92093-0001

Communicated by the Managing Editors

Received September 18, 1996

This work deals with the identity $B_\mu(q,t)=\sum_{\nu\to\mu} c_{\mu\nu}(q,t)$, where $B_\mu(q,t)$ denotes the biexponent generator of a partition μ. That is, $B_\mu(q,t)=\sum_{s\in\mu} q^{a'(s)}t^{l'(s)}$, with $a'(s)$ and $l'(s)$ the co-arm and co-leg of the lattice square s in μ. The coefficients $c_{\mu\nu}(q,t)$ are closely related to certain rational functions occuring in one of the Pieri rules for the Macdonald polynomials and the symbol $\nu\to\mu$ is used to indicate that the sum is over partitions ν which immediately precede μ in the Young lattice. This identity has an indirect manipulatorial proof involving a number of deep identities established by Macdonald. We show here that it may be given an elementary probabilistic proof by a mechanism which emulates the Greene–Nijehuis–Wilf proof of the hook formula. © 1998 Academic Press

INTRODUCTION

Given a partition μ we shall represent it as customary by a Ferrers diagram. We shall use the French convention here and, given that the parts of μ are $\mu_1\geqslant\mu_2\geqslant\cdots\geqslant\mu_k>0$, we let the corresponding Ferrer's diagram have μ_i lattice squares in the ith row (counting from the bottom up). We shall also adopt the Macdonald convention of calling the *arm*, *leg*, *co-arm*, and *co-leg* of a lattice square s the parameters $a(s)$, $l(s)$, $a'(s)$ and $l'(s)$, giving the number of cells of μ that are respectively *strictly* East, North, West, and South of s in μ. We recall that Macdonald in [13] defines the symmetric function basis $\{P_\mu(x;q,t)\}_\mu$ as the unique family of polynomials satisfying the following conditions

(a) $P_\lambda=S_\lambda+\sum_{\mu<\lambda} S_\mu \xi_{\mu\lambda}(q,t)$

(b) $\langle P_\lambda, P_\mu\rangle_{q,t}=0 \qquad \text{for} \quad \lambda\neq\mu,$

* Work by the authors was carried out under NSF grant support.

0097-3165/98 $25.00

where $\langle\ ,\ \rangle_{q,t}$ denotes the scalar product of symmetric polynomials defined by setting for the power basis $\{p_\rho\}$

$$\langle p_{\rho^{(1)}}, p_{\rho^{(2)}}\rangle_{q,t} = \begin{cases} z_\rho \prod_i \dfrac{1-q^{\rho_i}}{1-t^{\rho_i}} & \text{if} \quad \rho^{(1)}=\rho^{(2)}=\rho \\ 0 & \text{otherwise,} \end{cases}$$

where z_ρ is the integer that makes $n!/z_\rho$ the number of permutations with cycle structure ρ. Macdonald shows that the basis $\{Q_\lambda(x; q, t)\}_\mu$, dual to $\{P_\mu(x; q, t)\}_\mu$ with respect to this scalar product, is given by the formula

$$Q_\lambda(x; q, t) = d_\lambda(q, t)\, P_\lambda(x; q, t),$$

where

$$d_\lambda(q, t) = \frac{h_\lambda(q, t)}{h'_\lambda(q, t)}$$

and

$$h_\lambda(q, t) = \prod_{s\in\lambda} (1 - q^{a_\lambda(s)} t^{l_\lambda(s)+1}), \qquad h'_\lambda(q, t) = \prod_{s\in\lambda} (1 - q^{a_\lambda(s)+1} t^{l_\lambda(s)}) \tag{I.1}$$

Macdonald sets

$$J_\mu(x; q, t) = h_\mu(q, t)\, P_\mu(x; q, t) = h'_\mu(q, t)\, Q_\mu(x; q, t), \tag{I.2}$$

and then defines his q, t-analogues of the Kostka coefficients by means of an expansion that in λ-ring notation may be written as

$$J_\mu(x; q, t) = \sum_\lambda S_\lambda[X(1-t)]\, K_{\lambda\mu}(q, t). \tag{I.3}$$

This note is concerned with the modified basis $\{\tilde{H}_\mu(x; q, t)\}_\mu$ defined by setting

$$\tilde{H}_\mu(x; q, t) = \sum_\lambda S_\lambda(x)\, \tilde{K}_{\lambda\mu}(q, t), \tag{I.4}$$

where we have set

$$\tilde{K}_{\lambda\mu}(q, t) = K_{\lambda\mu}(q, 1/t)\, t^{n(\mu)} \tag{I.5}$$

with

$$n(\mu) = \sum_{s\in\mu} l_\mu(s).$$

To this date it is still an open problem to prove that the $K_{\lambda\mu}(q,t)$ (and the $\tilde{K}_{\lambda\mu}(q,t)$ as well) are polynomials with positive integer coefficients. In [2] we have conjectured that $\tilde{H}_{\mu}(x;q,t)$ is in fact (for a given $\mu \vdash n$) the bivariate Frobenius characteristic of a certain S_n-module $\mathbf{H}_{\mu}$ yielding a bigraded version of the left regular representation of S_n. In particular this would imply that the expression

$$F_{\mu}(q,t)=\sum_{\lambda} f_{\lambda}\tilde{K}_{\lambda\mu}(q,t)$$

should be the Hilbert series of $\mathbf{H}_{\mu}$. Here, f_{λ} denotes the number of standard tableaux of shape λ. Since Macdonald proved that

$$K_{\lambda\mu}(1,1)=f_{\lambda}, \tag{I.6}$$

we see that we must necessarily have

$$F_{\mu}(1,1)=\sum_{\lambda} f_{\lambda}^2=n! \tag{I.7}$$

According to our conjectures in [2] the polynomial

$$\partial_{p_1}\tilde{H}_{\mu}(x;q,t)$$

should give the Frobenius characteristic of the action of S_{n-1} on $\mathbf{H}_{\mu}$. Using the fact that the operator ∂_{p_1} is in a sense[1] dual to multiplication by the elementary symmetric function e_1, we can transform one of the Pieri rules given by Macdonald in [14] into the expansion of $\partial_{p_1}\tilde{H}_{\mu}(x;q,t)$ in terms of the polynomials $\tilde{H}_{\nu}(x;q,t)$ whose index ν immediately precedes μ in the Young partial order. More precisely, we obtain

$$\partial_{p_1}\tilde{H}_{\mu}(x;q,t)=\sum_{\nu\to\mu} c_{\mu\nu}(q,t)\,\tilde{H}_{\nu}(x;q,t) \tag{I.8}$$

with

$$c_{\mu\nu}(q,t)=\prod_{s\in\mathcal{R}_{\mu/\nu}} \frac{t^{l_{\mu}(s)}-q^{a_{\mu}(s)+1}}{t^{l_{\nu}(s)}-q^{a_{\nu}(s)+1}} \prod_{s\in\mathcal{C}_{\mu/\nu}} \frac{q^{a_{\mu}(s)}-t^{l_{\mu}(s)+1}}{q^{a_{\nu}(s)}-t^{l_{\nu}(s)+1}}, \tag{I.9}$$

where $\mathcal{R}_{\mu/\nu}$ (resp. $\mathcal{C}_{\mu/\nu}$) denotes the set of lattice squares of ν that are in the same row (resp. same column) as the square we must remove from μ to

[1] This will be made precise in section 1.

obtain ν. This given, an application of $\partial_{p_1}^{n-1}$ to both sides of (I.8) yields the recursion

$$F_\mu(q,t) = \sum_{\nu \to \mu} c_{\mu\nu}(q,t)\, F_\nu(q,t), \tag{I.10}$$

which together with the initial condition $F_{(1)}(q,t) = 1$ permits the computation of extensive tables of $F_\mu(q,t)$. Of course, all the data so obtained not only confirm the polynomiality and positive integrality of the coefficients of $F_\mu(q,t)$ but exhibit some truly remarkable symmetries under various transformations of the variables μ, q, and t. The temptation is strong to try and deduce some of these properties directly from the recursion in (I.10). In particular, we should want to construct a pair of statistics $\alpha_\mu(\sigma)$, $\beta_\mu(\sigma)$ on permutations $\sigma \in S_n$ yielding

$$F_\mu(q,t) = \sum_{\sigma \in S_n} q^{\alpha_\mu(\sigma)} t^{\beta_\mu(\sigma)}. \tag{I.11}$$

Unfortunately, the complexity of the coefficients $c_{\mu\nu}(q,t)$ turns this into an arduous task. The present work results from a systematic effort to understand as much as possible about the mechanism which results in the positive polynomiality of $F_\mu(q,t)$ in spite of the intricate rationality of the recursion.

The idea that a "hook walk" of sorts is involved here stems from noting what takes place if we successively make the substitutions $t \to 1/t$ and then $t \to q$. To this end, setting

$$G_\mu(q) = (F_\mu(q, 1/t)\, t^{n(\mu)})|_{t \to q},$$

routine manipulations yield that the recursion in (I.10) becomes

$$G_\mu(q) = \sum_{\nu \to \mu} \gamma_{\mu\nu}(q)\, G_\nu(q), \tag{I.12}$$

with

$$\gamma_{\mu\nu}(q) = \prod_{s \in \mathscr{R}_{\mu/\nu}} \frac{1 - q^{h_\mu(s)}}{1 - q^{h_\nu(s)}} \prod_{s \in \mathscr{C}_{\mu/\nu}} \frac{1 - q^{h_\mu(s)}}{1 - q^{h_\nu(s)}}$$

where

$$h_\mu(s) = l_\mu(s) + a_\mu(s) + 1 \qquad \text{and} \qquad h_\nu(s) = l_\nu(s) + a_\nu(s) + 1.$$

However, now these coefficients $\gamma_{\mu\nu}(q)$ may be given a very revealing form. Indeed, since when s is not in $\mathscr{R}_{\mu/\nu}$ or $\mathscr{C}_{\mu/\nu}$ we have

$$h_\mu(s) = l_\mu(s) + a_\mu(s) + 1 = l_\nu(s) + a_\nu(s) + 1 = h_\nu(s),$$

we may write

$$\prod_{s \in \mathscr{R}_{\mu/\nu} + \mathscr{C}_{\mu/\nu}} \frac{1-q^{h_\mu(s)}}{1-q^{h_\nu(s)}} = \frac{1}{1-q} \frac{\prod_{s \in \mu} (1-q^{h_\mu(s)})}{\prod_{s \in \nu} (1-q^{h_\nu(s)})}$$

where the divisor $1-q$ compensates for the fact that μ differs from ν by a corner square (of hook length $=1$). Using the notation

$$[m]_q = 1 + q + \cdots + q^{m-1} = \frac{1-q^m}{1-q},$$

we can finally rewrite the recursion in (I.12) in the form

$$\frac{G_\mu(q)}{\prod_{s \in \mu} [h_\mu(s)]_q} = \sum_{\nu \to \mu} \frac{G_\nu(q)}{\prod_{s \in \nu} [h_\nu(s)]_q}.$$

This means that the expression $G_\mu(q)/\prod_{s \in \mu} [h_\mu(s)]_q$ satisfies the same recursion as the number of of standard tableaux f_μ. Since the initial condition is $G_{(1)} = 1$, we deduce that for all partitions μ we must have

$$G_\mu(q) = f_\mu \prod_{s \in \mu} [h_\mu(s)]_q. \tag{I.14}$$

This identity, which was noted by Macdonald in [14], points out the order of difficulty of finding a pair of statistics yielding (I.11). Indeed, once that is done, the specialization that sends $F_\mu(q, t)$ to $G_\mu(q)$ would deliver a q-analogue of the hook formula.

The derivation of (I.14) suggests that the coefficient $c_{\mu\nu}(q)$ is some sort of q, t-analogue of the ratio h_μ/h_ν, where h_μ and h_ν denote the hook products for μ and ν respectively. This given, the recursion in (I.10) may be viewed as a q, t-analogue of the identity

$$n! = \sum_{\nu \to \mu} \frac{h_\mu}{h_\nu} (n-1)!.$$

Dividing both sides of this identity by $n!$ we get

$$1 = \frac{1}{n} \sum_{\nu \to \mu} \frac{h_\mu}{h_\nu},$$

which is precisely what Greene, Nijenhuis, and Wilf prove by means of their random hook walk. We shall show here that an appropriate q, t-extension of their argument yields a probabilistic proof of the identity

$$1 = \frac{1}{B_\mu(q, t)} \sum_{\nu \to \mu} c_{\mu\nu}(q), \tag{I.15}$$

where

$$B_\mu(q, t) = \sum_{s \in \mu} t^{l'_\mu(s)} q^{a'_\mu(s)}. \tag{I.16}$$

The contents of this note are divided into three sections. In the first section we give the original argument that led us to discover this identity. We also give an alternate proof which indicates the close relationship that (I.15) has to certain special properties of the coefficients $\tilde{K}_{\lambda\mu}(q, t)$. In the second section we introduce our q, t-hook walk and show that it yields (I.15) as desired. In the final section we state a number of closely related identities and suggest possible extensions of the present work.

We should mention that some of the computer experimentation that was suggested by the the present work was the starting point of a development which culminated into the proofs given in [9] and [10] that the $K_{\lambda\mu}(q, t)$ are in fact polynomials with integer coefficients.

1. MANIPULATIONS

Our presentation here relies heavily on λ-ring notation and we shall begin with a brief description of this device. The reader is referred to [1] and [8] for further details. If P and Q are symmetric polynomials and Q has positive integer coefficients, then by $P[Q]$ we mean the symmetric polynomial obtained by interpreting Q as a multiset of monomials A and literally substituting the elements of A for the variables of P. Note that if $P = p_k$[2] this operation reduces to setting

$$p_k[Q] = \sum_{a \in A} a^k.$$

This given, to compute $P[Q]$ in full generality we simply expand P in terms of the power basis

$$P = \sum_\rho c_\rho p_\rho,$$

and then set

$$P[Q] = \sum_\rho c_\rho \prod_i p_{\rho_i}[Q]. \tag{1.1}$$

[2] The kth power symmetric function.

This is usually referred to as the *plethysm of Q into P*. λ-Ring notation simply extends plethysm to the case when Q is allowed to have negative as well as positive integer coefficients. To do this we simply decompose Q as a difference of two multisets $Q=A-B$ and then set

$$p_k[Q]=\sum_{a\in A} a^k-\sum_{b\in B} b^k. \tag{1.2}$$

This given, the computation of $P[Q]$ may again be carried out according to formula (1.1). We should note that the definition in (1.2) is motivated by the requirement that for any two polynomials Q_1 and Q_2 we should have the two basic properties

$$p_k[Q_1+Q_2]=p_k[Q_1]+p_k[Q_2], \qquad p_k[Q_1Q_2]=p_k[Q_1]\,p_k[Q_2]. \tag{1.3}$$

This definition can clearly be extended to the case when P as well as Q are symmetric formal Laurent series. The convenience of this notation is mainly due to the fact that, because of the properties in (1.3), many of the manipulations that are natural in the context of substitution are still correct for λ-ring substitutions.

To carry out calculations in Macdonald theory by this device we need to start by giving a λ-ring expression to the Macdonald kernel. To this end we define the basic *Cauchy* rational function Ω by setting

$$\Omega=\sum_{\rho} p_\rho/z_\rho=\exp\left(\sum_{k\geqslant 1}\frac{p_k}{k}\right), \tag{1.4}$$

where for a partition $\rho=1^{\alpha_1}2^{\alpha_2}\cdots n^{\alpha_n}$ we let $z_\rho=1^{\alpha_1}2^{\alpha_2}\cdots n^{\alpha_n}\alpha_1!\,\alpha_2!\cdots\alpha_n!$. This given, it is easy to deduce from (1.1) and (1.3) that for any difference $A-B$ of two Laurent multisets of monomials we have

$$\Omega[A-B]=\prod_{a\in A}\frac{1}{1-a}\prod_{b\in B}(1-b). \tag{1.5}$$

In particular, this gives that the Macdonald kernel

$$\Omega_{q,t}(x;y)=\prod_{m\geqslant 0}\prod_{i,j}\frac{1-tx_iy_jq^m}{1-x_iy_jq^m}$$

may simply be written as

$$\Omega_{q,t}(x;y)=\Omega\left[XY\frac{1-t}{1-q}\right], \tag{1.6}$$

where we set $X = x_1 + x_2 + \cdots$ and $Y = y_1 + y_2 + \cdots$. Then the duality of the two bases $\{P_\mu(x; q, t)\}_\mu$ and $\{Q_\mu(x; q, t)\}_\mu$ with respect to the scalar product $\langle\ ,\ \rangle_{q,t}$ translates into the identity

$$\Omega\left[XY\frac{1-t}{1-q}\right] = \sum_\mu P_\mu(x; q, t)\, Q_\mu(x; q, t). \tag{1.7}$$

Using (I.2), this identity may be rewritten as

$$\Omega\left[XY\frac{1-t}{1-q}\right] = \sum_\mu \frac{J_\mu(x; q, t)\, J_\mu(y; q, t)}{h_\mu(q, t)\, h'_\mu(q, t)}.$$

Making the λ-ring substitutions $X \to X/(1-t)$ and $Y \to Y/(1-t)$ then yields that

$$\Omega\left[\frac{XY}{(1-t)(1-q)}\right] = \sum_\mu \frac{J_\mu\left[\frac{X}{1-t}; q, t\right] J_\mu\left[\frac{Y}{1-t}; q, t\right]}{h_\mu(q, t)\, h'_\mu(q, t)}.$$

Setting for convenience

$$J_\mu\left[\frac{X}{1-t}; q, t\right] = H_\mu(x; q, t), \tag{1.8}$$

we get

$$\Omega\left[\frac{XY}{(1-t)(1-q)}\right] = \sum_\mu \frac{H_\mu(x; q, t)\, H_\mu(y; q, t)}{h_\mu(q, t)\, h'_\mu(q, t)}.$$

Note next that from (I.3) we deduce the Schur function expansion

$$H_\mu(x; q, t) = \sum_\lambda S_\lambda(x)\, K_{\lambda\mu}(q, t). \tag{1.9}$$

Thus extracting the terms of total degree $2n$ in the variables x_i, y_j we derive that

$$h_n\left[\frac{XY}{(1-t)(1-q)}\right] = \sum_{\mu \vdash n} \frac{H_\mu(x; q, t)\, H_\mu(y; q, t)}{h_\mu(q, t)\, h'_\mu(q, t)}. \tag{1.10}$$

This leads us to our first basic identity.

THEOREM 1.1.

$$e_n\left[\frac{XY}{(1-t)(1-q)}\right]=\sum_{\mu\vdash n}\frac{\tilde{H}_\mu(x;q,t)\,\tilde{H}_\mu(y;q,t)}{\tilde{h}_\mu(q,t)\,\tilde{h}'_\mu(q,t)}. \tag{1.11}$$

where we have set

$$\tilde{h}_\mu(q,t)=\prod_{s\in\mu}(q^{a_\mu(s)}-t^{l_\mu(s)+1}),\qquad \tilde{h}'_\mu(q,t)=\prod_{s\in\mu}(t^{l_\mu(s)}-q^{a_\lambda(s)+1}). \tag{1.12}$$

Proof. From (I.4) and (I.5) we get that

$$\tilde{H}_\mu(x;q,t)=H_\mu(x;q,1/t)\,t^{n(\mu)}. \tag{1.13}$$

Note further that the definitions in (I.1) give

$$h_\mu(q,1/t)=(-1)^n\,\tilde{h}_\mu(q,t)/t^{n(\mu)+n}$$
$$h_\mu(q,1/t)=\tilde{h}_\mu(q,t)/t^{n(\mu)}.$$

This given, (1.7) follows immediately from (1.6) by replacing t with $1/t$ and noting that

$$h_n\left[\frac{XY}{(1-1/t)(1-q)}\right]=(-1)^n\,t^n e_n\left[\frac{XY}{(1-t)(1-q)}\right].$$

COROLLARY 1.1.

$$e_n\left[\frac{X}{(1-t)(1-q)}\right]=\sum_{\mu\vdash n}\frac{\tilde{H}_\mu(x;q,t)}{\tilde{h}_\mu(q,t)\,\tilde{h}'_\mu(q,t)}. \tag{1.14}$$

Proof. We simply evaluate both sides of (1.11) at an alphabet Y containing a single letter y_1 and note that we have

$$e_n\left[\frac{Xy_1}{(1-t)(1-q)}\right]=y_1^n e_n\left[\frac{X}{(1-t)(1-q)}\right]\qquad\text{and}\qquad \tilde{H}_\mu(y_1;q,t)=y_1^n.$$

Thus (1.14) is obtained by canceling the common factor y_1^n from both sides of the resulting identity.

The basic result that ties formula (I.8) to the Stanley–Macdonald Pieri rules may be stated as follows:

THEOREM 1.2. *For any $v\vdash n-1$ we have*

$$e_1(x)\,\tilde{H}_v(x;q,t)=\sum_{\mu\vdash n}\tilde{H}_\mu(x;q,t)\,d_{\mu v}(q,t)\,\chi(v\to\mu), \tag{1.15}$$

where the symbol $\chi(\nu \to \mu)$ is to indicate that the sum is to be carried out over partitions μ which immediately follow ν in the Young lattice, and

$$d_{\mu\nu}(q, t) = \prod_{s \in \mathscr{R}_{\mu/\nu}} \frac{q^{a_\nu(s)} - t^{l_\nu(s)+1}}{q^{a_\mu(s)} - t^{l_\mu(s)+1}} \prod_{s \in \mathscr{C}_{\mu/\nu}} \frac{t^{l_\nu(s)} - q^{a_\nu(s)+1}}{t^{l_\mu(s)} - q^{a_\mu(s)+1}}, \tag{1.16}$$

with $\mathscr{R}_{\mu/\nu}$ (resp. $\mathscr{C}_{\mu/\nu}$) denoting as before the set of lattice squares of ν that are in the same row (resp. same column) as the square we must remove from μ to obtain ν.

Proof. This identity is obtained by taking one of the Pieri rules for the basis $P_\lambda(x; q, t)$ given by Macdonald in [14] and translating it to the present setting by means of (I.2), (I.3), (1.8), and (1.13). The details of this computation are given in [4] (see Theorem 2.1 there).

COROLLARY 1.1. *With the same conventions as above, and for any $\mu \vdash n$,*

$$\partial_{p_1} \tilde{H}_\mu(x; q, t) = \sum_{\nu \to \mu} c_{\mu\nu}(q, t)\, \tilde{H}_\nu(x; q, t), \tag{1.17a}$$

where the coefficients $c_{\mu\nu}(q, t)$ are as given in (I.9). *We also have*

$$c_{\mu\nu}(q, t) = \frac{d_{\mu\nu}(q, t)}{(1-t)(1-q)} \frac{\tilde{h}_\mu(q, t)\, \tilde{h}'_\mu(q, t)}{\tilde{h}_\nu(q, t)\, \tilde{h}'_\nu(q, t)}. \tag{1.17b}$$

Proof. Note that we also have the expansion

$$e_n\left[\frac{XY}{(1-t)(1-q)}\right] = \sum_{\rho \vdash n} \frac{p_\rho(x)\, p_\rho(y)}{z_\rho} \frac{(-1)^{n-k(\rho)}}{p_\rho[(1-t)(1-q)]},$$

where $k(\rho)$ denotes the number of parts of ρ. Combining this with (1.11) we deduce that the two bases $\{\tilde{H}_\mu(x; q, t)/\tilde{h}_\mu(q, t)\}_\mu$ and $\{\tilde{H}_\mu(x; q, t)/\tilde{h}'_\mu(q, t)\}_\mu$ are dual with respect to the scalar product $\langle\ ,\ \rangle_*$ defined by setting for the power basis elements

$$\langle p_{\rho^{(1)}}, p_{\rho^{(2)}} \rangle_* = \begin{cases} 0 & \text{if } \rho^{(1)} \neq \rho^{(2)} \\ z_\rho p_\rho[(1-t)(1-q)](-1)^{n-k(\rho)} & \text{if } \rho^{(1)} = \rho^{(2)} = \rho \end{cases}.$$

Now a simple manipulation shows that we have

$$\langle \partial_{p_1} p_{\rho^{(1)}}, p_{\rho^{(2)}} \rangle_* = \frac{1}{(1-t)(1-q)} \langle p_{\rho^{(1)}}, p_1 p_{\rho^{(2)}} \rangle_*.$$

In other words, the operator ∂_{p_1} is the adjoint of multiplication by $p_1/((1-t)(1-q))$ with respect to the scalar product $\langle\ ,\ \rangle_*$. This means that the action of ∂_{p_1} on the kernel

$$e_n\left[\frac{XY}{(1-t)(1-q)}\right]$$

as a symmetric function of the x_i's has the same effect as multiplication of

$$e_{n-1}\left[\frac{XY}{(1-t)(1-q)}\right]$$

by $p_1(y)/((1-t)(1-q))$. Using (1.11), this results in the identity

$$\sum_{\mu\vdash n}\frac{\partial_{p_1}\tilde{H}_\mu(x;q,t)\,\tilde{H}_\mu(y;q,t)}{\tilde{h}_\mu(q,t)\,\tilde{h}'_\mu(q,t)} = \frac{1}{(1-t)(1-q)}\sum_{\nu\vdash n-1}\frac{\tilde{H}_\nu(x;q,t)\,p_1(y)\,\tilde{H}_\nu(y;q,t)}{\tilde{h}_\nu(q,t)\,\tilde{h}'_\nu(q,t)}.$$

Since p_1 and e_1 are one and the same we can use the Pieri rule in (1.15) and rewrite the right-hand side of this relation in the form

$$\text{RHS}=\frac{1}{(1-t)(1-q)}\sum_\nu \tilde{H}_\nu(x;q,t)\,\frac{1}{\tilde{h}_\nu(q,t)\,\tilde{h}'_\nu(q,t)}\sum_{\mu\leftarrow\nu}\tilde{H}_\mu(y;q,t)\,d_{\mu\nu}(q,t).$$

Substituting this in the equation above and equating coefficients of $\tilde{H}_\mu(y;q,t)$ on both sides gives

$$\partial_{p_1}\tilde{H}_\mu(x;q,t)\,\frac{1}{\tilde{h}_\mu(q,t)\,\tilde{h}'_\mu(q,t)} = \frac{1}{(1-t)(1-q)}\sum_{\nu\rightarrow\mu}\tilde{H}_\nu(x;q,t)\,\frac{1}{\tilde{h}_\nu(q,t)\,\tilde{h}'_\nu(q,t)}\,d_{\mu\nu}(q,t).$$

This establishes the recursion in (1.17a) with

$$c_{\mu\nu}(q,t)=\frac{d_{\mu\nu}(q,t)}{(1-t)(1-q)}\,\frac{\tilde{h}_\mu(q,t)\,\tilde{h}'_\mu(q,t)}{\tilde{h}_\nu(q,t)\,\tilde{h}'_\nu(q,t)}. \tag{1.18}$$

We leave it to the reader to verify that the expression on the right-hand side of this formula simplifies to the right-hand side of formula (I.9).

Macdonald established the existence of the basis $\{P_\lambda(x;q,t)\}_\lambda$ by characterizing it as the eigensystem of a certain difference operator ∂_1. Our polynomials $\tilde{H}_\mu(x;q,t)$ have an analogous characterization in terms of the difference operator Δ_1 which in λ-ring notation is given by setting for any symmetric polynomial $P(x)$

$$\Delta_1 P = P - P\left[X + \frac{(1-t)(1-q)}{z} \right] \Omega[-Xz]\big|_{z^o}. \tag{1.19}$$

Here the symbol $|_{z^o}$ represents the operation of taking a constant term in a formal Laurent series in the variable z.

More precisely, it is shown in [5, Theorem 2.2] that we have

THEOREM 1.3.

$$\Delta_1 \tilde{H}_\mu(x; q, t) = (1-t)(1-q)\, B_\mu(q, t)\, \tilde{H}_\mu(x; q, t). \tag{1.20}$$

Sketch of Proof. We first rewrite the Macdonald operator ∂_1 in λ-ring notation and compute its effect on the basis $J_\mu(x; q, t)$. Using the relations (1.8) and (1.13) the result is then transformed into an identity involving $\tilde{H}_\mu(x; q, t)$. This done, formula (1.20) is obtained after a few straightforward manipulations. This computation is carried out in full detail in [5] (see the proofs of Theorems 2.1 and 2.2 there).

The λ-ring formula in (1.19) makes it convenient to compute the action of Δ_1 in a number of special cases. In particular, we can easily derive the following result which is basic in the present treatment.

PROPOSITION 1.1.

$$\Delta_1 e_n\left[\frac{X}{(1-t)(1-q)}\right] = e_1(x)\, e_{n-1}\left[\frac{X}{(1-t)(1-q)}\right] \tag{1.21}$$

Proof. Note that for any two multisets of monomials A, B we have the addition formula

$$e_n[A+B] = \sum_{k=0}^{n} e_k[A]\, e_{n-k}[B].$$

Using this with $A = X/((1-t)(1-q))$ and $B = 1/z$, from the definition (1.19) we immediately obtain

$$\Delta_1 e_n\left[\frac{X}{(1-t)(1-q)}\right]$$
$$= e_n\left[\frac{X}{(1-t)(1-q)}\right] - \sum_{k=0}^{n} e_k\left[\frac{X}{(1-t)(1-q)}\right] e_{n-k}[1/z]\, \Omega\left[-\frac{X}{z}\right]\Big|_{z^o}. \tag{1.22}$$

However, since

$$e_{n-k}[1/z]=\begin{cases}1 & \text{for} \quad n-k=0,\\ 1/z & \text{for} \quad n-k=1,\\ 0 & \text{for} \quad n-k\geqslant 2,\end{cases}$$

(1.22) reduces to

$$\begin{aligned}\Delta_1 e_n\left[\frac{X}{(1-t)(1-q)}\right] &= -e_{n-1}\left[\frac{X}{(1-t)(1-q)}\right]\Omega\left[-\frac{X}{z}\right]\Big|_z\\ &= e_{n-1}\left[\frac{X}{(1-t)(1-q)}\right] e_1(x)\end{aligned}$$

as desired.

An immediate application of this result is our

First Proof of $B_\mu(q,t)=\sum_{\nu\to\mu} c_{\mu\nu}(q,t)$. Using (1.20) and (1.14) we can rewrite the left-hand side of (1.21) as

$$\text{LHS}=\sum_{\mu\vdash n}\frac{\tilde{H}_\mu(x;q,t)(1-t)(1-q)\,B_\mu(q,t)}{\tilde{h}_\mu(q,t)\,\tilde{h}'_\mu(q,t)}.$$

On the other hand, the right-hand side may be written as

$$\text{RHS}=\sum_{\nu\vdash n-1}\frac{e_1(x)\,\tilde{H}_\nu(x;q,t)}{\tilde{h}_\nu(q,t)\,\tilde{h}'_\nu(q,t)}.$$

By applying (1.15) we can transform this into

$$\text{RHS}=\sum_{\nu\vdash n-1}\frac{1}{\tilde{h}_\nu(q,t)\,\tilde{h}'_\nu(q,t)}\sum_\mu \tilde{H}_\mu(x;q,t)\,d_{\mu\nu}(q,t)\,\chi(\nu\to\mu).$$

Equating the LHS and the RHS we derive the identity

$$\begin{aligned}&\sum_{\mu\vdash n}\frac{\tilde{H}_\mu(x;q,t)(1-t)(1-q)\,B_\mu(q,t)}{\tilde{h}_\mu(q,t)\,\tilde{h}'_\mu(q,t)}\\ &\quad=\sum_\mu \tilde{H}_\mu(x;q,t)\sum_{\nu\vdash n-1}\frac{1}{\tilde{h}_\nu(q,t)\,\tilde{h}'_\nu(q,t)}\,d_{\mu\nu}(q,t)\,\chi(\nu\to\mu).\end{aligned}$$

Equating coefficients of $\tilde{H}_\mu(x; q, t)$ yields

$$\frac{(1-t)(1-q)}{\tilde{h}_\mu(q, t)\, \tilde{h}'_\mu(q, t)}\, B_\mu(q, t) = \sum_{\nu \to \mu} \frac{1}{\tilde{h}_\nu(q, t)\, \tilde{h}'_\nu(q, t)}\, d_{\mu\nu}(q, t),$$

and our desired identity follows from (1.17b).

An alternate proof of the identity is based on one of the specializations of $P_\mu(x; q, t)$ given by Macdonald in the original paper [13]. When this result is translated into a specialization of the polynomial $\tilde{H}_\mu(x; q, t)$ we obtain an identity which in λ-ring notation can be stated as follows.

PROPOSITION 1.2.

$$\tilde{H}_\mu[1-u; q, t] = \prod_{s \in \mu} (1 - u t^{l'_\mu(s)} q^{a'_\mu(s)}) \tag{1.23}$$

A detailed proof of this result can be found in [5, Corollary 2.1].

COROLLARY 1.2. *For* $k = 0, ..., n-1$ *and all* $\mu \vdash n$ *we have*

$$\tilde{K}_{1^k n-k, \mu}(q, t) = e_k[B_\mu(q, t) - 1], \tag{1.24}$$

in particular,

$$\begin{cases} \text{(a)} & \tilde{K}_{n, \mu}(q, t) = 1 \\ \text{(b)} & \tilde{K}_{1\, n-1, \mu}(q, t) = B_\mu(q, t) - 1 \\ \text{(c)} & \tilde{K}_{1^n, \mu}(q, t) = t^{n(\mu)} q^{n(\mu')}. \end{cases} \tag{1.25}$$

Proof. The identity in (1.23) combined with the expansion in (I.4) gives

$$\sum_\lambda S_\lambda[1-u]\, \tilde{K}_{\lambda\mu}(q, t) = \prod_{s \in \mu} (1 - u t^{l'_\mu(s)} q^{a'_\mu(s)}). \tag{1.26}$$

Now it is easily shown that $S_\lambda[1-u]$ fails to vanish only when λ is a hook. More precisely, we have

$$S_\lambda[1-u] = \begin{cases} (-u)^k (1-u) & \text{if } \lambda = 1^k n-k \text{ for some } k < n \\ 0 & \text{otherwise.} \end{cases}$$

Using this in (1.26) and cancelling the factor $1-u$ from both sides we get

$$\sum_{k=0}^{n-1} (-u)^k\, \tilde{K}_{1^k n-k, \mu}(q, t) = \prod_{s \in \mu}{}^{(oo)} (1 - u t^{l'_\mu(s)} q^{a'_\mu(s)}),$$

where the superscript $^{(oo)}$ is to indicate that the product omits the factor corresponding to the corner cell with $a'=l'=0$. This given, (1.24) follows by equating coefficients of u^k.

We are thus in a position to give our

Second Proof of $B_\mu(q,t)=\sum_{\nu\to\mu} c_{\mu\nu}(q,t)$. Note that (I.4) and (a) of (1.25) (with n replaced by $n-1$) give that for any $\nu\vdash n-1$ we have

$$\langle \tilde{H}_\nu, S_{n-1}\rangle = 1$$

where here the angles $\langle\ ,\ \rangle$ are to represent the customary Hall inner product of symmetric polynomials. Thus, using (I.8) we may write

$$\sum_{\nu\to\mu} c_{\mu\nu}(q,t)=\langle \partial_{p_1}\tilde{H}_\mu, S_{n-1}\rangle. \tag{1.27}$$

Now it is well known and easy to show that the adjoint of the operator ∂_{p_1} with respect to the Hall inner product is multiplication by p_1. From this and (I.4) we finally deduce that

$$\begin{aligned}\sum_{\nu\to\mu} c_{\mu\nu}(q,t)&=\langle \tilde{H}_\mu, p_1 S_{n-1}\rangle = \langle \tilde{H}_\mu, S_n + S_{n-1,1}\rangle\\ &=\tilde{K}_{n,\mu}(q,t)+\tilde{K}_{n-1,1,\mu}(q,t)\end{aligned}$$

and our identity follows from (1.25a and b).

2. THE q,t-HOOK WALK

We shall start with a brief review of the Greene–Nijenhuis–Wilf proof of the identity

$$1=\frac{1}{n}\sum_{\nu\to\mu}\frac{h_\mu}{h_\nu}. \tag{2.1}$$

To simplify our language we need to make some notational conventions. To begin with we shall hereafter identify a partition μ with its Ferrers diagram. We should also recall that the *hook* of a cell s of μ consists of s together with its *arm*, whose length we have denoted by $a_\mu(s)$ and its *leg* whose length we have denoted by $l_\mu(s)$. Since we use the French convention of depicting Ferrers diagrams, the arm of s consists of the cells of μ which are strictly east of s and the leg consists of the cells of μ which are strictly north of s. Likewise, the co-arm and co-leg consist of the cells respectively strictly west and strictly south. We shall often use the words *arm*, *co-arm*, *leg*, and *co-leg* to refer to their respective lengths. We set $h_\mu(s) = 1 + a_\mu(s) + l_\mu(s)$ and refer to it as the *hook length* of s in μ. We shall also set $h_\mu = \prod_{s \in \mu} h_\mu(s)$. When v immediately precedes μ (which we have expressed by writing $v \to \mu$) it will be convenient to denote by μ/v the corner cell we must remove from μ to obtain v. A cell s with coarm a' and coleg l' will be represented by the pair $(a'+1, l'+1)$. If $s = (x, y)$ and $s' = (x', y')$ we shall write $s \ll s'$ if and only if $x < x'$ and $y < y'$ and $s \ll = s'$ if and only if $x \leqslant x'$ and $y \leqslant y'$. The collection of cells that are weakly northeast of s will be denoted by NE(s) and will be referred to as the *shadow* of s. That is,

$$\mathrm{NE}(s) = \{s' : s \ll = s'\}.$$

We shall also express the inequality $s \ll s'$ by saying that *s is covered by s'*. Here the symbols $R_{\mu/v}$ and $C_{\mu/v}$ will have the same meaning as in the Introduction, but in addition, for a given cell s we shall denote by $R_{\mu/v}(s)$ and $C_{\mu/v}(s)$ the cells of $R_{\mu/v}$ and $C_{\mu/v}$ that are *strictly* northeast of s. Note that both $R_{\mu/v}(s)$ and $C_{\mu/v}(s)$ are empty when μ/v is not in the shadow of s. When μ/v is in the shadow of s we shall denote by $r[s]$ the element of $R_{\mu/v} \cup \{\mu/v\}$ that is directly North of s. Likewise $c[s]$ will denote the element of $C_{\mu/v} \cup \{\mu/v\}$ that is directly East of s.

Given $\mu \vdash n$, the basic ingredient in [11] is a random walk $Z_1, Z_2, ..., Z_m, ...$ over the cells of μ which is constructed according to the following mechanism.

(1) The initial point $Z_1 = (x_1, y_1)$ is obtained by selecting one of the cells of μ at random and with probability $1/n$.

(2) After k steps, given that $Z_k = s$,

(a) the walk stops if s is a corner cell of μ;

(b) if s is not a corner cell, then Z_{k+1} is obtained by selecting at random and with equal probability $1/(a_\mu(s) + l_\mu(s)) = 1/(h_\mu(s) - 1)$ one of the cells of the arm or the leg of s in μ.

Greene–Nijenhuis–Wilf establish (2.1) by showing that for any $v \to \mu$ the quantity $(1/n)(h_\mu/h_v)$ gives the probability that the random walk ends at the corner cell μ/v. Denoting by Z_{end} the ending position of the random walk, we may express this by writing

$$P[Z_{\text{end}} = \mu/v] = \frac{1}{n} \frac{h_\mu}{h_v}. \tag{2.2}$$

Clearly, if the random walk starts at the cell s then it can only end on a corner cell that is in the shadow of s. In fact, the G-N-W proof yields that for $s \ll \mu/\nu$

$$P[Z_{\text{end}} = \mu/\nu \mid Z_1 = s]$$
$$= \frac{1}{h_\mu(r[s]) - 1} \frac{1}{h_\mu(c[s]) - 1} \prod_{r \in R_{\mu/\nu}(s)} \frac{h_\mu(r)}{h_\nu(r)} \prod_{c \in C_{\mu/\nu}(s)} \frac{h_\mu(c)}{h_\nu(c)}, \tag{2.3}$$

for $r[s] = \mu/\nu$

$$P[Z_{\text{end}} = \mu/\nu \mid Z_1 = s] = \frac{1}{h_\mu(c[s]) - 1} \prod_{c \in C_{\mu/\nu}(s)} \frac{h_\mu(c)}{h_\nu(c)}, \tag{2.3r}$$

and for $c[s] = \mu/\nu$

$$P[Z_{\text{end}} = \mu/\nu \mid Z_1 = s] = \frac{1}{h_\mu(r[s]) - 1} \prod_{r \in R_{\mu/\nu}(s)} \frac{h_\mu(r)}{h_\nu(r)}. \tag{2.3c}$$

This given, (2.2) follows from the identity

$$P[Z_{\text{end}} = \mu/\nu] = \sum_{s \ll = \mu/\nu} P[Z_1 = s]\, P[Z_{\text{end}} = \mu/\nu \mid Z_1 = s]. \tag{2.4}$$

Remarkably, all of this has a complete q, t-analog in our setting. As we shall see, our proof of

$$B_\mu(q, t) = \sum_{\nu \to \mu} c_{\mu\nu}(q, t) \tag{2.5}$$

brings to light the finer combinatorial mechanism that underlies the G-N-W argument.

In order to use the probabilistic jargon in our argument, it is necessary to view the parameters q and t as positive numbers. In fact, it will be convenient to let $0 < q < 1$ and $t > 1$. However, the trained combinatorial eye should have no difficulty seeing that this condition is totally artificial. In fact, it can be done without completely by viewing each random walk as a *lattice path* and its probability as the *weight* of the path. In this setting q and t may be left as they should be, namely as two independent indeterminates. From this point of view our proof may viewed as a modification of the G-N-W proof obtained by simply *changing weights*. Nevertheless, the probabilistic jargon is too convenient to give up at this point and we shall use it at first, leaving the combinatorial implications to our final comments.

Following the G-N-W scheme our random walk may be described as follows.

(1) The initial point Z_1 is obtained by selecting the cell (x, y) of μ with probability $(q^{x-1}t^{y-1})/B_\mu(q, t)$.

(2) After k steps, given that $Z_k = s = (x, y)$,

(a) the walk stops if s is a corner cell of μ

(b) if s is not a corner cell then Z_{k+1} is obtained by selecting

(i) the cell $(x, y+j)$ of the leg of s with probability $t^{j-1}((q^{a_\mu(s)}(t-1))/(t^{l_\mu(s)} - q^{a_\mu(s)}))$

(ii) the cell $(x+i, y)$ of the arm of s with probability $q^{i-1}((t^{l_\mu(s)}(1-q))/(t^{l_\mu(s)} - q^{a_\mu(s)}))$.

Note that the probability of Z_{k+1} landing anywhere in the leg of s is given by the sum

$$\sum_{j=1}^{l_\mu(s)} t^{j-1} \frac{q^{a_\mu(s)}(t-1)}{t^{l_\mu(s)} - q^{a_\mu(s)}} = q^{a_\mu(s)} \frac{t^{l_\mu(s)} - 1}{t^{l_\mu(s)} - q^{a_\mu(s)}},$$

and the probability of Z_{k+1} landing anywhere in the arm of s is given by

$$\sum_{i=1}^{a_\mu(s)} q^{i-1} \frac{t^{l_\mu(s)}(1-q)}{t^{l_\mu(s)} - q^{a_\mu(s)}} = t^{l_\mu(s)} \frac{1 - q^{a_\mu(s)}}{t^{l_\mu(s)} - q^{a_\mu(s)}}$$

and we see that we do have, as necessary,

$$q^{a_\mu(s)} \frac{t^{l_\mu(s)} - 1}{t^{l_\mu(s)} - q^{a_\mu(s)}} + t^{l_\mu(s)} \frac{1 - q^{a_\mu(s)}}{t^{l_\mu(s)} - q^{a_\mu(s)}} = 1.$$

Our plan is to prove (2.5) by establishing that for any $\nu \to \mu$,

$$P[Z_{\text{end}} = \mu/\nu] = \frac{1}{B_\mu(q, t)} c_{\mu\nu}(q, t). \tag{2.6}$$

It will be convenient to set for any cell $s \in \mu$

$$A(s) = \frac{t^{l_\mu(s)}(1-q)}{t^{l_\mu(s)} - q^{a_\mu(s)}} \quad \text{and} \quad B(s) = \frac{q^{a_\mu(s)}(t-1)}{t^{l_\mu(s)} - q^{a_\mu(s)}}. \tag{2.7}$$

This given, in complete analogy with (2.3) we shall show that

$$P[Z_{\text{end}}=\mu/\nu \mid Z_1=s]$$
$$=A(r[s])\,B(c[s]) \prod_{r\in R_{\mu/\nu}(s)} \frac{t^{l_\mu(r)}-q^{a_\mu(r)+1}}{t^{l_\nu(r)}-q^{a_\nu(r)+1}} \prod_{c\in C_{\mu/\nu}(s)} \frac{q^{a_\mu(c)}-t^{l_\mu(c)+1}}{q^{a_\nu(c)}-t^{l_\nu(c)+1}}. \tag{2.8}$$

The identity in (2.8) as well as that in (2.5) is almost an immediate consequence of two elementary combinatorial lemmas which are at the root of the G-N-W argument. The first of these is a lattice path result which is interesting in its own right. Let $\mathscr{L}(h,k)$ denote the collection of lattice points

$$\mathscr{L}(h,k)=\{(i,j)\colon 1\leqslant i\leqslant h+1,\ 1\leqslant j\leqslant k+1\}.$$

Let $\mathscr{P}(h,k)$ denote the collection of lattice paths in $\mathscr{L}(h,k)$ which start at (1, 1), end at $(h+1, k+1)$, and proceed by East and North steps. To be precise, a path $\pi\in\mathscr{P}(h,k)$ is given by a sequence of $m=h+k+1$ lattice points

$$\pi=\{(i_1,i_1),(i_2,j_2),\ldots,(i_m,j_m)\}, \tag{2.9}$$

with

$$\begin{cases}(i_1,j_1)=(1,1)\\ (i_{s+1},j_{s+1})=\begin{cases}(i_s+1,j_s)\\ \text{or}\\ (i_s,j_s+1)\end{cases} & \text{for}\quad 1\leqslant s\leqslant h+k\\ (i_m,j_m)=(h+1,k+1)\end{cases}$$

Given two sets of weights, $\{a_{i,j}\}_{i=1\cdots h+1,\,j=1\cdots k+1}$ and $\{b_{i,j}\}_{i=1\cdots h+1,\,j=1\cdots k+1}$, we define the weight of a step $(i,j)\to(i',j')$ by setting

$$w[(i,j)\to(i',j')]=\begin{cases}\dfrac{1}{a_{i,j}} & \text{if}\quad (i',j')=(i+1,j)\quad \begin{pmatrix}\text{An East}\\ \text{Step}\end{pmatrix}\\[2ex] \dfrac{1}{b_{i,j}} & \text{if}\quad (i',j')=(i,j+1)\quad \begin{pmatrix}\text{A North}\\ \text{Step}\end{pmatrix}\end{cases}, \tag{2.10}$$

then define the weight $w[\pi]$ of a path $\pi\in\mathscr{P}(h,k)$ to be the product of the weights of each of its steps. More precisely, if π is as given in (2.9) we set

$$w[\pi]=\prod_{s=1}^{h+k} w[(i_s,j_s)\to(i_{s+1},j_{s+1})].$$

LEMMA 2.1. *Let $a_1, a_2, ..., a_h$ and $b_1, b_2, ..., b_k$ be fixed indeterminates and u be a fixed parameter. Let the weights $a_{i,j}$ and $b_{i,j}$ be given by setting*

$$a_{i,j} = \frac{a_i + b_j}{1 + ub_j}, \qquad b_{i,j} = \frac{a_i + b_j}{1 - ua_i} \tag{2.11}$$

with $a_{h+1} = b_{k+1} = 0$. Then

$$w[a_1, .., a_h; b_1, .., b_k] = \sum_{\pi \in \mathscr{P}(h,k)} w[\pi] = \frac{1}{a_1 a_2 \cdots a_h \ b_1 b_2 \cdots b_k}. \tag{2.12}$$

Proof. Note that our choice of weights $\{a_{i,j}\}$, $\{b_{i,j}\}$ assures that we have

$$\frac{a_i}{a_{i,j}} + \frac{b_j}{b_{i,j}} = 1. \tag{2.13}$$

This given, for $h = k = 1$ we have

$$w[a_1; b_1] = \frac{1}{a_{1,1}} \frac{1}{b_1} + \frac{1}{b_{1,1}} \frac{1}{a_1} = \left(\frac{a_1}{a_{1,1}} + \frac{b_1}{b_{1,1}} \right) \frac{1}{a_1 b_1} = \frac{1}{a_1 b_1}.$$

So we may proceed by induction on $m = h + k$. Let it be true for $m - 1$ and for any set of indeterminates. Since any path in $\mathscr{P}(h, k)$ must start with one of the two steps $(1, 1) \rightarrow (2, 1)$ or $(1, 1) \rightarrow (1, 2)$, we must have

$$w[a_1, .., a_h; b_1, .., b_k] = \frac{1}{a_{1,1}} w[a_2, .., a_h; b_1, .., b_k] + \frac{1}{b_{1,1}} w[a_1, .., a_h; b_2, .., b_k],$$

so by the induction hypothesis

$$\begin{aligned} w[a_1, .., a_h; b_1, .., b_k] &= \frac{1}{a_{1,1}} \frac{1}{a_2 \cdots a_h \ b_1 b_2 \cdots b_k} \frac{1}{b_{1,1}} \frac{1}{a_1 a_2 \cdots a_h \ b_2 \cdots b_k} \\ &= \left(\frac{a_1}{a_{1,1}} + \frac{b_1}{b_{1,1}} \right) \frac{1}{a_1 a_2 \cdots a_h b_1 \ b_2 \cdots b_k} \\ &= \frac{1}{a_1 a_2 \cdots a_h b_1 \ b_2 \cdots b_k}. \end{aligned}$$

This completes the induction and the proof.

LEMMA 2.2. *For any indeterminates* $a_1, a_2, ..., a_n$ *and* $b_1, b_2, ..., b_n$ *we have*

$$\prod_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} b_1 b_2 \cdots b_{i-1} a_i \prod_{i' > i} (a_{i'} + b_{i'}) + b_1 b_2 \cdots b_n. \quad (2.14)$$

Proof. This identity is obtained by expanding the product on the left-hand side and then combining into the ith summand all the monomials which contain a_i and do not contain $a_1, a_2, ..., a_{i-1}$. We may also prove (2.14) by an obvious induction argument.

To apply these two results to our q, t-hook walks we need to introduce further notation. To begin with it will be convenient to briefly denote a hook walk $Z_1, Z_2, ..., Z_{\text{end}}$ by the symbol HW. To distinguish between the random variable HW and its values, we need to introduce the notion of a *hook path*. By this we mean a sequence of cells of μ

$$\pi = \{s_1 \to s_2 \to \cdots \to s_{m+1}\} \quad (2.15)$$

with

$$\begin{cases} s_{i+1} & \text{in the arm or leg of } s_i \quad \text{for} \quad i = 1, ..., m \\ s_{m+1} = \mu/\nu & \text{for some } \nu \to \mu \end{cases}$$

For a fixed pair $\nu \to \mu$, set $\mu/\nu = (a' + 1, l' + 1)$. We will find it convenient to represent the cells $s \ll = \mu/\nu$ by their projections in $R_{\mu/\nu} \cup \{\mu/\nu\}$ and $C_{\mu/\nu} \cup \{\mu/\nu\}$. More precisely, we shall write

$$s = [r, c] \Leftrightarrow \begin{cases} r[s] = r \\ \text{and} \\ c[s] = c. \end{cases}$$

Now let

$$R' = \{r_1, r_2, ..., r_h\} \subseteq R_{\mu/\nu} \quad \text{and} \quad C' = \{c_1, c_2, ..., c_k\} \subseteq C_{\mu/\nu} \quad (2.16)$$

where the elements of R' are given in the left-to-right order and those in C' are given in the bottom-to-top order. The set of cells $s \in \mu$ such that $r[s] \in R' \cup \{\mu/\nu\}$ and $c[s] \in C' \cup \{\mu/\nu\}$ will be denoted by $\text{gr}[R', C']$ and will be referred to as the *grid* determined by the pair R', C'.

Note that when C' is empty then $\text{gr}[R', C']$ is simply $R' \cup \{\mu/\nu\}$, and likewise, when R' is empty then $\text{gr}[R', C']$ reduces to $C' \cup \{\mu/\nu\}$. Now it is easily seen that to each hook path

$$\pi = \{s_1 = (x_1, y_1) \to s_2 = (x_2, y_2) \to \cdots \to s_m = (x_m, y_m) \to \mu/\nu\}$$

we can associate a unique grid $\mathrm{gr}[\pi]$ which may be defined as the smallest grid which contains all the vertices of π. To do this we simply set

$$\mathrm{gr}[\pi]=\mathrm{gr}[R', C'] \quad \text{with} \quad \begin{cases} R'=\{r\in R_{\mu/\nu}: r=r[x_i] \ \text{ for some } 1\leqslant i\leqslant m\} \\ C'=\{c\in R_{\mu/\nu}: c=c[y_j] \ \text{ for some } 1\leqslant j\leqslant m\}. \end{cases} \tag{2.17}$$

Note that if we let n_k be the number of cells of μ that are between s_k and s_{k+1} then the probability of the transition $Z_k\to Z_{k+1}$ according to our definition is given by

$$P[Z_{k+1}=s_{k+1} \mid Z_k=s_k]=\begin{cases} q^{n_k}A(s_k) & \text{if} \quad s_k\to s_{k+1} \text{ is an East step} \\ t^{n_k}B(s_k) & \text{if} \quad s_k\to s_{k+1} \text{ is a North step}. \end{cases} \tag{2.18}$$

Let us now define the weight of a hook path π (given in (2.15)) as the product

$$w[\pi]=\prod_{k=1}^{m} w[s_k\to s_{k+1}]$$

with

$$w[s_k\to s_{k+1}]=\begin{cases} \dfrac{A(s_k)}{1-q} & \text{if} \quad s_k\to s_{k+1} \text{ is an East step} \\ \dfrac{B(s_k)}{t-1} & \text{if} \quad s_k\to s_{k+1} \text{ is a North step}. \end{cases} \tag{2.19}$$

Comparing (2.19) and (2.18) we see that if $\mathrm{gr}[\pi]=\mathrm{gr}[R', C']$, with R', C' as given in (2.16), and $\mu/\nu=(a'+1, l'+1)$ then the probability of a hook walk resulting in π, given that $Z_1=s_1=[r_1, c_1]$, may be written as

$$P[\mathrm{HW}=\pi \mid Z_1=s_1]=q^{a_\mu(r_1)-|R'|}\, t^{l_\mu(c_1)-|C'|}(1-q)^{|R'|}\,(t-1)^{|C'|}\, w[\pi]. \tag{2.20}$$

This places us in a position to establish our first basic identity.

THEOREM 2.1. *If R' and C' are as given in* (2.16) *and $s_1=[r_1, c_1]$ then*

$$P[\mathrm{gr}[HW]=\mathrm{gr}[R', C'] \mid Z_1=s_1] = q^{a_\mu(r_1)-|R'|}\, t^{l_\mu(c_1)-|C'|} \prod_{r'\in R'} A(r') \prod_{c'\in C'} B(c'). \tag{2.21}$$

Proof. Since

$$P[\mathrm{gr}[\mathrm{HW}]=\mathrm{gr}[R', C'] \mid Z_1=s_1]=\sum_{\mathrm{gr}[\pi]=\mathrm{gr}[R', C']} P[\mathrm{HW}=\pi \mid Z_1=s_1],$$

from (2.20) we get that

$$P[\mathrm{gr}[\mathrm{HW}]=\mathrm{gr}[R', C'] \mid Z_1=s_1]$$
$$=q^{a_\mu(r_1)-|R'|}t^{l_\mu(c_1)-|C'|}(1-q)^{|R'|}(t-1)^{|C'|}\sum_{\mathrm{gr}[\pi]=\mathrm{gr}[R', C']} w[\pi]. \quad (2.22)$$

To evaluate the sum on the right-hand side we need only show that the assignment of weights in (2.19) satisfies the conditions required by Lemma 2.1. To this end let $s \ll \mu/v$ and set $r[s]=r$ and $c[s]=c$ (see Fig. 1).

Now a look at the figure above should reveal that the following identities hold true for any triplet r, c, s with $s=[r, c]$:

$$\begin{aligned} l_\mu(s)&=l_\mu(r)+l_\mu(c),\\ a_\mu(s)&=a_\mu(r)+a_\mu(c). \end{aligned} \quad (2.23)$$

For convenience set

$$\begin{aligned} \xi_s&=\frac{1-q}{A(s)}=1-\frac{q^{a_\mu(s)}}{t^{l_\mu(s)}}; \qquad \eta_s=\frac{t-1}{B(s)}=\frac{t^{l_\mu(s)}}{q^{a_\mu(s)}}-1\\ \xi_r&=\frac{1-q}{A(r)}=1-\frac{q^{a_\mu(r)}}{t^{l_\mu(r)}}; \qquad \eta_c=\frac{t-1}{B(c)}=\frac{t^{l_\mu(c)}}{q^{a_\mu(c)}}-1. \end{aligned} \quad (2.24)$$

This given, we have

$$\frac{1}{1+\eta_s}=1-\xi_s=\frac{q^{a_\mu(s)}}{t^{l_\mu(s)}}=\frac{q^{a_\mu(r)}}{t^{l_\mu(r)}}\times\frac{q^{a_\mu(c)}}{t^{l_\mu(c)}}=\frac{1-\xi_r}{1+\eta_c},$$

from which we derive that

$$\xi_s=\frac{\xi_r+\eta_c}{1+\eta_c}, \qquad \eta_s=\frac{\xi_r+\eta_c}{1-\xi_r}.$$

This shows that the assignment of weights in (2.19) satisfies the conditions in (2.11) with $u=1$

$$a_i=\xi_{r_i}, \qquad b_j=\eta_{c_j}$$

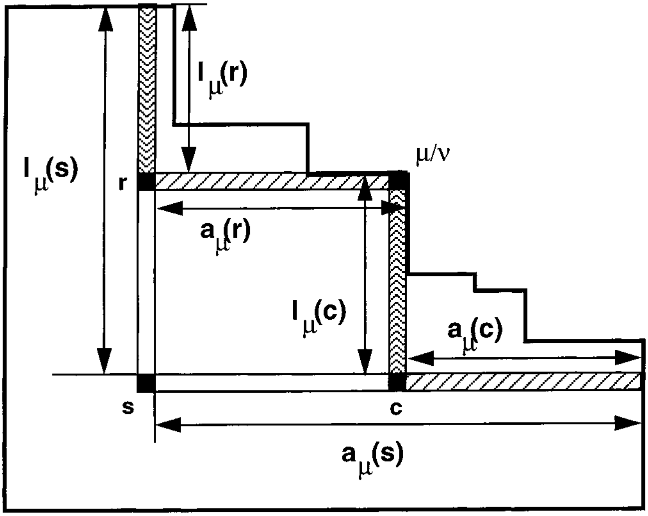

Figure 1

and

$$a_{i,j} = \xi_{[r_i, c_j]} = \frac{\xi_{r_i} + \eta_{c_j}}{1 + \eta_{c_j}}, \qquad b_{i,j} = \eta_{[r_i, c_j]} = \frac{\xi_{r_i} + \eta_{c_j}}{1 - \xi_{r_i}}.$$

So Lemma 2.1 gives

$$\sum_{\mathrm{gr}[\pi] = \mathrm{gr}[R', C']} w[\pi] = \frac{1}{\xi_{r_1} \xi_{r_2} \cdots \xi_{r_h} \eta_{c_1} \eta_{c_2} \cdots \eta_{c_k}}.$$

Substituting this into (2.22) and using the relations in (2.24) we finally obtain

$$P[\mathrm{gr}[\mathrm{HW}] = \mathrm{gr}[R', C'] \mid Z_1 = s_1] = q^{a_\nu(r_1) - |R'|} t^{l_\nu(c_1) - |C'|} \prod_{i=1}^{h} A(r_i) \prod_{j=1}^{k} B(c_j),$$

which is another way of writing (2.21).

Remark 2.1. We should note that (2.21) is the q, t-analogue of the G-N-W identity

$$P[\mathrm{gr}[\mathrm{HW}] = \mathrm{gr}[R', C'] | Z_1 = s_1] = \prod_{r' \in R'} \frac{1}{h_{r'} - 1} \prod_{c' \in C'} \frac{1}{h_{c'} - 1}. \tag{2.25}$$

Since in the G-N-W case the assignment of weight in (2.19) reduces to

$$w[s_k \rightarrow s_{k+1}] = \frac{1}{h_\mu(s_k) - 1}$$

and the relations in (2.23) give

$$h_\mu(s) - 1 = h_\mu(r) - 1 + h_\mu(c) - 1,$$

we see that (2.25) is another instance of (2.12) with

$$a_i = h_\mu(r_i) - 1, \qquad b_j = h_\mu(c_j) - 1, \qquad \text{and} \qquad a_{i,j} = b_{i,j} = h_\mu([r_i, c_j]) - 1.$$

Let us set for any partition μ and any $s \in \mu$,

$$\tilde{h}'_\mu(s) = t^{l_\mu(s)} - q^{a_\mu(s)+1}, \qquad \tilde{h}_\mu(s) = q^{a_\mu(s)} - t^{l_\mu(s)+1}.$$

Note that for any $r \in R_{\mu/\nu}$ we have

$$q + A(r) = q + \frac{t^{l_\mu(r)}(1-q)}{t^{l_\mu(r)} - q^{a_\mu(r)}} = \frac{t^{l_\mu(r)} - q^{a_\mu(r)+1}}{t^{l_\mu(r)} - q^{a_\mu(r)}} = \frac{\tilde{h}'_\mu(r)}{\tilde{h}'_\nu(r)}. \tag{2.26}$$

Similarly, for any $c \in C_{\mu/\nu}$ we have

$$t + B(c) = t + \frac{q^{a_\mu(c)}(1-t)}{q^{a_\mu(c)} - t^{l_\mu(c)}} = \frac{q^{a_\mu(c)} - t^{l_\mu(c)+1}}{q^{a_\mu(c)} - t^{l_\mu(c)}} = \frac{\tilde{h}_\mu(c)}{\tilde{h}_\nu(c)}. \tag{2.27}$$

We can thus state the following beautiful corollary of Theorem 2.1:

THEOREM 2.2. *For any $s \ll \mu/\nu$,*

$$P[Z_{\text{end}} = \mu/\nu \mid Z_1 = s] = A(r[s])\, B(c[s]) \prod_{r \in R_{\mu/\nu}(s)} (q + A(r)) \prod_{c \in C_{\mu/\nu}(s)} (t + B(c)). \tag{2.28a}$$

Or, which is the same,

$$P[Z_{\text{end}} = \mu/\nu \mid Z_1 = s] = A(r[s])\, B(c[s]) \prod_{r \in R_{\mu/\nu}(s)} \frac{\tilde{h}'_\mu(r)}{\tilde{h}'_\nu(r)} \prod_{c \in C_{\mu/\nu}(s)} \frac{\tilde{h}_\mu(c)}{\tilde{h}_\nu(c)}. \tag{2.28b}$$

Proof. The first equality follows immediately by summing (2.21) over subsets $R' \subseteq R_{\mu/\nu}(s)$ and $C' \subseteq C_{\mu/\nu}(s)$ and the second follows from the first because of (2.26) and (2.27).

The identities in (2.26) and (2.27) allow us to rewrite the coefficient $c_{\mu\nu}(q, t)$ in a rather revealing form.

PROPOSITION 2.1. *Let $r_1, r_2, ..., r_{a'}$ be the elements of $R_{\mu/\nu}$ in the left-to-right order and $c_1, c_2, ..., c_{l'}$ be the elements of $C_{\mu/\nu}$ in the bottom-to-top order. Then*

$$c_{\mu\nu}(q, t) = \left(\sum_{i=1}^{a'} q^{i-1} A(r_i) \prod_{i'>i} (q + A(r_{i'})) + q^{a'} \right) \times \left(\sum_{j=1}^{l'} t^{j-1} B(c_j) \prod_{j'>j} ((t + B(c_{j'})) + t^{l'} \right). \quad (2.29)$$

Proof. Using (2.26) we may write

$$\prod_{r \in R_{\mu/\nu}} \frac{\tilde{h}'_\mu(r)}{\tilde{h}'_\nu(r)} = \prod_{i=1}^{a'} (q + A(r_i)).$$

Using Lemma 2.2 with $a_i = A(r_i)$ and $b_i = q$ gives

$$\prod_{r \in R_{\mu/\nu}} \frac{\tilde{h}'_\mu(r)}{\tilde{h}'_\nu(r)} = \sum_{i=1}^{a'} q^{i-1} A(r_i) \prod_{i'>i} (q + A(r_{i'})) \ + \ q^{a'}.$$

Similarly, the relations in (2.27) and Lemma 2.2 with $a_j = B(c_j)$ and $b_i = t$ give

$$\prod_{c \in C_{\mu/\nu}} \frac{\tilde{h}_\mu(c)}{\tilde{h}_\nu(c)} = \sum_{j=1}^{l'} t^{i-1} B(c_j) \prod_{j'>j} ((t + B(c'_j)) \ + \ t^{l'}.$$

Multiplying these two identities and using the definition (I.9) of $c_{\mu\nu}(q, t)$ gives (2.29) as desired.

This identity may be converted into the following hook walk interpretation for the $c_{\mu\nu}(q, t)$:

THEOREM 2.3.

$$c_{\mu\nu}(q, t) = \sum_{s \ll = \mu/\nu} q^{a'(s)} t^{l'(s)} P[Z_{\text{end}} = \mu/\nu \mid Z_1 = s]. \quad (2.30)$$

In particular, we derive that

$$\sum_{\nu \to \mu} c_{\mu\nu}(q, t) = B_\mu(q, t).$$

Proof. Note that Theorem 2.2 gives

$$\sum_{i=1}^{a'} \sum_{j=1}^{l'} q^{i-1} t^{j-1} A(r_i)\, B(c_j) \prod_{i'>i} (q + A(r_{i'})) \prod_{j'>j} ((t + B(c_{j'}))$$

$$= \sum_{i=1}^{a'} \sum_{j=1}^{l'} q^{i-1} t^{j-1} P[Z_{\text{end}} = \mu/\nu \mid Z_1 = [r_i, c_j]]$$

$$= \sum_{s \ll \mu/\nu} q^{a'(s)} t^{l'(s)} P[Z_{\text{end}} = \mu/\nu \mid Z_1 = s]. \tag{2.31}$$

Similarly, we derive that

$$\sum_{i=1}^{a'} q^{i-1} t^{l'} A(r_i) \prod_{i'>i} (q + A(r_{i'})) = \sum_{r \in R_{\mu/\nu}} q^{a'(r)} t^{l'(r)} P[Z_{\text{end}} = \mu/\nu \mid Z_1 = r] \tag{2.31r}$$

and

$$\sum_{j=1}^{l'} q^{a'} t^{j-1} B(c_j) \prod_{i'>i} (c + B(c_{j'})) = \sum_{c \in C_{\mu/\nu}} q^{a'(c)} t^{l'(c)} P[Z_{\text{end}} = \mu/\nu \mid Z_1 = c], \tag{2.31c}$$

which are our q, t-analogues of (2.3), (2.3r), and (2.3c). Since

$$P[Z_{\text{end}} = \mu/\nu \mid Z_1 = \mu/\nu] = 1,$$

expanding the left-hand side of (2.29) and using (2.31), (3.31r), and (3.31c) yields (2.30) precisely as asserted. The last assertion follows immediately from (2.30) and and the fact that for any $s \in \mu$ we must have

$$\sum_{\nu \to \mu} P[Z_{\text{end}} = \mu/\nu] = 1.$$

The identity in (2.30) may be given a suggestive reformulation which brings to light a number of remarkable properties of the coefficients $c_{\mu\nu}(q, t)$. To see this let ϕ_ν be a function of the partitions $\nu \to \mu$. Multiplying (2.30) by ϕ_ν and summing over all $\nu \to \mu$ we get

$$\sum_{\nu \to \mu} c_{\mu\nu}(q, t)\, \phi(\nu) = \sum_{s \in \mu} q^{a'(s)} t^{l'(s)} \sum_{\nu \to \mu} \phi(\nu)\, P[Z_{\text{end}} = \mu/\nu \mid Z_1 = s]. \tag{2.32}$$

Now if (by a slight abuse of notation) we set $\phi(Z_{\text{end}}) = \phi(\nu)$ when $Z_{\text{end}} = \mu/\nu$, then we can write

$$\sum_{\nu \to \mu} \phi(\nu)\, P[Z_{\text{end}} = \mu/\nu \mid Z_1 = s] = E[\phi(Z_{\text{end}}) \mid Z_1 = s], \tag{2.33}$$

where the right-hand side may be referred to as the *conditional expectation* of $\phi(Z_{\text{end}})$ given that $Z_1 = s$. Combining (2.32) and (2.33) we obtain the following corollary of Theorem 2.3:

THEOREM 2.4. *For any function ϕ of the partitions $\nu \to \mu$ we have*

$$\sum_{\nu \to \mu} c_{\mu\nu}(q, t)\, \phi(\nu) = \sum_{s \in \mu} q^{a'(s)} t^{l'(s)} E[\phi(Z_{\text{end}}) \,|\, Z_1 = s]. \tag{2.34}$$

In particular, letting $\phi(\nu) = F_\nu(q, t)$, from (I.10) *we derive that*

$$F_\mu(q, t) = \sum_{s \in \mu} q^{a'(s)} t^{l'(s)} E[F_{Z_{\text{end}}}(q, t) \,|\, Z_1 = s]. \tag{2.35}$$

Remark 2.2. Although the recursion in (2.35) was derived from (I.10), it should be considered as an interesting alternate to (I.10). In fact, by iterating the latter we end up expressing F_μ as a sum of certain rational functions $R_T(q, t)$ indexed by standard tableaux T of shape μ. In contrast, iterating on (2.35), and suitably grouping the terms thus obtained, we obtain a formula for F_μ as a sum of certain rational functions $R_\sigma(q, t)$ indexed by permutations $\sigma \in S_n$. We should mention that Maple computations lead us to conjecture that the expression $E[F_{Z_{\text{end}}}(q, t) \,|\, Z_1 = s]$ is actually, for all $s \in \mu$, a polynomial in q, t with integer coefficients. It develops that the validity of this conjecture can be easily derived from the identity expressed by Theorem 2.2 of [9]. This given, it would be interesting to find a representation theoretical interpretation of this fact in terms of the action of S_{n-1} on the bigraded modules $\mathbf{H}_\mu$ studied in [6]. We hope to return to these questions in later work.

Formula (2.35) may yet be rewritten in a compacted form using certain constancy properties of the expression in (2.33). This follows from a q, t-analogue of another result of G-N-W. To state it we need some notation. Let μ be a partition with m corners, and let (a'_i, l'_i) for $i = 1 \cdots m$ be the co-arm and co-leg of the corners of μ in the left-to-right order. For any pair $i, j \in [1, m]$ set

$$A_{i,j} = \{ s \in \mu : a'_{i-1} < a'(s) \leqslant a'_i;\ l'_{j+1} < l'(s) \leqslant l'_j \}, \tag{2.36}$$

where for convenience we set $a'_0 = l_{m+1} = -1$. In words, A_{i_o, j_o} is the subset of μ consisting of the cells which have in their shadow only the corner cells with coordinates,

$$(a'_i + 1, l'_i + 1) \qquad \text{for} \quad i_o \leqslant i \leqslant j_o.$$

Now in complete analogy with a result (and its proof) given by G-N-W in [11] we have

THEOREM 2.5. *For any* $\nu \to \mu$ *and any* $s \ll = \nu/\mu$ *we have*

$$P[Z_{\text{end}} = \mu/\nu \mid Z_1 = s] = P[Z_{\text{end}} = \mu/\nu \mid Z_1 = r[s]] \times P[Z_{\text{end}} = \mu/\nu \mid Z_1 = c[s]]. \tag{2.37}$$

In particular, if μ *has* m *corners and the* $A_{i,j}$ (*for* $i, j \in \neq [1, m]$) *are as given in* (2.36), *then the conditional probability*

$$P[Z_{\text{end}} = \mu/\nu \mid Z_1 = s]$$

remains constant as s *varies in* $A_{i,j}$.

Proof. By specializing (2.28a) to the case when $s = r[s] = r$ we obtain

$$P[Z_{\text{end}} = \mu/\nu \mid Z_1 = r] = A(r) \prod_{r' \in R_{\mu/\nu}(r)} (q + A(r')). \tag{2.38}$$

Similarly, when $s = c[s] = c$ we get

$$P[Z_{\text{end}} = \mu/\nu \mid Z_1 = c] = B(c) \prod_{c' \in C_{\mu/\nu}(s)} (t + B(c')). \tag{2.39}$$

This given, the identity in (2.37) is simply another way of writing (2.28a). Now let again $r_1, r_2, ..., r_{a'}$ be the elements of $R_{\mu/\nu}$ in the left-to-right order and $c_1, c_2, ..., c_{l'}$ be the elements of $C_{\mu/\nu}$ as they they are read from bottom to top. Set

$$P_i = P[Z_{\text{end}} = \mu/\nu \mid Z_1 = r_i], \qquad Q_j = P[Z_{\text{end}} = \mu/\nu \mid Z_1 = c_j].$$

Note that if, for convenience, we set $A(r_i) = A_i$ and $B(c_j) = B_j$, then from (2.38) we get the recursion

$$P_i = \frac{A_i}{A_{i+1}} (q + A_{i+1}) P_{i+1},$$

or better

$$\frac{1}{A_i} P_i = \left(\frac{q}{A_{i+1}} + 1 \right) P_{i+1}.$$

Note that if $a(r_i) = a$ then $a(r_{i+1}) = a - 1$ and thus when $l(r_{i+1}) = l(r_i) = l$ this recursion reduces to

$$\frac{t^l - q^a}{t^l(1-q)} P_i = \left(\frac{t^l - q^{a-1}}{t^l(1-q)} q + 1 \right) P_{i+1},$$

which gives

$$P_i = P_{i+1}.$$

In other words,

$$l(r_i) = l(r_{i+1}) \rightarrow P[Z_{\text{end}} = \mu/\nu \,|\, Z_1 = r_i] = P[Z_{\text{end}} = \mu/\nu \,|\, Z_1 = r_{i+1}]. \tag{2.40}$$

Similarly, we show that

$$a(c_j) = a(c_{j+1}) \rightarrow P[Z_{\text{end}} = \mu/\nu \,|\, Z_1 = c_j] = P[Z_{\text{end}} = \mu/\nu \,|\, Z_1 = c_{j+1}]. \tag{2.41}$$

Since when s varies in a subset A_{ij} both $l(r[s])$ and $a(c[s])$ remain constant, our last assertion is an immediate consequence of (2.40), (2.41), and the factorization in (2.37).

3. FURTHER q, t-ANALOGUES

In their second paper [12], Green, Nijenhuis, and Wilf show that their hook walk mechanism can be used to give a probabilist proof of the so-called *upper recursion* for the number of standard tableaux. This is an identity due to A. Young [18], which is obtained by summing f_μ over partitions which immediately follow a fixed partition ν. More precisely, for a given $\nu \vdash n-1$ we have

$$nf_\nu = \sum_{\mu \leftarrow \nu} f_\mu. \tag{3.1}$$

This identity was used by Rutherford [17] to give a proof of Young's formula

$$n! = \sum_{\mu \vdash n} f_\mu^2. \tag{3.2}$$

We show in this section that the theory of Macdonald polynomials produces several q, t-analogues of (3.1) and (3.2). All this suggests that the q, t-hook walk mechanism should have an extension that yields proofs of these further identities.

Our first three q, t-analogues may be stated as follows:

PROPOSITION 3.1. *For every $\nu \vdash n-1$ we have*

$$\begin{aligned}
&\text{(a)}\quad 1=\sum_{\mu\leftarrow\nu} d_{\mu\nu}(q,t)\\
&\text{(b)}\quad 1=\sum_{\mu\leftarrow\nu} d_{\mu\nu}(q,t)\,T_{\mu/\nu}\\
&\text{(c)}\quad nF_\nu(q,t)=\sum_{\mu\leftarrow\nu} F_\mu(q,t)\,d_{\mu\nu}(q,t)
\end{aligned}\tag{3.3}$$

where the coefficients $d_{\mu\nu}(q,t)$ are as given in (1.16) *and $T_{\mu/\nu}=t^{r_\nu}q^{s_\nu}$ with r_ν, s_ν the co-leg and co-arm of the corner cell μ/ν.*

Proof. Plethystically substituting $1-u$ into both sides of (1.15) we obtain

$$(1-u)\,\tilde{H}_\nu[1-u;q,t]=\sum_{\mu\leftarrow\nu}\tilde{H}_\mu[1-u;q,t]\,d_{\mu\nu}(q,t).$$

Using (1.23) this may be rewritten as

$$(1-u)\prod_{s\in\nu}(1-ut^{l'_\nu(s)}q^{a'_\nu(s)})=\sum_{\mu\leftarrow\nu}\prod_{s\in\mu}(1-ut^{l'_\mu(s)}q^{a'_\mu(s)})\,d_{\mu\nu}(q,t).$$

Cancelling the common factor $\prod_{s\in\nu}(1-ut^{l'_\nu(s)}q^{a'_\nu(s)})$ yields

$$(1-u)=\sum_{\mu\leftarrow\nu}(1-uT_{\mu/\nu})\,d_{\mu\nu}(q,t),$$

from which we derive (3.3a) and (3.3b) by equating coefficients of u^0 and u^1. Note next that applying $\partial^n_{p_1}$ to both sides of (1.15) and using the relation

$$\partial^n_{p_1}\tilde{H}_\mu(x;q,t)=F_\mu(q,t)\tag{3.4}$$

we get

$$\partial^n_{p_1}(e_1(x)\,\tilde{H}_\nu(x;q,t))=\sum_{\mu\leftarrow\nu}F_\mu(q,t)\,d_{\mu\nu}(q,t).\tag{3.5}$$

However, Leibnitz formula and (3.4) (with μ replaced by ν) yield that

$$\partial^n_{p_1}(e_1(x)\,\tilde{H}_\nu(x;q,t))=n(\partial_{p_1}e_1(x))(\partial^{n-1}_{p_1}\tilde{H}_\nu(x;q,t))=nF_\nu(q,t)).$$

Combining this with (3.5) gives (3.3c) as desired.

It develops that (3.3a), (3.3b), and (3.3c) are but three different variants of the upper recursion. To see this note that, by dividing both sides of (3.1) by nf_ν, the resulting identity may be rewritten in the form

$$1 = \sum_{\mu \leftarrow \nu} h_\nu / h_\mu. \tag{3.6}$$

On the other hand, from the definition (1.16), we can deduce (as we did for $c_{\mu\nu}$) that making the replacement $t \to 1/q$ and then letting $q = 1$ reduces $d_{\mu\nu}$ to the ratio h_ν / h_μ. Thus we see that the same replacements reduce (3.3a) and (3.3b) to (3.6) and (3.1c) to

$$n(n-1)! = \sum_{\mu \to \nu} n! \frac{h_\nu}{h_\mu},$$

which is yet another way of writing (3.6).

The same reasoning shows that the following identities are variants of (3.2).

PROPOSITION 3.2.

$$\begin{aligned}
&\text{(a)} \quad \frac{1}{(1-t)^n (1-q)^n} = \sum_{\mu \vdash n} \frac{F_\mu(q,t)}{\tilde{h}_\mu(q,t)\, \tilde{h}'_\mu(q,t)} \\
&\text{(b)} \quad \frac{n}{(1-t)^n (1-q)^n} = \sum_{\mu \vdash n} \frac{F_\mu(q,t)\, B_\mu(q,t)}{\tilde{h}_\mu(q,t)\, \tilde{h}'_\mu(q,t)} \\
&\text{(c)} \quad \frac{n!}{(1-t)^n (1-q)^n} = \sum_{\mu \vdash n} \frac{F^2_\mu(q,t)}{\tilde{h}_\mu(q,t)\, \tilde{h}'_\mu(q,t)}.
\end{aligned} \tag{3.7}$$

Proof. Note that the power sum expansion of $e_n[X/((1-t)(1-q))]$ can be written as

$$e_n\left[\frac{X}{(1-t)(1-q)}\right] = \frac{1}{n!} \frac{p_1^n(x)}{(1-t)^n (1-q)^n} + R,$$

where the remainder R contains no terms in which $p_1(x)$ is raised to the nth power. Thus (3.7a) can be obtained by applying $\partial^n_{p_1}$ to both sides of (1.14).

Note next that if we use (1.14) and (1.20) we can rewrite (1.21) in the form

$$e_1(x)\, e_{n-1}\left[\frac{X}{(1-t)(1-q)}\right] = \sum_{\mu \vdash n} \frac{(1-t)(1-q)\, B_\mu(q,t)\, \tilde{H}_\mu(x;q,t)}{\tilde{h}_\mu(q,t)\, \tilde{h}'_\mu(q,t)}.$$

Applying $\partial_{p_1}^n$ to both sides and using the Leibnitz rule on the left-hand side gives (3.7b) upon division by $(1-t)(1-q)$.

Finally, (3.7c) is obtained by applying the operator $\partial_{p_1(x)}^n \partial_{p_1(y)}^n$ to (1.11) and using the fact that the only term which contributes to the left-hand side of the resulting identity is

$$\frac{1}{n!} \frac{p_1^n(x)\, p_1^n(y)}{(1-t)^n\,(1-q)^n}.$$

Remark 3.1. We should point out that Rutherford in [17] derives (3.1) by a multiple use of the *lower recursion*

$$f_\mu = \sum_{\nu \to \mu} f_\nu. \tag{3.8}$$

This done, he derived (3.2) from (3.1) and (3.8) by an induction argument based on the following steps:

$$\begin{aligned}(n+1)\,n! &= \sum_{\mu \vdash n} f_\mu (n+1)\, f_\mu = \sum_{\mu \vdash n} f_\mu \sum_{\lambda \leftarrow \mu} f\lambda \\ &= \sum_{\lambda \vdash n+1} f_\lambda \sum_{\mu \to \lambda} f_\mu = \sum_{\lambda \vdash n+1} f_\lambda^2.\end{aligned} \tag{3.9}$$

Now it develops that we have several q, t-analogues of this derivation. For instance, (3.7c) follows from (3.3c), (1.18), and (I.10), and by induction according to the following sequence of steps:

$$\begin{aligned}\frac{(n+1)\,n!}{(1-t)^{n+1}\,(1-q)^{n+1}} &= \sum_{\mu \vdash n} \frac{F_\mu(q,t)}{\tilde{h}_\mu(q,t)\,\tilde{h}'_\mu(q,t)} \frac{(n+1)\,F_\mu(q,t)}{(1-t)(1-q)} \\ &= \sum_{\mu \vdash n} \frac{F_\mu(q,t)}{\tilde{h}_\mu(q,t)\,\tilde{h}'_\mu(q,t)} \sum_{\lambda \leftarrow \mu} \frac{F_\lambda(q,t)\, d_{\lambda\mu}(q,t)}{(1-t)(1-q)} \\ &= \sum_{\lambda \vdash n+1} \frac{F_\lambda(q,t)}{\tilde{h}_\lambda(q,t)\,\tilde{h}'_\lambda(q,t)} \sum_{\mu \to \lambda} c_{\lambda\mu}(q,t)\, F_\mu(q,t) \\ &= \sum_{\lambda \vdash n+1} \frac{F_\lambda^2(q,t)}{\tilde{h}_\lambda(q,t)\,\tilde{h}'_\lambda(q,t)}.\end{aligned} \tag{3.10}$$

Curiously, if we apply the same reasoning to (3.7a) or (3.7b) we are led to a whole family of identities interpolating between (3.7a) and (3.7c). To be precise, let $B_\mu^{(k)}(q, t)$ (for $k \geqslant 1$) be the rational(?) function defined by the recursion

$$B_\mu^{(k)}(q, t) = \sum_{\nu \to \mu} c_{\mu\nu}(q, t)\, B_\nu^{(k-1)}(q, t) \qquad (\text{with } B_\mu^{(0)}(q, t) = 1). \tag{3.11}$$

Then, starting from (3.7a), after k iterations of the same sequence of steps we carried out in (3.10), we end up with the following further variant of the upper recursion

$$\frac{n(n-1)\cdots(n-k+1)}{(1-t)^n\,(1-q)^n} = \sum_{\mu \vdash n} \frac{F_\mu(q, t)\, B_\mu^{(k)}(q, t)}{\tilde{h}_\mu(q, t)\, \tilde{h}'_\mu(q, t)} \qquad (\text{for } k \geqslant 1). \tag{3.12}$$

Note that we have

$$B_\mu^{(1)}(q, t) = B_\mu(q, t) \qquad \text{and} \qquad B_\mu^{(n-1)}(q, t) = F_\mu(q, t).$$

Thus (3.12) reduces to (3.7b) for $k = 1$ and to (3.7c) for $k = n - 1$.

Remark 3.2. Rutherford's proof of the upper recursion may be viewed as a precursor of the Robinson–Schensted correspondence. Indeed, it is precisely by *bijectivating* Rutherford's argument that MacLarnan in [15] was led to the construction of his several variations of the correspondence. This given we get the feeling, especially from the steps in (3.10), that the solution of some of the combinatorial problems concerning the conjectured Hilbert series $F_\mu(q, t)$ as well as the coefficients $\tilde{K}_{\lambda\mu}(q, t)$ may depend on the discovery of a μ-depending or μ-weighted form of Jeu de Taquin.

Another problem which is suggested by these q, t-analogies is the construction of a μ-dependent version of the bijective proof of the hook formula given by Pak and Stoyanovskii in [16]. Their bijection would then be the special case $\mu = 1^n$. The desired μ-dependent bijection should combinatorially unravel the rationality of the recursion

$$F_\mu(q, t) = \sum_{\mu \to \mu} c_{\mu, \nu}(q, t)\, F_\nu(q, t),$$

which as we have seen is an amalgamated form of the hook formula.

It is interesting to see what becomes of our q, t-hook walk under the specializations $q = 0$ and $t = 1$. It develops that the identities we can derive from it tie in very well with the representation theoretical results obtained in [7] and [6].

For convenience let $P[\uparrow_s^j; q, t]$ denote the probability that the hook walk takes a North step from a cell s to a cell j rows above and likewise let $P[{}_s\!\rightarrow_i; q, t]$ denote the probability of an East step from s to a cell i columns to the right. We recall that in the general case we have

$$\text{(a)}\qquad P[\uparrow_s^j; q, t] = \frac{t^{j-1} q^{a_\mu(s)}(t-1)}{t^{l_\mu(s)} - q^{a_\mu(s)}},$$

$$\text{(b)}\quad P[{}_s\!\rightarrow_j; q, t] = \frac{q^{i-1} t^{l_\mu(s)}(1-q)}{t^{l_\mu(s)} - q^{a_\mu(s)}}. \tag{3.13}$$

For a given cell $s \in \mu$ let $c(s)$ denote the corner cell of μ that is in the shadow of s and has the least co-leg and let $v(s)$ be the partition obtained by removing $c(s)$ fom μ. This given we have

PROPOSITION 3.3. *For $q=0$ as well as for $t=1$ the hook walk starting from any cell s proceeds by East steps straight to the East boundary of μ (unless it is already at the start) then climbs by steps to the corner cell $c(s)$.*

Proof. Note first that in the general case when the walk reaches a cell s on the East boundary of μ it must climb with North steps with probability given by (3.13a) for $a_\mu(s)=0$. That is,

$$P[\uparrow_s^j; q, t] = \frac{t^{j-1}(t-1)}{t^{l_\mu(s)} - 1}.$$

This will be so even when $q=0$. When $t=1$ the probability of a North step reduces to

$$\lim_{t\rightarrow 1} \frac{t^{j-1}(t-1)}{t^{l_\mu(s)} - 1} = \frac{1}{l_\mu(s)}.$$

On the other hand, when the walk is at a cell s with $a_\mu(s) > 0$ from (3.13a) we get in either case

$$\lim_{q\rightarrow 0} P[\uparrow_s^j; q, t] = \lim_{t\rightarrow 1} P[\uparrow_s^j; q, t] = 0,$$

while (3.13b) for $a_\mu(s) > 0$ gives

$$\lim_{q\rightarrow 0} P[{}_s\!\rightarrow_j; q, t] = \begin{cases} 0 & \text{if } i>1 \\ 1 & \text{if } i=1 \end{cases} \quad \text{and} \quad \lim_{t\rightarrow 1} P[{}_s\!\rightarrow_j; q, t] = \frac{q^{i-1}(1-q)}{1-q^{a_\mu(s)}}.$$

Thus in either case the walk moves only by East steps whenever it can and when it can no more it goes by North steps. This establishes our assertion.

Let now μ be a k-corner partition and let $c_i = (\alpha_i, \beta_i)$ for $i = 1, .., k$ be its corner cells listed according to decreasing co-legs (that is, from left to right in the french way of depicting partitions). Let $\nu^{(i)}$ denote the partition obtained by removing c_i from μ. Note that by our previous notation we can also represent c_i by $\mu/\nu^{(i)}$. If ν is a predecessor of μ and $\nu=\nu^{(i)}$ then we let $S[\mu,\nu]$ denote the collection of cells of μ which are covered by c_i and (if $i<k$) not covered by c_{i+1}. Set

$$B_{\mu\nu}(q,t)=\sum_{s\in S[\mu,\nu]} q^{a'_\mu(s)}l^{l'_\mu(s)}. \tag{3.14}$$

Finally, let α_ν and β_ν respectively denote the co-arm and co-leg of the cell μ/ν and let γ_ν be the length of the vertical step of the boundary of μ that lies directly below the corner cell μ/ν. Note that when $\nu=\nu^{(i)}$ we have $\alpha_\nu=\alpha_i-1$ $\beta_\nu=\beta_i-1$ and

$$\gamma_i=\begin{cases}\beta_i-\beta_{i+1} & \text{if } i<k\\ \beta_k & \text{if } i=k\end{cases}.$$

This given, Proposition 3.3 has the following immediate corollary.

THEOREM 3.1. *In the limiting cases $q=0$ and $t=1$ the conjectured Hilbert series $F_\mu(q,t)$ reduces to the polynomials determined by the following recursions*

$$\begin{aligned}&\text{(a)}\quad F_\mu(0,t)=\sum_{\nu\to\mu}F_\nu(0,t)(t^{\beta_\nu}+t^{\beta_\nu-1}+\cdots+t^{\beta_\nu-\gamma_\nu+1})\\ &\text{(b)}\quad F_\mu(q,1)=\sum_{\nu\to\mu}F_\nu(q,1)\,\gamma_\nu(1+q+q^2+\cdots+q^{\alpha_\nu-1}).\end{aligned} \tag{3.15}$$

Proof. From Proposition 3.3 we derive that starting from a cell s the hook walk with probability 1, in either case, terminates at the cell $c(s)$. Thus the conditional expectation $E[F_{Z_{\text{end}}}(q,t)\,|\,Z_1=s]$ occurring in (2.35) reduces to $F_{\nu(s)}(0,t)$ or $F_{\nu(s)}(q,1)$, as the case may be. Thus from (2.35) we derive that

$$\begin{aligned}&\text{(a)}\quad F_\mu(0,t)=\sum_{\nu\to\mu}F_\nu(0,t)\,B_{\mu\nu}[0,t]\\ &\text{(b)}\quad F_\mu(q,1)=\sum_{\nu\to\mu}F_\nu(q,1)\,B_{\mu\nu}[q,1].\end{aligned} \tag{3.16}$$

Now it is easy to see from the definition (3.14) that

$$\begin{aligned}&\text{(a)}\quad B_{\mu\nu}[0,t]=(t^{\beta_\nu}+t^{\beta_\nu-1}+\cdots+t^{\beta_\nu-\gamma_\nu+1}),\\ &\text{(b)}\quad B_{\mu\nu}[q,1]=\gamma_\nu(1+q+q^2+\cdots+q^{\alpha_\nu-1}).\end{aligned}$$

Substituting this into (3.16) gives (3.15) as desired.

It is interesting to see how these identities reflect the results in [7] and [6] concerning the S_n-modules $\mathbf{R}_\mu$ and $\mathbf{H}_\mu$ respectively studied there. To begin with it was shown in [6] that the y-degree 0 portion of $\mathbf{H}_\mu$ is isomorphic to $\mathbf{R}_\mu$, and moreover we show there that the graded Frobenius characteristic of that portion of $\mathbf{H}_\mu$ is indeed given by the polymial $\tilde{H}_\mu(x; , 0, 1)$. In particular we can deduce from all this that $F_\mu(0, t)$ is in fact the Hilbert polynomial of $\mathbf{R}_\mu$. The curious thing is that the hook walk for $t=1$ is identical with the algorithm shown in [7][3] to yield the standard monomial basis of the ring $\mathbf{R}_\mu$ considered as a quotient of the polynomial ring $\mathbf{Q}[x_1, ..., x_n]$ by a certain canonical ideal $\mathscr{J}_\mu$. In [7] the latter algorithm results from studying what happens to the S_n-module $\mathbf{R}_\mu$ under restriction to S_{n-1}. This brings us to the problem of understanding in which way *restriction* in [7] corresponds to the specific *hook walk* we encounter here. We believe that such an understanding might lead to unraveling how restricting $\mathbf{H}_\mu$ to S_{n-1} is related to our general q, t-hook walk and ultimately to the recursion in (2.35).

The situation for $t=1$ is equally intriguing. Using some of the identities proved by Macdonald in [13], it is shown in [4][4] that for $\mu=(\mu_1, \mu_2, ..., \mu_k)$ we have

$$\tilde{H}_\mu(X; q, 1) = \prod_{i=1}^{k} \left((q)_{\mu_i} h_{\mu_i} \left[\frac{X}{(1-q)} \right] \right), \tag{3.17}$$

where for an integer $m \geqslant 0$ we set $(q)_m = (1-q)(1-q^2) \cdots (1-q^m)$. Applying the operator $\partial_{p_1}^n$ to both sides of this identity and using the Leibnitz formula we obtain that

$$F_\mu(q, 1) = \frac{n!}{\mu_1! \, \mu_2! \cdots \mu_k!} [\mu_1]_q! \, [\mu_2]_q! \cdots [\mu_k]_q!, \tag{3.18}$$

where as customary, for an integer $m \geqslant 0$, we set $[m]_q! = \prod_{s=1}^{m} (1+q+ \cdots + q^{s-1})$. We should note that in [6] we prove that the right-hand side of (3.17) gives the y-graded Frobenius characteristic of our module $\mathbf{H}_\mu$. Thus in particular we do know that the right-hand side of (3.18) is the Hilbert polynomial of $\mathbf{H}_\mu$ when the x-grading of $\mathbf{H}_\mu$ is not taken into account. And of course, in perfect agreement with all of this, we can easily verify that the polynomial in (3.18) satisfies the recursion in (3.16b). This brings us again to the conclusion that, somehow, restriction of $\mathbf{H}_\mu$ to S_{n-1} (as a y-graded S_n-module) must be performable by some algebraic mechanism that closely reflects the type of hook walk we obtain here for $t=1$. In conclusion, we

[3] See Eq. (1.2) there.

[4] See Chapter IV, Section 3, Theorem 3.9.

see that there are several promising avenues to pursue in trying to extend or sharpen the results obtained here, most particularly in the direction of proving the polynomiality of the conjectured Hilbert series $F_{\mu}(q, t)$. We hope to return to this and related questions in some later work.

REFERENCES

1. Y. M. Chen, A. M. Garsia, and J. Remmel, Algorithms for plethysm, *in* "Combinatorics and Algebra," Contemporary Mathematics, Vol. 34 (Curtis Greene, Ed.), pp. 109–153, American Math. Society, Providence, RI, 1984.
2. A. M. Garsia and M. Haiman, A graded representation module for Macdonald's polynomials, *Proc. Natl. Acad. Sci. U.S.A.* **90** (1993), 3607–3610.
3. A. M. Garsia and M. Haiman, Factorizations of Pieri rules for Macdonald polynomials, *Discrete Math.* **139** (1995), 219–256.
4. A. M. Garsia and M. Haiman, "Orbit Harmonics and Graded Representations," research monograph (S. Brlek, Ed.), Lab. de Comb. et Informatique Mathématique, Univ. Québec à Montréal, to appear.
5. A. Garsia and M. Haiman, A remarkable q, t-catalan sequence and q-Lagrange inversion, *J. Algebraic Combin.* **5** (1996), 191–244.
6. A. Garsia and M. Haiman, Some bigraded S_n-modules and the Macdonald q, t-Kostka coefficients, *Electron. J. Algebraic Combin.* **3**, No. 2 (1996), 561–620.
7. A. M. Garsia and C. Procesi, On certain graded S_n-modules and the q-Kostka polynomials, *Adv. in Math.* **94** (1992), 82–138.
8. A. Garsia, Orthogonality of Milne's polynomials and raising operators, *Discrete Math.* **99** (1992), 247–264.
9. A. Garsia and G. Tesler, Plethystic formulas for Macdonald q, t-Kostka coefficients, *Adv. in Math.* **123**, No. 2 (1996), 144–222.
10. A. Garsia and J. Remmel, Plethystic formulas and positivity for q, t-Kostka coefficients, *in* "Rotafest Volume," accepted for publication.
11. C. Greene, A. Nijenhuis, and H. Wilf, A probabilistic proof of a formula for the number of Young tableaux of a given shape, *Adv. Math.* **31**, No. 1 (1979), 104–109.
12. C. Greene, A. Nijenhuis, and H. Wilf, Another probabilistic method in the theory of Young tableaux, *J. Combin. Theory*, *Ser. A* **37** (1984), 127–135.
13. I. G. Macdonald, A new class of symmetric functions, *in* "Actes du 20^{e} Séminaire Lotharingien," Publ. I.R.M.A. Strasbourg, pp. 131–171.
14. I. G. Macdonald, "Symmetric Functions and Hall Polynomials," 2nd ed., Clarendon, Oxford, 1995.
15. T. J. McLarnan, "Tableaux Recursions and Symmetric Schensted Correspondences for Ordinary, Shifted, and Oscillating Tableaux," Ph.D. thesis, University of California, San Diego, 1985.
16. I. M. Pak and A. V. Stoyanovskii, A bijective proof of the Hook-length formula and its analogs, *Funct. Anal. Appl.* **26**, No. 3 (1992), 216–218.
17. D. E. Rutherford, "Substitutional Analysis," pp. 25–26, Edinburgh University Press, Edinburgh, 1948.
18. A. Young, On quantitative substitutional analysis, III, *Proc. London Math. Soc.* **28** (1928), 255–292.