

POITOU–TATE WITHOUT RESTRICTIONS ON THE ORDER

KEŠTUTIS ČESNAVIČIUS

ABSTRACT. The Poitou–Tate sequence relates Galois cohomology with restricted ramification of a finite Galois module M over a global field to that of the dual module under the assumption that $\#M$ is a unit away from the allowed ramification set. We remove the assumption on $\#M$ by proving a generalization that allows arbitrary “ramification sets” that contain the archimedean places. We also prove that restricted products of local cohomologies that appear in the Poitou–Tate sequence may be identified with derived functor cohomology of an adèle ring. In our proof of the generalized sequence we adopt this derived functor point of view and exploit properties of a natural topology carried by cohomology of the adèles.

1. INTRODUCTION

1.1. Poitou–Tate for Galois cohomology with restricted ramification. One of the core results of global arithmetic duality is the Poitou–Tate exact sequence of Galois cohomology groups:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & H^0(\mathrm{Gal}(K_\Sigma/K), M) & \longrightarrow & \prod'_{v \in \Sigma} H^0(K_v, M) & \longrightarrow & H^2(\mathrm{Gal}(K_\Sigma/K), M^D)^* \\
 & & & & \longleftarrow & & \\
 & & \longrightarrow & H^1(\mathrm{Gal}(K_\Sigma/K), M) & \longrightarrow & \prod'_{v \in \Sigma} H^1(K_v, M) & \longrightarrow & H^1(\mathrm{Gal}(K_\Sigma/K), M^D)^* & (\star) \\
 & & & & \longleftarrow & & \\
 & & \longrightarrow & H^2(\mathrm{Gal}(K_\Sigma/K), M) & \longrightarrow & \prod'_{v \in \Sigma} H^2(K_v, M) & \longrightarrow & H^0(\mathrm{Gal}(K_\Sigma/K), M^D)^* & \longrightarrow 0,
 \end{array}$$

where K is a global field, Σ is a possibly infinite nonempty set of places of K containing the archimedean places, $\mathrm{Gal}(K_\Sigma/K)$ is the Galois group of the maximal unramified outside Σ subextension K_Σ/K of a separable closure K^s/K (a choice of which is fixed), M is a finite discrete $\mathrm{Gal}(K_\Sigma/K)$ -module whose order is a unit at all places not in Σ , the restricted products $\prod'_{v \in \Sigma}$ are with respect to the unramified subgroups, and $M^D := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}(1))$ is the dual of M .

The Poitou–Tate sequence (\star) is proved in [Mil06, I.4.10] and is useful in many contexts. However, its requirement that $\#M$ be a unit outside Σ may be restrictive. The goal of this paper is to dispose of this requirement by proving a generalization of (\star) and to offer another, possibly more conceptual, interpretation of the restricted products appearing in (\star) . Our generalization allows new Σ even in the number field case, and it also gives a different proof of the exactness of (\star) .

1.2. Poitou–Tate without restrictions on $\#M$. As in §1.1, let Σ be a set of places containing the archimedean places. For simplicity, suppose that Σ also contains a finite place (see Theorem 5.1

Date: September 7, 2015.

2010 Mathematics Subject Classification. Primary 11R99; Secondary 11R34, 11R56.

Key words and phrases. Arithmetic duality, global field, Poitou–Tate, Cassels–Poitou–Tate, adèles.

Topological aspects, and hence also our work in §3, are crucial for the proof of the exactness of (★). However, a reader who is willing to accept the necessary topological properties of $H^n(\mathbb{A}_K^{\epsilon\Sigma}, -)$ as known may skip §3 and instead consult §4.1, which gathers together those conclusions of §§2–3 that are needed in §§4–6. Likewise, a reader who is interested in the proof of (★) but would prefer to take $H^n(\mathbb{A}_K^{\epsilon\Sigma}, -)$ to mean $\prod'_{v \in \Sigma} H^n(K_v, -)$ may skip §§2–3 entirely and start with §4.

In §4 we prove arithmetic duality results that are key for the proof of (★); §4 is also the place where the input from [Čes14a, §2] mentioned in Remark 1.4 enters the picture (through Lemma 4.4). In §5 we show that the sequence (★) is a quick consequence of the results of §4. In §6 we include a proof of the Cassels–Poitou–Tate sequence in the generality permitted by (★). This sequence with finite Σ has played instrumental roles in the proofs of a number of results on the arithmetic of abelian varieties, so it seems worthwhile to record in §6 its generalization that allows arbitrary Σ . We conclude the paper by showing in Appendix A that (★) indeed generalizes (★).

1.7. Notation. The following notation is in place throughout the paper: K is a global field; if $\text{char } K = 0$, then S is the spectrum of the ring of integers of K ; if $\text{char } K > 0$, then S is the proper smooth curve over a finite field such that the function field of S is K ; a place of K is denoted by v , and K_v is the resulting completion; if v is nonarchimedean, then \mathcal{O}_v and \mathbb{F}_v are the ring of integers and the residue field of K_v . We fix a set Σ of places of K that contains the archimedean places. For a finite subset $\Sigma_0 \subset \Sigma$ that contains the archimedean places, we set

$$\mathbb{A}_{K, \Sigma_0}^{\epsilon\Sigma} := \prod_{v \in \Sigma_0} K_v \times \prod_{v \in \Sigma \setminus \Sigma_0} \mathcal{O}_v,$$

to be interpreted as the zero ring if $\Sigma = \emptyset$. We let $\mathbb{A}_K^{\epsilon\Sigma}$ denote² the ring of “ Σ -adeles”, i.e.,

$$\mathbb{A}_K^{\epsilon\Sigma} := \varinjlim_{\Sigma_0 \subset \Sigma, \#\Sigma_0 < \infty} \mathbb{A}_{K, \Sigma_0}^{\epsilon\Sigma}.$$

Thus, if Σ is the set of all places, then $\mathbb{A}_K^{\epsilon\Sigma}$ is the adeles \mathbb{A}_K , and if Σ is finite, then $\mathbb{A}_K^{\epsilon\Sigma} = \prod_{v \in \Sigma} K_v$.

We identify the closed points of S with the nonarchimedean v . For an open $U \subset S$, the notation $v \notin U$ signifies that v does not correspond to a point of U (and hence could be archimedean). We denote by $U \cap \Sigma$ the set of those places in Σ that correspond to a point of U . We write $S - \Sigma$ for the Dedekind scheme that is the localization of S away from Σ , i.e., $S - \Sigma = \varprojlim U$ where U ranges over the nonempty opens of S with $S \setminus U \subset \Sigma$. We say that such a U *contains* $S - \Sigma$.

1.8. Conventions. For a field F , a choice of its algebraic closure \overline{F} is made implicitly. Likewise implicit are the choices of embeddings $\overline{K} \hookrightarrow \overline{K}_v$. All the cohomology other than profinite group cohomology is computed in the fppf topology, but we implicitly use [Gro68, 11.7 1°] (or [Čes15b, B.17] in the algebraic space case) to make identifications with étale cohomology when the coefficients are a smooth group scheme (or a smooth algebraic space); further identifications with Galois cohomology are likewise implicit. Derived functor H^n is taken to vanish for $n < 0$. The neutral class of a nonabelian H^1 is denoted by $*$.

All actions and homogeneous spaces (and hence also torsors) are right instead of left unless noted otherwise. Algebraic spaces are not assumed to be quasi-separated. We implicitly use [SP, 04SK], i.e., the fppf local nature of being an algebraic space, to ensure that sheaves at hand, e.g., torsors under group algebraic spaces, are representable by algebraic spaces. For a scheme T , a T -group algebraic space G , and an fppf cover T'/T , the set of isomorphism classes of G -torsors trivialized by T' is denoted by $H^1(T'/T, G)$; we implicitly use the cocycle point of view to identify $H^1(T'/T, G)$ with a subquotient of $G(T' \times_T T')$ under an action of $G(T')$.

²We fear that the shorter notation \mathbb{A}_K^Σ is confusing: should \mathbb{A}_K^Σ stand for $\mathbb{A}_K^{\epsilon\Sigma}$ or for the similarly constructed $\mathbb{A}_K^{\# \Sigma}$?

In this paper ‘locally compact’ means that every point has a compact neighborhood. Neither compact nor locally compact spaces are assumed to be Hausdorff. The Pontryagin dual of a locally compact Hausdorff abelian topological group A is denoted by A^* and is defined by $A^* := \mathrm{Hom}_{\mathrm{cont}}(A, \mathbb{R}/\mathbb{Z})$ (we will always discuss Pontryagin duality in the context of torsion groups, or even n -torsion groups for a fixed $n \in \mathbb{Z}_{\geq 1}$, so we could equivalently use $A^* := \mathrm{Hom}_{\mathrm{cont}}(A, \mathbb{Q}/\mathbb{Z})$ as our definition). Topology on cohomology of local fields (resp., on cohomology of adèle rings) is always the one defined in [Čes15b, 3.1–3.2] (resp., in §3.1).

Acknowledgements. I thank Brendan Creutz for questions and suggestions that led to Poitou–Tate and Cassels–Poitou–Tate sequences proved in this paper. I thank the referees for helpful comments and suggestions. I thank James Milne for pointing out Remark 2.15 and for sending me a scan of [Azu66]. I thank Brendan Creutz, Cristian González-Avilés, Bjorn Poonen, and Yunqing Tang for helpful conversations or correspondence regarding the material of this paper. Some of the results presented here were obtained while the author was a Research Fellow at the Miller Institute for Basic Research in Science at the University of California Berkeley. I thank the Miller Institute and UC Berkeley for excellent conditions for research.

2. COHOMOLOGY OF THE ADELES AS A RESTRICTED PRODUCT

The goal of this section is to prove in Theorem 2.20 that, under suitable hypotheses on the coefficients, the cohomology of the adèles $\mathbb{A}_K^{\epsilon\Sigma}$ decomposes as a restricted product of local cohomologies. Such a decomposition is most interesting when the coefficients are a K -group scheme G , for instance, in §§4–6 we will use it for a commutative finite G (see also Corollaries 2.6 and 2.14 for examples of concrete conclusions with $G = \mathrm{GL}_n$ or $G = \mathbb{G}_m$).

The proof begins by reducing to the case of a smooth G and then proceeds by induction on the degree of the cohomology in question, with some of the key steps happening in the proof of Lemma 2.12. In these key steps we apply the inductive hypothesis to coefficients that need not come from K by base change. More precisely, the relevant coefficients are quotients of restrictions of scalars along R'/R , where R is an infinite product of rings of integers of nonarchimedean completions of K and R' is an infinite product of rings of integers of unramified extensions of those completions.

In spite of not being base changes from K , these restrictions of scalars are nevertheless schemes due to the quasi-projectivity of G . The inductive argument therefore requires applying some of the intermediate results to a priori arbitrary $\mathbb{A}_K^{\epsilon\Sigma}$ -group schemes \mathcal{G} of finite presentation. However, allowing such \mathcal{G} leads to further complications: due to the loss of quasi-projectivity, we no longer know how to argue the representability by schemes of the new restrictions of scalars and of their quotients. (Restricting to affine \mathcal{G} would resolve some of these difficulties but would exclude the possibility of G being a semiabelian variety.) In contrast, the representability by algebraic spaces holds, but comes at the expense of also having to allow algebraic spaces in the intermediate results.

In conclusion, for purely technical reasons we formulate our results for coefficients that are $\mathbb{A}_K^{\epsilon\Sigma}$ -group algebraic spaces. This generality makes the overall proof work, but does not lead to new cases of interest because quasi-separated K -group algebraic spaces are automatically schemes (as is recalled in Lemma 2.8).

To avoid cluttering the subsequent arguments with repetitive citations, we begin by recording the following two lemmas that concern limit arguments for algebraic spaces. We recall that for a scheme T and a T -group algebraic space \mathcal{G} , viewed as the sheaf of groups on the big fppf site of T , the notation $H^n(T, \mathcal{G})$ stands for $\mathcal{G}(T)$ if $n = 0$, for the pointed set of isomorphism classes of \mathcal{G} -torsor fppf sheaves (that are necessarily T -algebraic spaces) if $n = 1$, and for the derived functor fppf

cohomology with coefficients \mathcal{G} if $n \geq 1$ and \mathcal{G} is commutative (in the commutative case, the two possible meanings of $H^1(T, \mathcal{G})$ agree by [Gir71, III.3.5.4]).

Lemma 2.1. *Let I be a directed partially ordered set, $(T_i)_{i \in I}$ an inverse system of quasi-compact and quasi-separated schemes, $i_0 \in I$, and \mathcal{G}_{i_0} a T_{i_0} -group algebraic space that is locally of finite presentation. Suppose that the maps $T_{i'} \rightarrow T_i$ are affine, set $T := \varprojlim T_i$ (which is a scheme by [EGA IV₃, 8.2.3]), and set $\mathcal{G} := (\mathcal{G}_{i_0})_T$ and $\mathcal{G}_i := (\mathcal{G}_{i_0})_{T_i}$ for $i \geq i_0$. The pullback maps induce isomorphisms*

$$\varinjlim_{i \geq i_0} H^n(T_i, \mathcal{G}_i) \cong H^n(T, \mathcal{G})$$

for $n \leq 1$, and also for $n \geq 2$ if \mathcal{G}_{i_0} is commutative.

Proof. We treat the case $n = 0$, the case $n = 1$, and the case $n \geq 2$ in (0), (1), and (2), respectively.

- (0) In the case of an affine T_{i_0} , the claim is a special case of the functor of points criterion [SP, 04AK] for being locally of finite presentation. To reduce the general case to the affine one, we cover T_{i_0} by finitely many affine opens, cover their pairwise intersections by finitely many further affine opens, use the description of T provided by [EGA IV₃, 8.2.2], and use the fact that filtered direct limits commute with finite inverse limits in the category of sets.
- (1) For a faithfully flat morphism $T'_{i_0} \rightarrow T_{i_0}$ of finite presentation and $i \geq i_0$, we set

$$T'_i := (T'_{i_0})_{T_i} \quad \text{and} \quad T' := (T'_{i_0})_T.$$

By the $n = 0$ case applied to $(T'_i)_{i \in I}$, to $(T'_i \times_{T_i} T'_i)_{i \in I}$, and to $(T'_i \times_{T_i} T'_i \times_{T_i} T'_i)_{i \in I}$,

$$\varinjlim_{i \geq i_0} H^1(T'_i/T_i, \mathcal{G}_i) \cong H^1(T'/T, \mathcal{G}), \quad (2.1.1)$$

where we have again used the fact that taking filtered direct limits is exact. Since each T_i (resp., T) is quasi-compact and every torsor under \mathcal{G}_i (resp., \mathcal{G}) is an algebraic space that is faithfully flat and locally of finite presentation over T_i (resp., T), every such torsor is trivialized by a single scheme morphism that is faithfully flat and of finite presentation. The claim therefore results by varying the i_0 and the T'_{i_0} in (2.1.1).

- (2) The proof of [SGA 4_{II}, VII, 5.9] based on topos-theoretic generalities of [SGA 4_{II}, VI] and written for étale cohomology with scheme coefficients continues to work for fppf cohomology with algebraic space coefficients (reasoning as in the $n = 0$ case for the limit formalism). \square

Lemma 2.2. *In the setup of Lemma 2.1, let X_{i_0} be a T_{i_0} -algebraic space of finite presentation. Set*

$$X := (X_{i_0})_T \quad \text{and} \quad X_i := (X_{i_0})_{T_i} \quad \text{for } i \geq i_0.$$

If $X \rightarrow T$ is separated (resp., is proper, resp., is flat, resp., is smooth, resp., has geometrically connected fibers), then the same holds for $X_i \rightarrow T_i$ for large enough i .

Proof. For all the properties except fibral geometric connectedness, the claim is a special case of [Ryd15, B.3]. For fibral geometric connectedness, we apply [Con12b, 5.1] and [EGA IV₃, 8.3.4]. \square

We now turn to the setup needed for the sought restricted product interpretations of cohomology.

2.3. The comparison maps. Let \mathcal{G} be an $\mathbb{A}_K^{\varepsilon \Sigma}$ -group algebraic space of finite presentation. Limit arguments supplied by [SP, 07SK] descend \mathcal{G} to a (commutative if so is \mathcal{G}) $\mathbb{A}_{K, \Sigma_0}^{\varepsilon \Sigma}$ -group algebraic space \mathcal{G}_{Σ_0} of finite presentation for some finite subset $\Sigma_0 \subset \Sigma$ containing all the infinite places and prove that \mathcal{G}_{Σ_0} is unique up to base change to $\mathbb{A}_{K, \Sigma'_0}^{\varepsilon \Sigma}$ with $\Sigma_0 \subset \Sigma'_0 \subset \Sigma$ and $\#\Sigma'_0 < \infty$ and up to isomorphism.

The main example is $\mathcal{G} = G_{\mathbb{A}_K^\Sigma}$ for a K -group scheme G of finite type. For such \mathcal{G} we may choose a U -model of G of finite presentation for a nonempty open $U \subset S$, let Σ_0 contain all the places of Σ outside U , and let \mathcal{G}_{Σ_0} be the base change of the chosen U -model.

By Lemma 2.1, for $n \leq 1$, and also for $n \geq 2$ if \mathcal{G} is commutative, the pullback maps

$$H^n(\mathbb{A}_{K, \Sigma'_0}^{\Sigma}, \mathcal{G}_{\Sigma_0}) \rightarrow \prod_{v \in \Sigma'_0} H^n(K_v, \mathcal{G}) \times \prod_{v \in \Sigma \setminus \Sigma'_0} H^n(\mathcal{O}_v, \mathcal{G}_{\Sigma_0}) \quad (2.3.1)$$

for varying Σ'_0 give rise to the comparison map

$$H^n(\mathbb{A}_K^{\Sigma}, \mathcal{G}) \rightarrow \varinjlim_{\Sigma_0 \subset \Sigma'_0 \subset \Sigma, \#\Sigma'_0 < \infty} \left(\prod_{v \in \Sigma'_0} H^n(K_v, \mathcal{G}) \times \prod_{v \in \Sigma \setminus \Sigma'_0} H^n(\mathcal{O}_v, \mathcal{G}_{\Sigma_0}) \right), \quad (2.3.2)$$

whose target does not depend on the choices of Σ_0 and \mathcal{G}_{Σ_0} .

We seek to prove in Theorems 2.13 and 2.18 that the comparison map (2.3.2) is bijective in many cases, in particular, if $\mathcal{G} = G_{\mathbb{A}_K^\Sigma}$. Our methods will exploit the following lemma, whose case of local R_i would suffice for us.

Lemma 2.4 ([Bha14, Thm. 1.3]). *For a set of rings $\{R_i\}_{i \in I}$ and a quasi-compact and quasi-separated \mathbb{Z} -algebraic space X , the natural map*

$$X(\prod_i R_i) \rightarrow \prod_i X(R_i)$$

is bijective. □

The proofs in this section could be simplified significantly if Lemma 2.4 stayed true for X that are algebraic stacks instead of algebraic spaces. However, Lemma 2.4 fails for such X : see [Bha14, 8.4].

Lemma 2.4 allows us to settle some initial cases of the bijectivity of the comparison map.

Proposition 2.5. *For $n = 0$ (resp., $n = 1$), the maps (2.3.1)–(2.3.2) are bijective (resp., injective).*

Proof. It suffices to treat (2.3.1). The $n = 0$ case is immediate from Lemma 2.4.

In the $n = 1$ case, a standard twisting argument [Gir71, III.2.6.3] reduces us to proving that a torsor under an inner twist of $(\mathcal{G}_{\Sigma_0})_{\mathbb{A}_{K, \Sigma'_0}^{\Sigma}}$ is trivial as soon as its pullbacks to K_v for $v \in \Sigma'_0$ and to \mathcal{O}_v for $v \in \Sigma \setminus \Sigma'_0$ are trivial. Such triviality follows by applying Lemma 2.4 because inner twists of $(\mathcal{G}_{\Sigma_0})_{\mathbb{A}_{K, \Sigma'_0}^{\Sigma}}$ and their torsors are algebraic spaces by [SP, 04SK], and they inherit quasi-compactness and quasi-separatedness by [SP, 041L and 041N]. □

Corollary 2.6. *There are no nontrivial vector bundles on $\text{Spec } \mathbb{A}_K$.*

Proof. Proposition 2.5 allows us to infer the vanishing of $H^1(\mathbb{A}_K, \text{GL}_m)$ from the vanishing of $H^1(\mathcal{O}_v, \text{GL}_m)$ for $v \nmid \infty$ and of $H^1(K_v, \text{GL}_m)$. □

We start building up for Lemma 2.12, which is a key step towards proving the bijectivity of the comparison map for smooth \mathcal{G} . Lemma 2.7, whose part (b) extends Lang's theorem [Lan56, Thm. 2], will be helpful throughout §§2–3.

Lemma 2.7. *Let k be a field, G a finite type k -group scheme, and X a homogeneous space under G .*

(a) *If k is perfect and G is finite connected, then $\#X(k) = 1$, so that, in particular, every G -torsor is trivial.*

(b) *If k is finite and X is geometrically connected, then $X(k) \neq \emptyset$.*

- (c) If k is finite and G is connected, then $H^1(k, G) = \{*\}$.
- (d) If k is perfect and of dimension ≤ 1 (in the sense of [Ser02, II.§3.1]) and G fits into an exact sequence

$$1 \rightarrow G' \rightarrow G \rightarrow G'' \rightarrow 1 \quad (2.7.1)$$

of k -group schemes of finite type, then the map

$$H^1(k, G) \rightarrow H^1(k, G'')$$

is surjective.

- (e) If k is finite and G is commutative, then $H^n(k, G) = 0$ for $n \geq 2$.

Proof.

- (a) We use the étale sheaf property of X to reduce to the $X(k) \neq \emptyset$ case by passing to a finite, and hence separable, extension k' of k . Then we enlarge k further to reduce to the $k = \bar{k}$ case. If $k = \bar{k}$, then the singleton $G(k)$ acts transitively on $X(k)$, and the claim follows.
- (b) Due to geometric connectedness, X is a homogeneous space under the identity component G^0 , so we assume that G is connected. Then, by [SGA 3_{I new}, VII_A, 8.3], G is an extension of a smooth connected k -group scheme Q by a finite connected H . Since X/H is a homogeneous space under Q , [Ser02, III.§2.4, Thm. 3] and Lang's theorem [Lan56, Thm. 2] let us choose a $p \in (X/H)(k)$. The fiber X_p is a homogeneous space under H , so $X_p(k) \neq \emptyset$ by (a).
- (c) It suffices to apply (b) to principal homogeneous spaces (i.e., torsors) under G .
- (d) In the case when G and G'' are smooth, the claim is a corollary of a theorem of Springer, see [Ser02, III.§2.4, Cor. 2]. We will reduce the general case to this smooth case.

By [CGP10, C.4.1], there is a k -smooth closed k -subgroup scheme $\widetilde{G}'' \subset G''$ for which $\widetilde{G}''(L) = G''(L)$ for every finite (automatically separable) field extension L of k . The description of torsors through cocycles therefore gives the induced bijection

$$H^1(k, \widetilde{G}'') \xrightarrow{\sim} H^1(k, G'').$$

Thus, by pulling back (2.7.1) along $\widetilde{G}'' \subset G''$, we reduce to the case when G'' is smooth.

To reduce further to the case when G' is also smooth, we begin by applying [SGA 3_{I new}, VII_A, 8.3] to infer the existence of a finite connected subgroup $N \subset G'$ that is normal even in G and for which G'/N is smooth (N is either trivial or a large enough Frobenius kernel). Then, by (a),

$$G(L) = (G/N)(L)$$

for every finite field extension L of k , so we again get a bijection

$$H^1(k, G) \xrightarrow{\sim} H^1(k, G/N).$$

Passage to quotients by N therefore reduces us to the case when both G' and G'' are smooth. Then, by [SGA 3_{I new}, 9.2 (viii)], G is also smooth, which completes the promised reduction.

- (e) If G is smooth, then a cohomological dimension argument suffices. In general, [Čes15b, 6.4] embeds G into a commutative smooth \widetilde{G} , and (d) allows replacing G by the smooth \widetilde{G}/G . \square

We will sometimes use the following lemma implicitly.

Lemma 2.8 ([Art69, 4.2]). *For a field k , every quasi-separated k -group algebraic space locally of finite type is a scheme.* \square

Corollary 2.9. *For a Henselian local ring R with a finite residue field \mathbb{F} and a flat R -group algebraic space G of finite presentation, $H^n(R, G)$ vanishes if $n = 1$ and G is smooth with connected $G_{\mathbb{F}}$, and also if $n \geq 2$ and G is commutative.*

Proof. By [Čes15b, B.11] and [Čes15b, B.13] (which is based on [Toë11, 3.4]),

$$H^n(R, G) \rightarrow H^n(\mathbb{F}, G)$$

is injective, so Lemma 2.7 (c) and (e) (together with Lemma 2.8) give the claim. \square

Lemma 2.10. *Let $T' \rightarrow T$ be a finite étale morphism of schemes. For a T' -algebraic space X' , let X denote the restriction of scalars $\text{Res}_{T'/T} X'$ (which is a T -algebraic space by [SP, 05YF]). If a T' -morphism $f': X' \rightarrow Y'$ is quasi-compact (resp., quasi-separated), then so is $\text{Res}_{T'/T} f': X \rightarrow Y$.*

Proof. Since $\text{Res}_{T'/T}$ commutes with fiber products and base change in T , we only need to prove the quasi-compactness claim and may work étale locally on T . Limit arguments therefore permit us to assume that $T' = \bigsqcup_{j=1}^d T$, in which case $\text{Res}_{T'/T} f'$ is a product of d quasi-compact T -morphisms. \square

Lemma 2.11. *Let T be a quasi-compact scheme and X a T -algebraic space of finite presentation. There is an $m_X \in \mathbb{Z}_{>0}$ such that for each geometric fiber $X_{\bar{t}}$ of $X \rightarrow T$, the underlying topological space $|X_{\bar{t}}|$ has at most m_X connected components.*

Proof. We replace T by an affine open to reduce to the case when T is affine. In this case, we take an étale cover $Y \twoheadrightarrow X$ of finite presentation with Y a scheme and replace X by Y to reduce to the scheme case, which is settled by [EGA IV₃, 9.7.9] (see also [EGA III₁, 0.9.3.1]). \square

We are now ready to use the preceding auxiliary lemmas to settle a key step towards the bijectivity of the comparison map.

Lemma 2.12. *Fix a set I . For $i \in I$, let R_i be a Henselian local ring with finite residue field \mathbb{F}_i . For a smooth $(\prod_i R_i)$ -group algebraic space \mathcal{F} of finite presentation, the pullback map*

$$H^n(\prod_i R_i, \mathcal{F}) \rightarrow \prod_i H^n(R_i, \mathcal{F})$$

is bijective for $n \leq 1$, and also for $n \geq 2$ if \mathcal{F} is commutative. For $n \geq 2$, both

$$H^n(\prod_i R_i, \mathcal{F}) = 0 \quad \text{and} \quad \prod_i H^n(R_i, \mathcal{F}) = 0;$$

the same holds for $n = 1$ if each $\mathcal{F}_{\mathbb{F}_i}$ is connected.

Proof. The last sentence follows from the rest due to Corollary 2.9.

For a fixed $d \in \mathbb{Z}_{>0}$, let R'_i be the Henselian local finite étale R_i -algebra that corresponds to the residue field extension $\mathbb{F}'_i/\mathbb{F}_i$ of degree d . Set

$$R := \prod_i R_i \quad \text{and} \quad R' := \prod_i R'_i,$$

so that R'/R is finite étale of degree d by [Con12a, 7.5.5]. We consider the restriction of scalars

$$\mathcal{H} := \text{Res}_{R'/R}(\mathcal{F}_{R'}),$$

which is an R -group algebraic space by [SP, 05YF]. The adjunction map $\iota: \mathcal{F} \rightarrow \mathcal{H}$ is injective, so

$$\mathcal{Q} := \mathcal{H}/\iota(\mathcal{F})$$

is an R -algebraic space by [SP, 06PH].

Claim 2.12.1. Both \mathcal{H} and \mathcal{Q} are smooth and of finite presentation over R .

Proof. Smoothness of \mathcal{H} follows from [SP, 04AK and 04AM] and implies that of \mathcal{Q} by [SP, 0AHE]. Thus, \mathcal{H} is of finite presentation by Lemma 2.10, and hence so is \mathcal{Q} by [SP, 040W and 03KS]. \square

Claim 2.12.2. The map $H^1(\prod_i R_i, \mathcal{F}) \rightarrow \prod_i H^1(R_i, \mathcal{F})$ is surjective.

Proof. We choose $m_{\mathcal{F}}$ as in Lemma 2.11 (applied with $X = \mathcal{F}$) and set $d := m_{\mathcal{F}}!$ in the construction of \mathcal{H} and \mathcal{Q} . Thanks to Lang's theorem [Lan56, Thm. 2], $H^1(\mathbb{F}_i, \mathcal{F})$ and $H^1(\mathbb{F}'_i, \mathcal{F})$ inject into analogous H^1 's with the component group of $\mathcal{F}_{\mathbb{F}_i}$ as coefficients. Every torsor under this component group is a finite étale \mathbb{F}_i -scheme of degree at most $m_{\mathcal{F}}$, so it has an \mathbb{F}'_i -point. In conclusion,

$$H^1(\mathbb{F}_i, \mathcal{F}) \rightarrow H^1(\mathbb{F}'_i, \mathcal{F})$$

vanishes, and hence so does each

$$H^1(R_i, \mathcal{F}) \rightarrow H^1(R'_i, \mathcal{F})$$

by [Čes15b, B.11]. Therefore, [Gir71, V.3.1.3] gives the vanishing of b in

$$\mathcal{Q}(R_i) \xrightarrow{\delta_i} H^1(R_i, \mathcal{F}) \xrightarrow{b} H^1(R_i, \mathcal{H}), \quad (2.12.3)$$

and the surjectivity of δ_i follows. It remains to use Claim 2.12.1 to apply Lemma 2.4 to \mathcal{Q} and use the compatibility of the sequences (2.12.3) with their counterpart over $\prod_i R_i$. \square

As in Proposition 2.5, Lemma 2.4 and [Gir71, III.2.6.3] prove that the map

$$H^n(\prod_i R_i, \mathcal{F}) \rightarrow \prod_i H^n(R_i, \mathcal{F})$$

is bijective for $n = 0$ and injective for $n = 1$, so Claim 2.12.2 settles the $n \leq 1$ case.

Suppose from now on that $n \geq 2$ and \mathcal{F} is commutative, so $\prod_i H^n(R_i, \mathcal{F}) = 0$ by Corollary 2.9. We argue the remaining vanishing of $H^n(R, \mathcal{F})$ by induction on n (using the settled bijectivity in the $n = 1$ case to get started).

We fix an $\alpha \in H^n(R, \mathcal{F})$ and seek to prove that $\alpha = 0$. Lemma 2.1 applied to strict Henselizations of $\text{Spec } R$ provides an étale cover

$$p: \text{Spec } A \twoheadrightarrow \text{Spec } R \quad \text{with } \alpha|_A = 0.$$

By Lemma 2.11, the fibral degrees of p are bounded by some $m \in \mathbb{Z}_{>0}$. We set $d := m!$, and, as before, let R'_i be the finite étale R_i algebra corresponding to the residue field extension of degree d and set $R' := \prod_i R'_i$. Each $\text{Spec } R'_i \rightarrow \text{Spec } R$ factors through p , and hence so does $p': \text{Spec } R' \rightarrow \text{Spec } R$, so that $\alpha|_{R'} = 0$. The acyclicity of p'_* supplied by [SGA 4_{II}, VIII, 5.5] shows that the spectral sequence

$$H^n(R, \mathbf{R}^m p'_*(\mathcal{F}_{R'})) \Rightarrow H^{n+m}(R', \mathcal{F})$$

degenerates, so the equality $\alpha|_{R'} = 0$ proves that the map

$$H^n(R, \mathcal{F}) \rightarrow H^n(R, \mathcal{H}) \cong H^n(R', \mathcal{F})$$

kills α . Finally, Claim 2.12.1 and the inductive hypothesis provide the isomorphisms in

$$\begin{array}{ccccccc} H^{n-1}(R, \mathcal{H}) & \longrightarrow & H^{n-1}(R, \mathcal{Q}) & \longrightarrow & H^n(R, \mathcal{F}) & \longrightarrow & H^n(R, \mathcal{H}) \\ \downarrow \cong & & \downarrow \cong & & \downarrow & & \downarrow \\ \prod_i H^{n-1}(R_i, \mathcal{H}) & \longrightarrow & \prod_i H^{n-1}(R_i, \mathcal{Q}) & \longrightarrow & 0 & \longrightarrow & 0, \end{array}$$

so a diagram chase proves that $\alpha = 0$. \square

With Lemma 2.12 at hand, we are ready to prove the bijectivity of the comparison map for smooth \mathcal{G} .

Theorem 2.13. *For a smooth $\mathbb{A}_K^{\epsilon\Sigma}$ -group algebraic space \mathcal{G} of finite presentation, the comparison map*

$$H^n(\mathbb{A}_K^{\epsilon\Sigma}, \mathcal{G}) \rightarrow \varinjlim_{\Sigma'_0} \left(\prod_{v \in \Sigma'_0} H^n(K_v, \mathcal{G}) \times \prod_{v \in \Sigma \setminus \Sigma'_0} H^n(\mathcal{O}_v, \mathcal{G}_{\Sigma_0}) \right) \quad (2.13.1)$$

of (2.3.2) is bijective for $n \leq 1$, and also for $n \geq 2$ if \mathcal{G} is commutative. For $n \geq 2$, this reduces to

$$H^n(\mathbb{A}_K^{\epsilon\Sigma}, \mathcal{G}) \cong \bigoplus_{v \in \Sigma} H^n(K_v, \mathcal{G});$$

the same holds for $n = 1$ if $\mathcal{G} \rightarrow \text{Spec } \mathbb{A}_K^{\epsilon\Sigma}$ has connected fibers.

Proof. We apply Lemma 2.2 to ensure that $\mathcal{G}_{\Sigma_0} \rightarrow \text{Spec } \mathbb{A}_{K, \Sigma_0}^{\epsilon\Sigma}$ of §2.3 inherits smoothness or fibral connectedness from $\mathcal{G} \rightarrow \text{Spec } \mathbb{A}_K^{\epsilon\Sigma}$. For a fixed Σ'_0 , Lemma 2.12 and the decomposition

$$\text{Spec } \mathbb{A}_{K, \Sigma'_0}^{\epsilon\Sigma} = \left(\bigsqcup_{v \in \Sigma'_0} \text{Spec } K_v \right) \sqcup \text{Spec}(\prod_{v \in \Sigma \setminus \Sigma'_0} \mathcal{O}_v)$$

show that (2.3.1) is bijective, so the bijectivity of (2.3.2) (i.e., of (2.13.1)) follows. For the last claim, we apply the vanishing aspect of Lemma 2.12. \square

The following corollary, mentioned to us by Bjorn Poonen, was one of motivations to work out the general Theorems 2.13 and 2.18. We recall that the Brauer group $\text{Br}(T)$ of a scheme T is by definition the group of similarity classes of Azumaya algebras over T .

Corollary 2.14. *The Albert–Brauer–Hasse–Noether exact sequence from global class field theory may be rewritten as*

$$0 \rightarrow \text{Br}(K) \rightarrow \text{Br}(\mathbb{A}_K) \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0.$$

Proof. Theorem 2.13 with $\mathcal{G} = \mathbb{G}_m$ gives

$$H^2(\mathbb{A}_K, \mathbb{G}_m) \cong \bigoplus_v \text{Br}(K_v), \quad \text{so that} \quad H^2(\mathbb{A}_K, \mathbb{G}_m) = H^2(\mathbb{A}_K, \mathbb{G}_m)_{\text{tors}},$$

which allows us to apply [Gab81, Ch. II, Thm. 1 on p. 163] to conclude that also

$$\text{Br}(\mathbb{A}_K) \cong H^2(\mathbb{A}_K, \mathbb{G}_m). \quad \square$$

Remark 2.15. In [Azu66], assuming that K is a number field, Azumaya gave a direct computation of $\text{Br}(\mathbb{A}_K)$ via central simple algebras and also proved that $\text{Br}(\mathbb{A}_K) \cong \bigoplus_v \text{Br}(K_v)$.

We turn to the remaining input to our main result (Theorem 2.18) regarding the comparison map.

Lemma 2.16. *For a K -group scheme G of finite type, there is a $d \in \mathbb{Z}_{>0}$, a nonempty open $U \subset S$, and a model $\mathcal{G} \rightarrow U$ of $G \rightarrow \text{Spec } K$ such that for every $v \in U$ there is an extension K'_v/K_v of degree d for which the pullback map*

$$H^1(\mathcal{O}_v, \mathcal{G}) \rightarrow H^1(\mathcal{O}'_v, \mathcal{G})$$

vanishes (here \mathcal{O}'_v is the ring of integers of K'_v).

Proof. We begin with some preliminary observations.

- (1) For a nonarchimedean local field k , its ring of integers \mathfrak{o} , and a finite étale \mathfrak{o} -group scheme \mathcal{F} of order $\#\mathcal{F}$, there is an extension k'/k of degree $(\#\mathcal{F})!$ for which the pullback map

$$H^1(\mathfrak{o}, \mathcal{F}) \rightarrow H^1(\mathfrak{o}', \mathcal{F})$$

vanishes (where \mathfrak{o}' is the ring of integers of k').

Proof. Every torsor under \mathcal{F} is represented by a finite étale \mathfrak{o} -group scheme of order $\#\mathcal{F}$. Thus, if we let k' be the unramified extension of k of degree $(\#\mathcal{F})!$, then every such torsor has an \mathfrak{o}' -point. \square

- (2) If a finite \mathcal{F} in (1) is not assumed to be étale but instead k is of characteristic $p > 0$ and \mathcal{F}_k is connected, then the same conclusion holds with $(\#\mathcal{F})!$ replaced by $p^{(\#\mathcal{F})!}$.

Proof. By the valuative criterion of properness,

$$H^1(\mathfrak{o}, \mathcal{F}) \rightarrow H^1(k, \mathcal{F})$$

is injective. Thus, the claim results from the vanishing of

$$H^1(k, \mathcal{F}) \rightarrow H^1(k^{1/p^{(\#\mathcal{F})!}}, \mathcal{F})$$

supplied by [Čes15b, 5.7 (b)]. □

- (3) For k and \mathfrak{o} as in (1) and \mathcal{C} a smooth \mathfrak{o} -group scheme with connected fibers,

$$H^1(\mathfrak{o}, \mathcal{C}) = \{*\}.$$

Proof. This is a special case of Corollary 2.9. □

By [SGA 3I_{new}, VI_B, 1.6.1 et VII_A, 8.3], G is an extension of a smooth K -group \tilde{G} by a finite connected K -group N . In turn, \tilde{G} is an extension of its finite étale component group by the smooth identity component \tilde{G}^0 . A short exact sequence of finite type K -groups spreads out, as does smoothness, finiteness, étaleness, or fibral connectedness of its terms (the latter by [EGA IV₃, 9.7.8]), so the nonabelian cohomology exact sequences combine with (1)–(3) to prove the claim. □

We are now ready to address the bijectivity of the comparison map in the $n = 1$ case.

Proposition 2.17. *For $n = 1$ and $\mathcal{G} \simeq G_{\mathbb{A}_{K,\Sigma}^{\varepsilon\Sigma}}$ with G a K -group scheme of finite type, the comparison map (2.3.2) is bijective.*

Proof. Let $d \in \mathbb{Z}_{>0}$ and a model $\mathcal{G} \rightarrow U$ of $G \rightarrow \text{Spec } K$ be as provided by Lemma 2.16. We shrink U to assume that $\mathcal{G} \rightarrow U$ is of finite presentation. For a Σ_0 containing all the places of Σ outside U , we set $\mathcal{G}_{\Sigma_0} := \mathcal{G}_{\mathbb{A}_{K,\Sigma_0}^{\varepsilon\Sigma}}$. Due to Proposition 2.5, the surjectivity of (2.3.1) argued below will suffice.

To simplify the notation we may and do assume that $\Sigma'_0 = \Sigma_0$ in (2.3.1). Thanks to the decomposition

$$\text{Spec}(\mathbb{A}_{K,\Sigma_0}^{\varepsilon\Sigma}) = \left(\bigsqcup_{v \in \Sigma_0} \text{Spec } K_v \right) \sqcup \text{Spec}(\prod_{v \in \Sigma \setminus \Sigma_0} \mathcal{O}_v),$$

we may focus on the surjectivity of

$$H^1(\prod_{v \in \Sigma \setminus \Sigma_0} \mathcal{O}_v, \mathcal{G}) \rightarrow \prod_{v \in \Sigma \setminus \Sigma_0} H^1(\mathcal{O}_v, \mathcal{G}). \quad (2.17.1)$$

For $v \in \Sigma \setminus \Sigma_0$, we let $\mathcal{O}'_v/\mathcal{O}_v$ be a finite free extension of degree d provided by Lemma 2.16. We choose an \mathcal{O}_v -basis of \mathcal{O}'_v for each $v \in \Sigma \setminus \Sigma_0$ to see that $\prod_{v \in \Sigma \setminus \Sigma_0} \mathcal{O}'_v / \prod_{v \in \Sigma \setminus \Sigma_0} \mathcal{O}_v$ is a finite free extension of degree d and that the following natural homomorphisms are isomorphisms:

$$\prod_{v \in \Sigma \setminus \Sigma_0} \mathcal{O}'_v \otimes_{\prod_{v \in \Sigma \setminus \Sigma_0} \mathcal{O}_v} \prod_{v \in \Sigma \setminus \Sigma_0} \mathcal{O}'_v \xrightarrow{\sim} \prod_{v \in \Sigma \setminus \Sigma_0} \mathcal{O}'_v \otimes_{\mathcal{O}_v} \mathcal{O}'_v,$$

$$\prod_{v \in \Sigma \setminus \Sigma_0} \mathcal{O}'_v \otimes_{\prod_{v \in \Sigma \setminus \Sigma_0} \mathcal{O}_v} \prod_{v \in \Sigma \setminus \Sigma_0} \mathcal{O}'_v \otimes_{\prod_{v \in \Sigma \setminus \Sigma_0} \mathcal{O}_v} \prod_{v \in \Sigma \setminus \Sigma_0} \mathcal{O}'_v \xrightarrow{\sim} \prod_{v \in \Sigma \setminus \Sigma_0} \mathcal{O}'_v \otimes_{\mathcal{O}_v} \mathcal{O}'_v \otimes_{\mathcal{O}_v} \mathcal{O}'_v.$$

For $v \in \Sigma \setminus \Sigma_0$, let $\mathcal{T}_v \rightarrow \text{Spec } \mathcal{O}_v$ be a $\mathcal{G}_{\mathcal{O}_v}$ -torsor. By the choice of \mathcal{O}'_v , there is a descent datum

$$g_v \in \mathcal{G}(\mathcal{O}'_v \otimes_{\mathcal{O}_v} \mathcal{O}'_v)$$

that gives rise to \mathcal{T}_v . Lemma 2.4 and the displayed bijections assemble the g_v to a descent datum

$$g \in \mathcal{G}(\prod_{v \in \Sigma \setminus \Sigma_0} \mathcal{O}'_v \otimes_{\prod_{v \in \Sigma \setminus \Sigma_0} \mathcal{O}_v} \prod_{v \in \Sigma \setminus \Sigma_0} \mathcal{O}'_v).$$

By construction, the class of the \mathcal{G} -torsor that results from g maps under (2.17.1) to the tuple whose v -component is the class of \mathcal{T}_v . In conclusion, (2.17.1) is surjective, as desired. \square

With all ingredients in place, we now turn to the promised bijectivity of the comparison map.

Theorem 2.18. *For a K -group scheme G of finite type, the comparison map*

$$H^n(\mathbb{A}_K^{\varepsilon\Sigma}, G) \rightarrow \varinjlim_{\Sigma'_0} \left(\prod_{v \in \Sigma'_0} H^n(K_v, G) \times \prod_{v \in \Sigma \setminus \Sigma'_0} H^n(\mathcal{O}_v, \mathcal{G}_{\Sigma_0}) \right)$$

of (2.3.2) is bijective for $n \leq 1$, and also for $n \geq 2$ if G is commutative. For $n \geq 2$, this reduces to

$$H^n(\mathbb{A}_K^{\varepsilon\Sigma}, G) \cong \bigoplus_{v \in \Sigma} H^n(K_v, G);$$

the same holds for $n = 1$ if G is connected and smooth.

Proof. Propositions 2.5 and 2.17 supply the bijectivity in the $n \leq 1$ case, so we assume that $n \geq 2$.

By [Ces15b, 6.4], G is a closed subgroup of a K -group scheme H that is of finite type, smooth, and commutative. By [SGA 3_I new, VI_A, 3.2], H/G is a K -group scheme that inherits these properties of H . The exact sequence

$$0 \rightarrow G \rightarrow H \rightarrow H/G \rightarrow 0$$

spreads out to an exact sequence of finite type models over a nonempty open $U \subset S$, so the five lemma allows us to deduce the claimed bijectivity for G from the bijectivity in the smooth case established in Theorem 2.13.

For the last sentence, Theorem 2.13 suffices in the $n = 1$ case. In the $n \geq 2$ case, we shrink U to assume that the U -models of H and H/G are smooth. We then combine [Gro68, 11.7 2°] with the five lemma to get

$$H^n(\mathcal{O}_v, \mathcal{G}) \cong H^n(\mathbb{F}_v, \mathcal{G}) \quad \text{for } v \in U,$$

where \mathcal{G} is the chosen U -model of G . It remains to note that $H^n(\mathbb{F}_v, \mathcal{G}) = 0$ by Lemma 2.7 (e). \square

In the following lemma we record conditions under which the target of the comparison map is a restricted product.

Lemma 2.19. *Let \mathcal{G} be a finitely presented $\mathbb{A}_K^{\varepsilon\Sigma}$ -group algebraic space, $n \in \mathbb{Z}_{\geq 0}$, and assume that*

- (1) $n = 0$, and \mathcal{G} is separated; or
- (2) $n = 1$, and \mathcal{G} is either smooth with connected fibers or proper; or
- (3) $n \geq 2$, and \mathcal{G} is commutative.

Use Lemma 2.2 to enlarge a Σ_0 of §2.3 to ensure that $\mathcal{G}_{\Sigma_0} \rightarrow \text{Spec } \mathbb{A}_{K, \Sigma_0}^{\varepsilon\Sigma}$ inherits the assumed properties of $\mathcal{G} \rightarrow \text{Spec } \mathbb{A}_K^{\varepsilon\Sigma}$. Then the pullback map

$$H^n(\mathcal{O}_v, \mathcal{G}_{\Sigma_0}) \rightarrow H^n(K_v, \mathcal{G})$$

is injective for all $v \in \Sigma \setminus \Sigma_0$, the resulting restricted product $\prod'_{v \in \Sigma} H^n(K_v, \mathcal{G})$ does not depend on the choices of Σ_0 and \mathcal{G}_{Σ_0} , and $\prod'_{v \in \Sigma} H^n(K_v, \mathcal{G})$ identifies with the target of the comparison map (2.3.2).

Proof. The injectivity follows from the valuative criterion of separatedness [SP, 03KU], Corollary 2.9, and the valuative criterion of properness [SP, 0A40] applied to the algebraic space parametrizing isomorphisms between two fixed $(\mathcal{G}_{\Sigma_0})_{\mathcal{O}_v}$ -torsors for $v \in \Sigma \setminus \Sigma_0$ (the fppf sheaf parametrizing torsor isomorphisms is fppf locally isomorphic to $(\mathcal{G}_{\Sigma_0})_{\mathcal{O}_v}$ itself, so it is representable by an algebraic space due to [SP, 04SK]). The independence follows from the second sentence of §2.3. The last claim follows from the rest. \square

We conclude §2 by combining the previous results to get the promised restricted product interpretation of the cohomology of $\mathbb{A}_K^{\varepsilon\Sigma}$.

Theorem 2.20. *Let \mathcal{G} be a finitely presented $\mathbb{A}_K^{\varepsilon\Sigma}$ -group algebraic space, $n \in \mathbb{Z}_{\geq 0}$, and assume that*

- (1) $n = 0$, and \mathcal{G} is separated; or
- (2) $n = 1$, and \mathcal{G} satisfies any of the following: is isomorphic to $G_{\mathbb{A}_K^{\varepsilon\Sigma}}$ for a proper K -group scheme G , is smooth with connected fibers, or is smooth and proper; or
- (3) $n \geq 2$, and \mathcal{G} is commutative and either smooth or isomorphic to $G_{\mathbb{A}_K^{\varepsilon\Sigma}}$ for a K -group scheme G .

Then the comparison map (2.3.2) reduces to a bijection towards the restricted product of Lemma 2.19:

$$H^n(\mathbb{A}_K^{\varepsilon\Sigma}, \mathcal{G}) \xrightarrow{\sim} \prod'_{v \in \Sigma} H^n(K_v, \mathcal{G}).$$

Proof. The claim follows by combining Lemma 2.19 with Proposition 2.5 and with Theorems 2.13 and 2.18. \square

3. TOPOLOGY ON COHOMOLOGY OF THE ADELES

The results of §2 are useful in the study of a natural topology carried by cohomology of the adèles. This topology plays an important role in the arithmetic duality proofs of §§4–6, so in §3 we gather the topological input needed for those proofs. For completeness, we also include several results that are not used in this paper but that seem indispensable to a robust topological theory.

We formulate the results in the setting where the coefficients are group algebraic spaces because this does not lead to any complications in the proofs and is convenient for §3.10 and Proposition 3.11, where we would otherwise need to impose additional assumptions to guarantee the representability of \mathcal{Q} by a scheme. However, if desired, the reader may restrict to the case of schemes throughout.

3.1. Topology on $H^n(\mathbb{A}_K^{\varepsilon\Sigma}, \mathcal{G})$. We suppose that \mathcal{G} of §2.3 is $\mathbb{A}_K^{\varepsilon\Sigma}$ -flat (e.g., that $\mathcal{G} \simeq G_{\mathbb{A}_K^{\varepsilon\Sigma}}$) and use Lemma 2.2 to ensure the $\mathbb{A}_{K, \Sigma_0}^{\varepsilon\Sigma}$ -flatness of \mathcal{G}_{Σ_0} .

For $n \leq 1$, and also for $n \geq 2$ if \mathcal{G} is commutative, each $H^n(K_v, \mathcal{G})$ with $v \in \Sigma$ and each $H^n(\mathcal{O}_v, \mathcal{G}_{\Sigma_0})$ with $v \in \Sigma \setminus \Sigma_0$ carries a “ v -adic” topology defined as in [Čes15b, 3.1–3.2]. Each pullback map

$$H^n(\mathcal{O}_v, \mathcal{G}_{\Sigma_0}) \rightarrow H^n(K_v, \mathcal{G}) \quad \text{with } v \in \Sigma \setminus \Sigma_0 \quad (3.1.1)$$

is continuous and open by [Čes15b, 3.5 (c), 2.7 (viii), and 2.9 (e) (with A.2 (b))] (with Lemma 2.8 for the required separatedness of $(\mathcal{G}_{\Sigma_0})_{\mathbb{F}_v}$). Thus,

$$\varinjlim_{\Sigma'_0} \left(\prod_{v \in \Sigma'_0} H^n(K_v, \mathcal{G}) \times \prod_{v \in \Sigma \setminus \Sigma'_0} H^n(\mathcal{O}_v, \mathcal{G}_{\Sigma_0}) \right),$$

which carries the direct limit topology of product topologies, has open transition maps, and the maps

$$\prod_{v \in \Sigma'_0} H^n(K_v, \mathcal{G}) \times \prod_{v \in \Sigma \setminus \Sigma'_0} H^n(\mathcal{O}_v, \mathcal{G}_{\Sigma_0}) \rightarrow \varinjlim \left(\prod_{v \in \Sigma'_0} H^n(K_v, \mathcal{G}) \times \prod_{v \in \Sigma \setminus \Sigma'_0} H^n(\mathcal{O}_v, \mathcal{G}_{\Sigma_0}) \right) \quad (3.1.2)$$

are open, too. We endow $H^n(\mathbb{A}_K^{\varepsilon\Sigma}, \mathcal{G})$ with the preimage topology via the comparison map

$$H^n(\mathbb{A}_K^{\varepsilon\Sigma}, \mathcal{G}) \rightarrow \varinjlim_{\Sigma_0 \subset \Sigma'_0 \subset \Sigma, \#\Sigma'_0 < \infty} \left(\prod_{v \in \Sigma'_0} H^n(K_v, \mathcal{G}) \times \prod_{v \in \Sigma \setminus \Sigma'_0} H^n(\mathcal{O}_v, \mathcal{G}_{\Sigma_0}) \right) \quad (3.1.3)$$

of (2.3.2). By the second sentence of §2.3, the topology on the target of (3.1.3) does not depend on the choices of Σ_0 and \mathcal{G}_{Σ_0} , and hence neither does the topology on $H^n(\mathbb{A}_K^{\varepsilon\Sigma}, \mathcal{G})$.

Remarks.

3.2. See Theorems 2.13 and 2.18 for some settings in which the map (3.1.3) is bijective.

3.3. For $n = 0$ and $\mathcal{G} = G_{\mathbb{A}_K}$ with G a K -group scheme of finite type, the topology of §3.1 recovers the usual adelic topology on $G(\mathbb{A}_K)$ discussed, for instance, in [Con12b, §4].

We record conditions, under which the target of (3.1.3) is homeomorphic to a restricted product.

Proposition 3.4. *Let \mathcal{G} be a flat $\mathbb{A}_K^{\varepsilon\Sigma}$ -group algebraic space of finite presentation and $n \in \mathbb{Z}_{\geq 0}$.*

(a) *If either of the conditions (1)–(3) of Lemma 2.19 holds, then the restricted product topology on $\prod'_{v \in \Sigma} H^n(K_v, \mathcal{G})$ does not depend on the choice of Σ_0 and \mathcal{G}_{Σ_0} and the resulting identification*

$$\prod'_{v \in \Sigma} H^n(K_v, \mathcal{G}) \cong \varinjlim_{\Sigma_0 \subset \Sigma'_0 \subset \Sigma, \#\Sigma'_0 < \infty} \left(\prod_{v \in \Sigma'_0} H^n(K_v, \mathcal{G}) \times \prod_{v \in \Sigma \setminus \Sigma'_0} H^n(\mathcal{O}_v, \mathcal{G}_{\Sigma_0}) \right)$$

is a homeomorphism.

(b) *If either of the conditions (1)–(3) of Theorem 2.20 holds, then the resulting identification*

$$H^n(\mathbb{A}_K^{\varepsilon\Sigma}, \mathcal{G}) \cong \prod'_{v \in \Sigma} H^n(K_v, \mathcal{G})$$

is a homeomorphism.

Proof. The claims follow by combining the openness of the maps (3.1.1)–(3.1.2) with their injectivity supplied by Lemma 2.19. \square

In Propositions 3.5 to 3.9, we record several basic properties of the adelic topology of §3.1.

Proposition 3.5. *Let \mathcal{G} and \mathcal{G}' be flat $\mathbb{A}_K^{\varepsilon\Sigma}$ -group algebraic spaces of finite presentation.*

(a) *For $n \leq 1$, and also for $n \geq 2$ if \mathcal{G} and \mathcal{G}' are commutative, the map*

$$H^n(\mathbb{A}_K^{\varepsilon\Sigma}, \mathcal{G}) \rightarrow H^n(\mathbb{A}_K^{\varepsilon\Sigma}, \mathcal{G}')$$

induced by an $\mathbb{A}_K^{\varepsilon\Sigma}$ -homomorphism $\mathcal{G} \rightarrow \mathcal{G}'$ is continuous.

(b) *For $n \leq 1$, and also for $n \geq 2$ and \mathcal{G} and \mathcal{G}' are commutative, the maps of (a) induce a homeomorphism*

$$H^n(\mathbb{A}_K^{\varepsilon\Sigma}, \mathcal{G} \times \mathcal{G}') \xrightarrow{\sim} H^n(\mathbb{A}_K^{\varepsilon\Sigma}, \mathcal{G}) \times H^n(\mathbb{A}_K^{\varepsilon\Sigma}, \mathcal{G}').$$

(c) *For $n = 0$, and also for $n \geq 1$ if \mathcal{G} is commutative, $H^n(\mathbb{A}_K^{\varepsilon\Sigma}, \mathcal{G})$ is a topological group.*

Proof. We let \mathcal{G}_{Σ_0} and \mathcal{G}'_{Σ_0} be $\mathbb{A}_{K, \Sigma_0}^{\varepsilon\Sigma}$ -descents of \mathcal{G} and \mathcal{G}' as in §3.1.

(a) We enlarge Σ_0 to descend $\mathcal{G} \rightarrow \mathcal{G}'$ to an $\mathbb{A}_{K, \Sigma_0}^{\varepsilon\Sigma}$ -homomorphism $\mathcal{G}_{\Sigma_0} \rightarrow \mathcal{G}'_{\Sigma_0}$. Then it remains to apply [Čes15b, 3.3], which supplies the continuity of

$$H^n(\mathcal{O}_v, \mathcal{G}_{\Sigma_0}) \rightarrow H^n(\mathcal{O}_v, \mathcal{G}'_{\Sigma_0}) \quad \text{for } v \in \Sigma \setminus \Sigma_0$$

and of

$$H^n(K_v, \mathcal{G}) \rightarrow H^n(K_v, \mathcal{G}') \quad \text{for } v \in \Sigma.$$

- (b) The bijections in question are continuous by (a). Their openness reduces to that of similar bijections for the targets of (3.1.3). The openness of (3.1.2) reduces further to proving the openness of similar bijections for the sources of (3.1.2). For this, the openness of

$$\begin{aligned} H^n(\mathcal{O}_v, \mathcal{G}_{\Sigma_0} \times \mathcal{G}'_{\Sigma_0}) &\xrightarrow{\sim} H^n(\mathcal{O}_v, \mathcal{G}_{\Sigma_0}) \times H^n(\mathcal{O}_v, \mathcal{G}'_{\Sigma_0}) && \text{for } v \in \Sigma \setminus \Sigma_0 && \text{and of} \\ H^n(K_v, \mathcal{G} \times \mathcal{G}') &\xrightarrow{\sim} H^n(K_v, \mathcal{G}) \times H^n(K_v, \mathcal{G}') && \text{for } v \in \Sigma \end{aligned}$$

provided by [Čes15b, 3.5 (c) and 2.9 (b)] (with Lemma 2.8 as in §3.1) suffices.

- (c) To check the continuity of the multiplication and the inverse of $H^n(\mathbb{A}_K^{\varepsilon\Sigma}, \mathcal{G})$ we proceed similarly to the proof of (b) to reduce to the analogous continuity for $H^n(\mathcal{O}_v, \mathcal{G}_{\Sigma_0})$ with $v \in \Sigma \setminus \Sigma_0$ and for $H^n(K_v, \mathcal{G})$ with $v \in \Sigma$. To conclude we apply [Čes15b, 3.5 (c) and 3.6]. \square

Proposition 3.6. *Let \mathcal{G} be a flat $\mathbb{A}_K^{\varepsilon\Sigma}$ -group algebraic space of finite presentation, \mathcal{T} a right \mathcal{G} -torsor, and $\tau\mathcal{G} := \text{Aut}_{\mathcal{G}}(\mathcal{T})$ the resulting inner form of \mathcal{G} . Twisting by \mathcal{T} induces a homeomorphism*

$$H^1(\mathbb{A}_K^{\varepsilon\Sigma}, \tau\mathcal{G}) \cong H^1(\mathbb{A}_K^{\varepsilon\Sigma}, \mathcal{G}).$$

Proof. We enlarge Σ_0 to descend \mathcal{T} to a \mathcal{G}_{Σ_0} -torsor \mathcal{T}_{Σ_0} . For an $\mathbb{A}_{K, \Sigma_0}^{\varepsilon\Sigma}$ -algebra R , we consider the natural in R twisting by \mathcal{T}_{Σ_0} bijection

$$H^1(R, \tau_{\Sigma_0} \mathcal{G}_{\Sigma_0}) \cong H^1(R, \mathcal{G}_{\Sigma_0}),$$

which by [Čes15b, 3.4] is a homeomorphism for $R = \mathcal{O}_v$ with $v \in \Sigma \setminus \Sigma_0$ and for $R = K_v$ with $v \in \Sigma$. Thus, twisting by \mathcal{T}_{Σ_0} induces a bijection

$$\varinjlim_{\Sigma'_0} \left(\prod_{v \in \Sigma'_0} H^1(K_v, \tau\mathcal{G}) \times \prod_{v \in \Sigma \setminus \Sigma'_0} H^1(\mathcal{O}_v, \tau_{\Sigma_0} \mathcal{G}_{\Sigma_0}) \right) \cong \varinjlim_{\Sigma'_0} \left(\prod_{v \in \Sigma'_0} H^1(K_v, \mathcal{G}) \times \prod_{v \in \Sigma \setminus \Sigma'_0} H^1(\mathcal{O}_v, \mathcal{G}_{\Sigma_0}) \right)$$

that is a homeomorphism compatible with the maps (2.3.2). The claim follows. \square

Proposition 3.7. *For an $\mathbb{A}_K^{\varepsilon\Sigma}$ -group algebraic space \mathcal{G} of finite presentation, if*

- (1) $n = 1$, and \mathcal{G} is smooth with connected fibers; or
- (2) $n \geq 2$, and \mathcal{G} is commutative and either smooth or isomorphic to $G_{\mathbb{A}_K^{\varepsilon\Sigma}}$ for a K -group scheme G ,

then $H^n(\mathbb{A}_K^{\varepsilon\Sigma}, \mathcal{G})$ is discrete.

Proof. By Proposition 3.4 (b) and Corollary 2.9 (with Lemma 2.2),

$$H^n(\mathbb{A}_K^{\varepsilon\Sigma}, \mathcal{G}) \cong \bigoplus_{v \in \Sigma} H^n(K_v, \mathcal{G})$$

both algebraically and topologically. By [Čes15b, 3.5], every singleton of $\bigoplus_{v \in \Sigma} H^n(K_v, \mathcal{G})$ is open. \square

We do not mention the $n \geq 2$ cases explicitly in Propositions 3.8, 3.9 and 3.11—for such n we have nothing to add to the discreteness that Proposition 3.7 (2) offers.

Proposition 3.8. *For a flat $\mathbb{A}_K^{\varepsilon\Sigma}$ -group algebraic space \mathcal{G} of finite presentation, if*

- (1) $n = 0$, and \mathcal{G} is separated; or
- (2) $n = 1$, and \mathcal{G} is either smooth with connected fibers or proper,

then $H^n(\mathbb{A}_K^{\varepsilon\Sigma}, \mathcal{G})$ is Hausdorff.

Proof. By Proposition 2.5, (3.1.3) is injective. By Proposition 3.4 (a) and [Čes15b, 2.17 (3), 3.8 (1), and 3.8 (4)], the target of (3.1.3) is Hausdorff. \square

Proposition 3.9. *For a flat $\mathbb{A}_K^{\epsilon\Sigma}$ -group algebraic space \mathcal{G} of finite presentation, if*

- (1) $n = 0$, and \mathcal{G} is separated; or
- (2) $n = 1$, and \mathcal{G} satisfies any of the following: is isomorphic to $G_{\mathbb{A}_K^{\epsilon\Sigma}}$ for a proper K -group scheme G , is smooth with connected fibers, or is smooth and proper,

then $H^n(\mathbb{A}_K^{\epsilon\Sigma}, \mathcal{G})$ is locally compact.

Proof. By Proposition 3.4 (b), $H^n(\mathbb{A}_K^{\epsilon\Sigma}, \mathcal{G})$ is homeomorphic to $\prod'_{v \in \Sigma} H^n(K_v, \mathcal{G})$, which is locally compact by [Čes15b, 3.7 and 3.10] (with Lemma 2.8). \square

We conclude §3 with a study of some topological aspects of long exact cohomology sequences.

3.10. Continuity of maps in cohomology sequences. Let

$$\iota: \mathcal{H} \hookrightarrow \mathcal{G}$$

be a monomorphism of flat $\mathbb{A}_K^{\epsilon\Sigma}$ -group algebraic spaces of finite presentation, and

$$\mathcal{Q} := \mathcal{G}/\iota(\mathcal{H})$$

the resulting quotient left homogeneous space, which is a flat $\mathbb{A}_K^{\epsilon\Sigma}$ -algebraic space of finite presentation by [SP, 06PH, 06ET, 06EV, 040W, and 03KS]. We consider the cohomology sequence

$$\cdots \rightarrow \mathcal{Q}(\mathbb{A}_K^{\epsilon\Sigma}) \rightarrow H^1(\mathbb{A}_K^{\epsilon\Sigma}, \mathcal{H}) \rightarrow H^1(\mathbb{A}_K^{\epsilon\Sigma}, \mathcal{G}) \xrightarrow{x} H^1(\mathbb{A}_K^{\epsilon\Sigma}, \mathcal{Q}) \xrightarrow{y} H^2(\mathbb{A}_K^{\epsilon\Sigma}, \mathcal{H}) \xrightarrow{z} \cdots, \quad (\dagger)$$

in which x (resp., y) is defined if $\iota(\mathcal{H})$ is normal (resp., central) in \mathcal{G} , and z and the subsequent maps are defined if \mathcal{G} is commutative. We use [SP, 07SM] to enlarge a Σ_0 of §3.1 and descend ι to a monomorphism $\iota_{\Sigma_0}: \mathcal{H}_{\Sigma_0} \hookrightarrow \mathcal{G}_{\Sigma_0}$ and \mathcal{Q} to $\mathcal{Q}_{\Sigma_0} := \mathcal{G}_{\Sigma_0}/\iota_{\Sigma_0}(\mathcal{H}_{\Sigma_0})$; we enlarge Σ_0 further to ensure that $\iota_{\Sigma_0}(\mathcal{H}_{\Sigma_0})$ is normal (resp., central) in \mathcal{G}_{Σ_0} if $\iota(\mathcal{H})$ is normal (resp., central) in \mathcal{G} . By [Čes15b, 4.2], all the maps of the counterparts of (\dagger) over \mathcal{O}_v for $v \in \Sigma \setminus \Sigma_0$ and over K_v for $v \in \Sigma$ are continuous (whenever defined), and hence so are all the maps of (\dagger) .

Proposition 3.11. *Assume the setup of §3.10.*

- (a) *If \mathcal{H} is smooth with connected fibers, then the map*

$$\mathcal{G}(\mathbb{A}_K^{\epsilon\Sigma}) \rightarrow \mathcal{Q}(\mathbb{A}_K^{\epsilon\Sigma})$$

is open.

- (b) *If \mathcal{G} is smooth with connected fibers, and \mathcal{H} is either smooth or isomorphic to the base change of a K -group scheme, then the map*

$$\mathcal{Q}(\mathbb{A}_K^{\epsilon\Sigma}) \rightarrow H^1(\mathbb{A}_K^{\epsilon\Sigma}, \mathcal{H})$$

is open.

- (c) *If \mathcal{Q} is smooth with geometrically connected fibers, and each of \mathcal{H} and \mathcal{G} is either smooth or isomorphic to the base change of a K -group scheme, then the map*

$$H^1(\mathbb{A}_K^{\epsilon\Sigma}, \mathcal{H}) \rightarrow H^1(\mathbb{A}_K^{\epsilon\Sigma}, \mathcal{G})$$

is open.

(d) If \mathcal{H} is commutative, $\iota(\mathcal{H})$ is normal in \mathcal{G} , and each of \mathcal{G} and \mathcal{Q} is either smooth or isomorphic to the base change of a K -group scheme, then the map

$$H^1(\mathbb{A}_K^{\varepsilon\Sigma}, \mathcal{G}) \xrightarrow{x} H^1(\mathbb{A}_K^{\varepsilon\Sigma}, \mathcal{Q})$$

is open.

Proof. We apply Lemma 2.2 to enlarge Σ_0 to ensure that $\mathbb{A}_{K, \Sigma_0}^{\varepsilon\Sigma}$ -descents inherit smoothness or fibral geometric connectedness. By Lemma 2.4 and Theorems 2.13 and 2.18, the $H^n(\mathbb{A}_K^{\varepsilon\Sigma}, -)$ in (a)–(d) have bijective comparison maps (3.1.3). Thus, by [Čes15b, 4.3] (i.e., by an analogue of (a)–(d) with $\mathbb{A}_K^{\varepsilon\Sigma}$ replaced by \mathcal{O}_v with $v \in \Sigma \setminus \Sigma_0$ or by K_v with $v \in \Sigma$) and by the openness of (3.1.2), it remains to argue the following claims:

- For (a), the surjectivity of $\mathcal{G}_{\Sigma_0}(\mathcal{O}_v) \rightarrow \mathcal{Q}_{\Sigma_0}(\mathcal{O}_v)$ for $v \in \Sigma \setminus \Sigma_0$;
- For (b), the surjectivity of $\mathcal{Q}_{\Sigma_0}(\mathcal{O}_v) \rightarrow H^1(\mathcal{O}_v, \mathcal{H}_{\Sigma_0})$ for $v \in \Sigma \setminus \Sigma_0$;
- For (c), the surjectivity of $H^1(\mathcal{O}_v, \mathcal{H}_{\Sigma_0}) \rightarrow H^1(\mathcal{O}_v, \mathcal{G}_{\Sigma_0})$ for $v \in \Sigma \setminus \Sigma_0$;
- For (d), the surjectivity of $H^1(\mathcal{O}_v, \mathcal{G}_{\Sigma_0}) \rightarrow H^1(\mathcal{O}_v, \mathcal{Q}_{\Sigma_0})$ for $v \in \Sigma \setminus \Sigma_0$.

We turn to proving these surjectivity claims.

- (a)–(b) The surjectivity follows by combining Corollary 2.9 with the cohomology sequence over \mathcal{O}_v .
- (c) To lift a $(\mathcal{G}_{\Sigma_0})_{\mathcal{O}_v}$ -torsor \mathcal{T} to an $(\mathcal{H}_{\Sigma_0})_{\mathcal{O}_v}$ -torsor, we need to show that $\mathcal{T}/(\mathcal{H}_{\Sigma_0})_{\mathcal{O}_v}$, which is fppf locally isomorphic to $(\mathcal{Q}_{\Sigma_0})_{\mathcal{O}_v}$, has an \mathcal{O}_v -point. Since $\mathcal{T}/(\mathcal{H}_{\Sigma_0})_{\mathcal{O}_v}$ is a left homogeneous space under the inner form $\text{Aut}_{\mathcal{G}_{\Sigma_0}}(\mathcal{T})$ of \mathcal{G}_{Σ_0} , it has an \mathbb{F}_v -point by Lemma 2.7 (b). To lift an \mathbb{F}_v -point of the smooth \mathcal{O}_v -algebraic space $\mathcal{T}/(\mathcal{H}_{\Sigma_0})_{\mathcal{O}_v}$ to an \mathcal{O}_v -point, we apply [Čes15b, B.8].
- (d) Let $\mathbf{B}\mathcal{G}_{\Sigma_0}$ and $\mathbf{B}\mathcal{Q}_{\Sigma_0}$ be the classifying stacks of \mathcal{G}_{Σ_0} and \mathcal{Q}_{Σ_0} . By [Čes15b, A.4 (d) and B.8], $\mathbf{B}\mathcal{G}_{\Sigma_0} \rightarrow \mathbf{B}\mathcal{Q}_{\Sigma_0}$ is a smooth morphism of $\mathbb{A}_{K, \Sigma_0}^{\varepsilon\Sigma}$ -algebraic stacks whose essential surjectivity on \mathcal{O}_v -points follows from that on \mathbb{F}_v -points supplied by Lemma 2.7 (d). \square

4. DISCRETENESS OF THE IMAGE OF GLOBAL COHOMOLOGY INSIDE ADELIC COHOMOLOGY

The main goals of this section are Lemma 4.4 (c) and Propositions 4.10 and 4.12. These results go most of the way towards proving the sought Poitou–Tate sequence (\star), which is derived in §5. Topological arguments that involve the adelic topologies of §3 play an important role throughout §§4–6, so we begin the discussion by summarizing in §4.1 the necessary topological facts. We proceed to use these facts without explicit notice.

4.1. $H^n(\mathbb{A}_K^{\varepsilon\Sigma}, G)$ for a commutative finite G . Let G be a commutative finite K -group scheme and let $\mathcal{G} \rightarrow U$ be its commutative finite flat model over a nonempty open $U \subset S$. For $n \geq 0$, we endow $H^n(\mathbb{A}_K^{\varepsilon\Sigma}, G)$ with the topology of §3.1, so Proposition 3.4 (b) provides a homeomorphism

$$H^n(\mathbb{A}_K^{\varepsilon\Sigma}, G) \cong \prod'_{v \in \Sigma} H^n(K_v, G)$$

induced by the pullback maps, where the restricted product is with respect to the open subgroups

$$H^n(\mathcal{O}_v, \mathcal{G}) \subset H^n(K_v, G) \quad \text{with } v \in U \cap \Sigma.$$

By Propositions 3.5 (c), 3.7 (2), 3.8, and 3.9, each $H^n(\mathbb{A}_K^{\varepsilon\Sigma}, G)$ is a locally compact Hausdorff abelian topological group that is discrete if $n \geq 2$. If $n \geq 3$, then $H^n(\mathbb{A}_K^{\varepsilon\Sigma}, G)$ is finite, and, in fact, then

$$H^n(\mathbb{A}_K^{\varepsilon\Sigma}, G) \cong \prod_{v \text{ real}} H^n(K_v, G).$$

4.2. The \widehat{H}^0 modification. Throughout §4–6, if v is an archimedean place, then we implicitly make the Tate modification: $H^n(K_v, -)$ stands for $\widehat{H}^n(K_v, -)$ (this has no effect for $n \geq 1$). We therefore write $\widehat{H}^0(\mathbb{A}_K^{\infty\Sigma}, -)$ for $\prod'_{v \in \Sigma} H^0(K_v, -)$, where the Tate modification is in effect for archimedean v , and reserve $H^0(\mathbb{A}_K^{\infty\Sigma}, -)$ for honest global sections. In the setting of §4.1,

$$H^0(\mathbb{A}_K^{\infty\Sigma}, G) \cong \prod_{v \in \Sigma} H^0(K_v, G),$$

so there is a surjection

$$H^0(\mathbb{A}_K^{\infty\Sigma}, G) \rightarrow \widehat{H}^0(\mathbb{A}_K^{\infty\Sigma}, G).$$

The principal advantage of $\widehat{H}^0(\mathbb{A}_K^{\infty\Sigma}, -)$ over $H^0(\mathbb{A}_K^{\infty\Sigma}, -)$ is the perfectness of the pairings (4.9.4).

4.3. The setup. Throughout §4–6 we fix commutative Cartier dual finite flat $(S - \Sigma)$ -group schemes G and H and adopt the following notation (together with its analogue for H):

- (1) For $n \in \mathbb{Z}_{\geq 0}$, we let

$$\text{loc}^n(G): H^n(S - \Sigma, G) \rightarrow H^n(\mathbb{A}_K^{\infty\Sigma}, G)$$

denote the indicated pullback, where $H^n(\mathbb{A}_K^{\infty\Sigma}, G)$ stands for $\widehat{H}^0(\mathbb{A}_K^{\infty\Sigma}, G)$ if $n = 0$;

- (2) We set $\text{III}^n(G) := \text{Ker}(\text{loc}^n(G))$, so that we also have

$$\text{III}^n(G) = \text{Ker}(H^n(S - \Sigma, G) \rightarrow \prod_{v \in \Sigma} H^n(K_v, G)). \quad (4.3.1)$$

Likewise, for a commutative finite flat group scheme \mathcal{G} over a nonempty open $U \subset S$, we set

$$\text{III}^n(\mathcal{G}) := \text{Ker}\left(H^n(U, \mathcal{G}) \rightarrow \prod_{v \notin U} H^n(K_v, \mathcal{G})\right).$$

Lemma 4.4. *We fix an $n \in \mathbb{Z}_{\geq 0}$ and spread out G and H to Cartier dual commutative finite flat U -group schemes \mathcal{G} and \mathcal{H} for some open $U \subset S$ that contains $S - \Sigma$.*

- (a) *If $n \notin \{0, 1, 2, 3\}$ (resp., if $U \neq S$ and $n \notin \{1, 2\}$), then*

$$\text{III}^n(\mathcal{G}) = 0 \quad \text{and} \quad \text{III}^n(\mathcal{H}) = 0.$$

If $n \in \{0, 1, 2, 3\}$, then there is a perfect bilinear pairing of finite abelian groups

$$\text{III}^n(\mathcal{G}) \times \text{III}^{3-n}(\mathcal{H}) \xrightarrow{[\text{Ces14a, 2.16}]}} \mathbb{Q}/\mathbb{Z}. \quad (4.4.1)$$

- (b) *If U is small enough, then the pullback $H^n(U, \mathcal{G}) \rightarrow H^n(S - \Sigma, G)$ maps $\text{III}^n(\mathcal{G})$ isomorphically onto $\text{III}^n(G)$, and likewise with H in place of G . Thus, $\text{III}^n(G)$ and $\text{III}^n(H)$ are finite.*

- (c) *If $n \notin \{0, 1, 2, 3\}$ (resp., if $(S - \Sigma) \neq S$ and $n \notin \{1, 2\}$), then*

$$\text{III}^n(G) = 0 \quad \text{and} \quad \text{III}^n(H) = 0.$$

Moreover, for small enough U , (4.4.1) induces perfect pairings of finite abelian groups

$$\text{III}^1(G) \times \text{III}^2(H) \rightarrow \mathbb{Q}/\mathbb{Z} \quad \text{and} \quad \text{III}^2(G) \times \text{III}^1(H) \rightarrow \mathbb{Q}/\mathbb{Z}. \quad (4.4.2)$$

Parts (a) and (b) of Lemma 4.4 serve as buildup to the key part (c): part (a) proves (c) in the case when Σ is finite, whereas (b) facilitates reduction to this special case.

Remark 4.5. The pairings (4.4.2) a priori depend on the choice of a small enough U . Our reliance on the Artin–Verdier duality proved in [Mil06, III.§8] forces us to use flat cohomology with compact supports defined in [Mil06, III.0.6 (b)] using completions, and not Henselizations as in [Mil06, III.0.4], which makes the functorialities needed for the independence of U harder to check.

Proof of Lemma 4.4.

- (a) The perfectness of (4.4.1) was established in [Čes14a, 2.16] (and earlier in [GA09, 4.7] in a special case). Thus, the claimed vanishing for $n = 3$ follows from that for $n = 0$, which is evident from (4.3.1). The vanishing for $n \geq 4$ follows from [Mil06, III.0.6 (a), III.3.2, and III.8.2], according to which the map in (4.3.1) is an isomorphism for $n \geq 4$ if Σ is finite.
- (b) We let U' range over the open subsets of U that contain $S - \Sigma$.

Claim 4.5.1. For each U' , one has $\#\text{III}^n(\mathcal{G}_{U'}) \leq \#\text{III}^n(\mathcal{G})$.

Proof. Part (a) reduces to the $n = 1$ case, in which the Cartesian square

$$\begin{array}{ccc} H^1(U, \mathcal{G}) & \hookrightarrow & H^1(U', \mathcal{G}) \\ \downarrow & & \downarrow \\ \prod_{v \in U \setminus U'} H^1(\mathcal{O}_v, \mathcal{G}) & \hookrightarrow & \prod_{v \in U \setminus U'} H^1(K_v, \mathcal{G}) \end{array} \quad (4.5.2)$$

leads to the inclusion $\text{III}^1(\mathcal{G}_{U'}) \subset \text{III}^1(\mathcal{G})$ (for a proof of Cartesianness, see [Čes15a, 4.3]). \square

By Lemma 2.1,

$$H^n(S - \Sigma, G) = \varinjlim_{U'} H^n(U', \mathcal{G}_{U'}),$$

so a fixed $x \in \text{III}^n(G)$ comes from $\text{III}^n(\mathcal{G}_{U'})$ for every small enough U' . Thus, (a) and Claim 4.5.1 prove the finiteness of $\text{III}^n(G)$ and supply a U' that works for every x . We shrink this U' until $\#\text{III}^n(\mathcal{G}_{U'}) = \#\text{III}^n(G)$ to arrange $\text{III}^n(\mathcal{G}_{U'}) \cong \text{III}^n(G)$. We then rename the new U' to U and repeat the same steps with G replaced by H to arrive at a desired small enough U .

- (c) The claim follows by combining (a) and (b). \square

Remarks.

4.6. Lemma 4.4 (b) implies in particular that fixed $x, y \in H^n(S - \Sigma, G)$ have the same pullback to every $H^n(K_v, G)$ with $v \in \Sigma$ as soon as such pullbacks agree for v in some large finite subset of Σ . To see this, choose U in Lemma 4.4 (b) so that x and y come from $H^n(U, \mathcal{G})$ and let the places outside U comprise the subset in question.

4.7. See [Mil06, I.4.10 (a)] and [GA09, 4.6–4.8] for special cases of Lemma 4.4.

4.8. The invariant map. We define it to be the composite

$$\text{inv}: H^2(\mathbb{A}_K^{\epsilon\Sigma}, \mathbb{G}_m) \stackrel{2.13}{\cong} \bigoplus_{v \in \Sigma} H^2(K_v, \mathbb{G}_m) \xrightarrow{\sum_{v \in \Sigma} \text{inv}_v} \mathbb{Q}/\mathbb{Z},$$

where the isomorphism is induced by the pullbacks and the inv_v are the local invariant maps.

4.9. The duality pairings. Cup product and Cartier duality give the bilinear pairing

$$H^1(\mathbb{A}_K^{\epsilon\Sigma}, G) \times H^1(\mathbb{A}_K^{\epsilon\Sigma}, H) \rightarrow H^2(\mathbb{A}_K^{\epsilon\Sigma}, \mathbb{G}_m) \xrightarrow{\text{inv}} \mathbb{Q}/\mathbb{Z}, \quad (4.9.1)$$

as well as the analogous pairings

$$H^0(\mathbb{A}_K^{\epsilon\Sigma}, G) \times H^2(\mathbb{A}_K^{\epsilon\Sigma}, H) \rightarrow \mathbb{Q}/\mathbb{Z} \quad \text{and} \quad H^2(\mathbb{A}_K^{\epsilon\Sigma}, G) \times H^0(\mathbb{A}_K^{\epsilon\Sigma}, H) \rightarrow \mathbb{Q}/\mathbb{Z}. \quad (4.9.2)$$

The pairings (4.9.1) and (4.9.2) identify with the sum of the pairings

$$H^n(K_v, G) \times H^{2-n}(K_v, H) \rightarrow H^2(K_v, \mathbb{G}_m) \xrightarrow{\text{inv}_v} \mathbb{Q}/\mathbb{Z} \quad \text{for } v \in \Sigma, \quad (4.9.3)$$

in which the Tate modification may be taken to be in play for archimedean v . The pairings (4.9.2) therefore induce bilinear pairings

$$\hat{H}^0(\mathbb{A}_K^{\epsilon\Sigma}, G) \times H^2(\mathbb{A}_K^{\epsilon\Sigma}, H) \rightarrow \mathbb{Q}/\mathbb{Z} \quad \text{and} \quad H^2(\mathbb{A}_K^{\epsilon\Sigma}, G) \times \hat{H}^0(\mathbb{A}_K^{\epsilon\Sigma}, H) \rightarrow \mathbb{Q}/\mathbb{Z}. \quad (4.9.4)$$

Proposition 4.10.

- (a) If \mathbb{Q}/\mathbb{Z} is taken to be discrete, then the pairings (4.9.1)–(4.9.4) are continuous.
- (b) The pairings (4.9.1), (4.9.3) with the Tate modification in play, and (4.9.4) are perfect, i.e., they exhibit their terms as Pontryagin dual locally compact Hausdorff abelian topological groups.
- (c) The images of the following pairs of maps are orthogonal complements under (4.9.1) or (4.9.4):

$$\begin{aligned}
H^0(S - \Sigma, G) &\xrightarrow{\text{loc}^0(G)} \widehat{H}^0(\mathbb{A}_K^{\varepsilon\Sigma}, G) & \text{and} & & H^2(S - \Sigma, H) &\xrightarrow{\text{loc}^2(H)} H^2(\mathbb{A}_K^{\varepsilon\Sigma}, H), \\
H^1(S - \Sigma, G) &\xrightarrow{\text{loc}^1(G)} H^1(\mathbb{A}_K^{\varepsilon\Sigma}, G) & \text{and} & & H^1(S - \Sigma, H) &\xrightarrow{\text{loc}^1(H)} H^1(\mathbb{A}_K^{\varepsilon\Sigma}, H), \\
H^2(S - \Sigma, G) &\xrightarrow{\text{loc}^2(G)} H^2(\mathbb{A}_K^{\varepsilon\Sigma}, G) & \text{and} & & H^0(S - \Sigma, H) &\xrightarrow{\text{loc}^0(H)} \widehat{H}^0(\mathbb{A}_K^{\varepsilon\Sigma}, H).
\end{aligned}$$

Proof. We spread out G and H to Cartier dual commutative finite flat U -group schemes \mathcal{G} and \mathcal{H} for an open $U \subset S$ that contains $S - \Sigma$.

- (a) The invariant maps are continuous because their sources are discrete by Proposition 3.7 (2). Moreover, the first map of (4.9.3) is continuous by [Čes14a, A.3]. It remains to note that if $v \in U \cap \Sigma$, then the pairing (4.9.3) vanishes on $H^n(\mathcal{O}_v, \mathcal{G}) \times H^{2-n}(\mathcal{O}_v, \mathcal{H})$ (e.g., by Corollary 2.9).
- (b) Tate–Shatz local duality [Mil06, I.2.3, I.2.13 (a), III.6.10] ensures perfectness of (4.9.3).³ Perfectness of (4.9.1) and (4.9.4) then follows from [Tat50, Thm. 3.2.1], because

$$H^n(\mathcal{O}_v, \mathcal{G}) \subset H^n(K_v, G) \quad \text{and} \quad H^{2-n}(\mathcal{O}_v, \mathcal{H}) \subset H^{2-n}(K_v, H) \quad \text{for } v \in U \cap \Sigma \quad (4.10.1)$$

are compact open orthogonal complements by [Čes15b, 3.10] and [Mil06, III.1.4 and III.7.2].

- (c) In this proof, we abbreviate $\widehat{H}^0(\mathbb{A}_K^{\varepsilon\Sigma}, -)$ by $H^0(\mathbb{A}_K^{\varepsilon\Sigma}, -)$. By [Čes14b, 5.3], the images of
$$H^n(U, \mathcal{G}) \rightarrow \bigoplus_{v \notin U} H^n(K_v, G) \quad \text{and} \quad H^{2-n}(U, \mathcal{H}) \rightarrow \bigoplus_{v \notin U} H^{2-n}(K_v, H) \quad (4.10.2)$$
are orthogonal complements. Therefore, by shrinking U and combining the orthogonality of the subgroups in (4.10.1) with the limit formalism supplied by Lemma 2.1, we deduce the orthogonality part of the claim.

We use Lemma 4.4 (b) to shrink U to arrange that the pullback induces an isomorphism $\text{III}^n(\mathcal{G}_{U'}) \cong \text{III}^n(G)$ for every open $U' \subset U$ that contains $S - \Sigma$ and likewise for H . We let

$$x = (x_v)_{v \in \Sigma} \quad \text{in} \quad H^n(\mathbb{A}_K^{\varepsilon\Sigma}, G) = \prod'_{v \in \Sigma} H^n(K_v, G)$$

be orthogonal to $\text{Im}(\text{loc}^{2-n}(H))$ and we shrink U to ensure that $x_v \in H^n(\mathcal{O}_v, \mathcal{G})$ for $v \in U \cap \Sigma$.

Due to the settled orthogonality part, the following claim together with its analogue for H will finish the proof.

Claim 4.10.3. The element x lies in the image of

$$\text{loc}^n(G): H^n(S - \Sigma, G) \rightarrow H^n(\mathbb{A}_K^{\varepsilon\Sigma}, G).$$

Proof. The images of the maps in (4.10.2) are orthogonal complements, so $(x_v)_{v \notin U}$ is the pullback of some $y \in H^n(U, \mathcal{G})$. We replace x by $x - y|_{H^n(\mathbb{A}_K^{\varepsilon\Sigma}, G)}$ to reduce to the case when $x_v = 0$ for $v \notin U$. Due to Corollary 2.9, this settles the $n = 2$ case.

In the $n = 1$ case, we let $U' \subset U$ be an open that contains $S - \Sigma$, so that (4.10.2) also gives a $y' \in H^n(U', \mathcal{G})$ that pulls back to $(x_v)_{v \notin U'}$. Since the square (4.5.2) is Cartesian,

³The agreement of the topologies used here and in loc. cit. is guaranteed by [Čes15b, 3.5 (c), 5.11, and 6.5].

$y' \in H^1(U, \mathcal{G})$, and even $y' \in \text{III}^1(\mathcal{G})$. But then the choice of U forces $y' \in \text{III}^1(\mathcal{G}_{U'})$, so $x_v = 0$ for $v \notin U'$. We vary U' to conclude that $x = 0$.

In the $n = 0$ case, we proceed similarly: if Σ is finite, we choose $U = S - \Sigma$; else, we choose $U \neq S$, vary U' , and use the injectivity of

$$H^0(U', \mathcal{G}) \rightarrow H^0(K_v, G) \quad \text{for } v \nmid \infty$$

to prove that $x = 0$. □ □

Corollary 4.11. *If $n \geq 3$, then $\text{loc}^n(G)$ is a surjection*

$$H^n(S - \Sigma, G) \twoheadrightarrow \prod_{\text{real } v} H^n(K_v, G).$$

If either $n \geq 3$ with $(S - \Sigma) \neq S$ or $n \geq 4$, then $\text{loc}^n(G)$ is an isomorphism

$$H^n(S - \Sigma, G) \cong \prod_{\text{real } v} H^n(K_v, G).$$

Proof. If $\#\Sigma < \infty$, then the fact that the images of (4.10.2) are orthogonal complements (proved in [Čes14b, 5.3]) gives the surjectivity claim. Surjectivity in general then follows from a limit argument. To obtain the claim of the last sentence it remains to apply Lemma 4.4 (c). □

Proposition 4.12. *For $n \in \mathbb{Z}_{\geq 0}$, the image*

$$\text{Im}(\text{loc}^n(G)) \subset H^n(\mathbb{A}_K^{\varepsilon\Sigma}, G)$$

is closed, discrete, and cocompact (here and in the proof, H^n for $n = 0$ abbreviates \widehat{H}^0), and likewise with H in place of G .

Proof. If $n \geq 3$, then $H^n(\mathbb{A}_K^{\varepsilon\Sigma}, G)$ itself is finite and discrete, so we assume that $n \leq 2$.

Proposition 4.10 (a) and (c) imply closedness. Proposition 4.10 (c) and [HR79, 24.11] then identify

$$H^n(\mathbb{A}_K^{\varepsilon\Sigma}, G) / \text{Im}(\text{loc}^n(G)) \quad \text{and} \quad \text{Im}(\text{loc}^{2-n}(H))$$

as Pontryagin duals, so it remains to prove discreteness.

If $n = 2$, then $H^2(\mathbb{A}_K^{\varepsilon\Sigma}, G)$ itself is discrete. If $n = 0$, then the discreteness of $\text{Im}(\text{loc}^n(G))$ follows from its finiteness. If $n = 1$, then we first spread out G to a commutative finite flat U -group scheme \mathcal{G} for an open $U \subset S$ that contains $S - \Sigma$. Then we apply [Čes15a, 4.3] to deduce that

$$\text{Im}(\text{loc}^1(G)) \cap \left(\prod_{v \in U \cap \Sigma} H^1(\mathcal{O}_v, \mathcal{G}) \times \prod_{v \notin U} H^1(K_v, G) \right)$$

is the image of the pullback map

$$H^1(U, \mathcal{G}) \rightarrow \prod_{v \in U \cap \Sigma} H^1(\mathcal{O}_v, \mathcal{G}) \times \prod_{v \notin U} H^1(K_v, G).$$

The discreteness of $\text{Im}(\text{loc}^1(G))$ now follows from [Čes14a, 2.9], which ensures that the pullback

$$H^1(U, \mathcal{G}) \rightarrow \bigoplus_{v \notin U} H^1(K_v, G)$$

has a discrete image and a finite kernel. □

Question 4.13. *Is the image of the pullback map*

$$H^1(S - \Sigma, G) \rightarrow H^1(\mathbb{A}_K^{\varepsilon\Sigma}, G)$$

still closed and discrete if G is only assumed to be a separated $(S - \Sigma)$ -group scheme of finite type?

We do not know the answer to Question 4.13 even in the case when Σ consists of all the places.

5. POITOU–TATE

With the results of §4 at hand, we are ready for the sought Poitou–Tate sequence (5.1.1). We continue to assume the setup of §4.3 and adopt the convention introduced in §4.2.

Theorem 5.1. *For Cartier dual commutative finite flat $(S - \Sigma)$ -group schemes G and H , the sequence*

$$\begin{array}{c}
 H^0(S - \Sigma, G) \xrightarrow{\text{loc}^0(G)} \widehat{H}^0(\mathbb{A}_K^{\infty\Sigma}, G) \xrightarrow{\text{loc}^2(H)^*} H^2(S - \Sigma, H)^* \\
 \left. \begin{array}{c} \xrightarrow{\hspace{10em}} \\ \xrightarrow{\hspace{10em}} \end{array} \right\} \\
 H^1(S - \Sigma, G) \xrightarrow{\text{loc}^1(G)} H^1(\mathbb{A}_K^{\infty\Sigma}, G) \xrightarrow{\text{loc}^1(H)^*} H^1(S - \Sigma, H)^* \\
 \left. \begin{array}{c} \xrightarrow{\hspace{10em}} \\ \xrightarrow{\hspace{10em}} \end{array} \right\} \\
 H^2(S - \Sigma, G) \xrightarrow{\text{loc}^2(G)} H^2(\mathbb{A}_K^{\infty\Sigma}, G) \xrightarrow{\text{loc}^0(H)^*} H^0(S - \Sigma, H)^*
 \end{array} \tag{5.1.1}$$

is exact, where the $\text{loc}^n(H)^*$ are defined using the perfectness of (4.9.1) and (4.9.4) supplied by Proposition 4.10 (b), the curved arrows are defined using the perfectness of (4.4.2),⁴ and all the maps are continuous when $H^n(S - \Sigma, G)$ and $H^n(S - \Sigma, H)$ are endowed with the discrete topology. Moreover, if $(S - \Sigma) \neq S$, then $\text{loc}^0(G)$ is injective and $\text{loc}^0(H)^*$ is surjective.

Proof. In this proof, $H^n(\mathbb{A}_K^{\infty\Sigma}, -)$ for $n = 0$ abbreviates $\widehat{H}^n(\mathbb{A}_K^{\infty\Sigma}, -)$.

For $n \in \{0, 1, 2\}$, [HR79, 24.11] with Proposition 4.10 identifies the exact sequences

$$\begin{array}{l}
 0 \rightarrow \text{Im}(\text{loc}^n(G)) \rightarrow H^n(\mathbb{A}_K^{\infty\Sigma}, G) \rightarrow H^n(\mathbb{A}_K^{\infty\Sigma}, G)/\text{Im}(\text{loc}^n(G)) \rightarrow 0 \quad \text{and} \\
 0 \rightarrow (H^{2-n}(\mathbb{A}_K^{\infty\Sigma}, H)/\text{Im}(\text{loc}^{2-n}(H)))^* \rightarrow H^{2-n}(\mathbb{A}_K^{\infty\Sigma}, H)^* \rightarrow \text{Im}(\text{loc}^{2-n}(H))^* \rightarrow 0.
 \end{array} \tag{5.1.2}$$

Moreover, [HR79, 24.11] with the discreteness aspect of Proposition 4.12 gives the exact sequence

$$0 \rightarrow \text{Im}(\text{loc}^{2-n}(H))^* \rightarrow H^{2-n}(S - \Sigma, H)^* \rightarrow \text{III}^{2-n}(H)^* \rightarrow 0. \tag{5.1.3}$$

The exactness of (5.1.1) and the continuity of its maps follow. The claim about the injectivity of $\text{loc}^0(G)$ and the surjectivity of $\text{loc}^0(H)^*$ follows from Lemma 4.4 (c) and (5.1.3). \square

Remarks.

5.2. The maps in (5.1.1) are strict, i.e., the quotient and subspace topologies on their images agree: for $\text{loc}^n(G)$, this follows from Proposition 4.12; for $\text{loc}^n(H)^*$, this follows from (5.1.2) and (5.1.3); for the curved arrows, one also uses the finiteness of $\text{III}^{2-n}(H)^*$ supplied by Lemma 4.4 (c).

5.3. If $(S - \Sigma) = S$ and either K has no real places or $\#G$ is odd, then (5.1.1) recovers dualities between

$$H^1(S, G) \quad \text{and} \quad H^2(S, H)$$

and between

$$H^2(S, G) \quad \text{and} \quad H^1(S, H).$$

Example 5.4. We present a concrete cohomological computation that illustrates the expectation that a suitable version of the original Poitou–Tate sequence (\star) may hold without the assumption on the “ramification set” Σ (Theorem 5.1 fulfills this expectation). For this, we will compute the order of $H^2(S - \Sigma, \mu_p)$ in terms of other invariants of K and Σ , will observe that the computation does not need assumptions on the residue characteristics of the finite places in Σ , and will then show that the same order may also be computed from the Poitou–Tate sequence with $G = \mathbb{Z}/p\mathbb{Z}$ and $H = \mu_p$.

⁴If Σ is infinite, then the curved arrows inherit from (4.4.2) an a priori dependence on noncanonical choices, see Remark 4.5.

- Exactness at $\frac{H^1(\mathbb{A}_K^{\varepsilon\Sigma}, G)}{\text{Sel}(G_{\mathbb{A}_K^{\varepsilon\Sigma}})}$ follows from [HR79, 24.11], the analogue of Claim 6.2.3 for G , and the discreteness aspect of Proposition 4.12 (see the proof of Theorem 5.1 for an analogous argument).
- Exactness at $\text{Sel}(H)^*$ follows from the definition of $x(G)$ and the commutativity of (6.2.2).
- Exactness at $H^2(S - \Sigma, G)$ and at $H^2(\mathbb{A}_K^{\varepsilon\Sigma}, G)$ follows from the exactness of (5.1.1). \square

Remarks.

6.3. Similarly to Remark 5.2, the maps in (6.2.1) are strict: for all the maps except $y(G)$ and $x(G)$, this follows from Remark 5.2 and [BouTG, III.28, Cor. 3]; for $x(G)$, one also uses the bottom row of (6.2.2); for $y(G)$, this follows from the discreteness of the image of $\text{loc}^1(H)|_{\text{Sel}(H)}$ supplied by Proposition 4.12.

6.4. With the choice

$$\text{Sel}(G_{\mathbb{A}_K^{\varepsilon\Sigma}}) = 0 \quad \text{and} \quad \text{Sel}(H_{\mathbb{A}_K^{\varepsilon\Sigma}}) = H^1(\mathbb{A}_K^{\varepsilon\Sigma}, H),$$

the sequence (6.2.1) specializes to a segment of the Poitou–Tate sequence (5.1.1). The choice

$$\text{Sel}(G_{\mathbb{A}_K^{\varepsilon\Sigma}}) = H^1(\mathbb{A}_K^{\varepsilon\Sigma}, G) \quad \text{and} \quad \text{Sel}(H_{\mathbb{A}_K^{\varepsilon\Sigma}}) = 0$$

recovers the duality between $\text{III}^2(G)$ and $\text{III}^1(H)$ proved in Lemma 4.4 (c).

6.5. See [Cre12, Prop. 4.6] for another proof of the exactness of (6.2.1) at the $\frac{H^1(\mathbb{A}_K^{\varepsilon\Sigma}, G)}{\text{Sel}(G_{\mathbb{A}_K^{\varepsilon\Sigma}})}$ term in the case when K is a number field, Σ is the set of all places, and $\text{Sel}(H_{\mathbb{A}_K^{\varepsilon\Sigma}}) = \prod_{v \in T} U_v \times \prod_{v \notin T} \{0\}$ with T a finite subset of Σ and U_v a subgroup of $H^1(K_v, H)$ for $v \in T$.

APPENDIX A. THE SEQUENCE (\star) RECOVERS THE SEQUENCE (\star)

We prove that (\star) is indeed a special case of (\star) , as promised in §1.2. We begin with a well-known Proposition A.1. For its statement we use [BLR90, 1.2/1] as our definition of a Néron model.

Proposition A.1. *Let D be a connected (and hence nonempty) Dedekind scheme, k its function field.*

(a) *A finite étale D -scheme X is a Néron model of its generic fiber.*

Let F be a finite étale k -group scheme and k^s a separable closure of k .

(b) *The Néron model $\mathcal{F} \rightarrow D$ of F exists and is separated quasi-finite étale. It is finite if and only if $F(k^s)$ is unramified at all the nongeneric $d \in D$ (i.e., if and only if $G_{k_d^{\text{sh}}}$ is constant for all such d , where k_d^{sh} is the fraction field of the strict Henselization of the local ring $\mathcal{O}_{D,d}$).*

(c) *The functor $\mathcal{G} \mapsto \mathcal{G}_k$ is an equivalence between the category of étale group Néron models and that of finite étale k -group schemes.*

(d) *Commutative finite étale D -group schemes \mathcal{G} form a full abelian subcategory of the category of abelian sheaves on $D_{\text{ét}}$. This category is equivalent by the functor $\mathcal{G} \mapsto \mathcal{G}(k^s)$ to the category of finite discrete $\text{Gal}(k^s/k)$ -modules that are unramified at all the nongeneric points of D .*

Proof.

(a) In checking the Néron property of X , étale descent reduces to the case of a constant X .

- (b) For the existence, (a), spreading-out, and [BLR90, 1.4/1 and 6.5/3] reduce to the case of a strictly local D , in which, by [BLR90, 7.1/1], $\mathcal{F} \rightarrow D$ is obtained from F by extending the constant subgroup $\underline{F(k)}_k \subset F$ to a constant subgroup over D . The other claims are immediate from this construction.
- (c) Essential surjectivity follows from (b). Full faithfulness follows from the Néron property.
- (d) For the abelian subcategory claim, it suffices to note that the subcategory is closed under direct sums, kernels, and quotients. The rest follows from (a), (b), and (c). \square

Proposition A.1 (d) allows us to identify the module M of (\star) with a commutative finite étale $(S - \Sigma)$ -group scheme \mathcal{M} . Since $\#\mathcal{M}$ is a unit on $(S - \Sigma)$, [TO70, p. 17, Lemma 5] applies and proves that the Cartier dual \mathcal{M}^D of \mathcal{M} is $(S - \Sigma)$ -étale. Thus, $\mathcal{M}^D(k^s)$ may be identified with M^D . To conclude that the sequence (\star) applied to $G = \mathcal{M}$ and $H = \mathcal{M}^D$ recovers (\star) , it remains to note:

- (1) One has

$$H^n(S - \Sigma, \mathcal{M}) \cong H^n(\text{Gal}(K_\Sigma/K), M),$$

and similarly for \mathcal{M}^D ;

- (2) One has

$$H^n(\mathbb{A}_K^{\epsilon\Sigma}, \mathcal{M}) \cong \prod'_{v \in \Sigma} H^n(K_v, M),$$

where the restricted product is taken with respect to the unramified cohomology subgroups and the Tate modifications are in play for $n = 0$.

Proof of (1). If Σ is finite, then a claimed isomorphism is provided by [Mil06, II.2.9]. To reduce to this case, we spread out \mathcal{M} to a finite étale $(S - \Sigma_0)$ -group scheme $\widetilde{\mathcal{M}}$ for some finite $\Sigma_0 \subset \Sigma$ for which $\#\mathcal{M}$ is a unit on $S - \Sigma_0$ and combine the isomorphism

$$H^n(\text{Gal}(K_\Sigma/K), M) \cong \varinjlim_{\Sigma_0 \subset \Sigma'_0 \subset \Sigma, \#\Sigma'_0 < \infty} H^n(\text{Gal}(K_{\Sigma'_0}/K), M)$$

provided by [Ser02, I.§2, Prop. 8] with the analogous isomorphism provided by Lemma 2.1. \square

Proof of (2). Let Σ_0 and $\widetilde{\mathcal{M}}$ be as in the proof of (1), so Theorem 2.20 identifies $H^n(\mathbb{A}_K^{\epsilon\Sigma}, \mathcal{M})$ with the restricted product $\prod'_{v \in \Sigma} H^n(K_v, M)$ with respect to the subgroups

$$H^n(\mathcal{O}_v, \widetilde{\mathcal{M}}) \subset H^n(K_v, \widetilde{\mathcal{M}}) \quad \text{for } v \in \Sigma \setminus \Sigma_0.$$

Moreover, [Gro68, 11.7 1°] identifies $H^n(K_v, \mathcal{M})$ with Galois cohomology $H^n(K_v, M)$. It remains to argue that this identification restricts to an isomorphism

$$H^n(\mathcal{O}_v, \widetilde{\mathcal{M}}) \cong H_{\text{nr}}^n(K_v, M) \quad \text{for } v \in \Sigma \setminus \Sigma_0.$$

For $n = 0$, this follows from the definitions and from the equality $\widetilde{\mathcal{M}}(\mathcal{O}_v) = \widetilde{\mathcal{M}}(K_v)$. For $n \geq 2$, both sides vanish due to Corollary 2.9 and [Ser02, II.§5.5, Prop. 18 (c)]. For $n = 1$, one combines Proposition A.1 (a) with [Čes15a, A.3 and A.4]. \square

REFERENCES

- [Art69] M. Artin, *The implicit function theorem in algebraic geometry*, Algebraic Geometry (Internat. Colloq., Tata Inst. Fund. Res., Bombay, 1968), Oxford Univ. Press, London, 1969, pp. 13–34. MR0262237 (41 #6847)
- [Azu66] Goro Azumaya, *The Brauer Group of a Commutative Ring*, National Science Foundation Advanced Science Seminar, Bowdoin College (1966). Mimeographed notes by J. E. Schneider.
- [Bha14] Bhargav Bhatt, *Algebraization and Tannaka duality*, preprint (2014). Available at <http://arxiv.org/abs/1404.7483>.

- [BLR90] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud, *Néron models*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 21, Springer-Verlag, Berlin, 1990. MR1045822 (91i:14034)
- [BouTG] Nicolas Bourbaki, *Éléments de mathématique. Topologie generale*, chap. I-IV, Hermann (1971); chap. V-X, Hermann (1974) (French).
- [Čes14a] Kęstutis Česnavičius, *p-Selmer growth in extensions of degree p*, preprint (2014). Available at <http://arxiv.org/abs/1408.1151>.
- [Čes14b] ———, *The ℓ -parity conjecture over the constant quadratic extension*, preprint (2014). Available at <http://arxiv.org/abs/1402.2939>.
- [Čes15a] ———, *Selmer groups as flat cohomology groups*, J. Ramanujan Math. Soc., to appear (2015). Available at <http://arxiv.org/abs/1301.4724>.
- [Čes15b] ———, *Topology on cohomology of local fields*, Forum Math. Sigma **3** (2015), e16, 55, DOI 10.1017/fms.2015.18.
- [CGP10] Brian Conrad, Ofer Gabber, and Gopal Prasad, *Pseudo-reductive groups*, New Mathematical Monographs, vol. 17, Cambridge University Press, Cambridge, 2010. MR2723571 (2011k:20093)
- [Con12a] Brian Conrad, *Finiteness theorems for algebraic groups over function fields*, Compos. Math. **148** (2012), no. 2, 555–639, DOI 10.1112/S0010437X11005665. MR2904198
- [Con12b] ———, *Weil and Grothendieck approaches to adelic points*, Enseign. Math. (2) **58** (2012), no. 1-2, 61–97. MR2985010
- [Cre12] Brendan Creutz, *A Grunwald-Wang type theorem for abelian varieties*, Acta Arith. **154** (2012), no. 4, 353–370, DOI 10.4064/aa154-4-2. MR2949874
- [EGA III₁] A. Grothendieck and J. Dieudonné, *Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. I*, Inst. Hautes Études Sci. Publ. Math. **11** (1961), 167. MR0217085 (36 #177c)
- [EGA IV₃] ———, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. III*, Inst. Hautes Études Sci. Publ. Math. **28** (1966), 255. MR0217086 (36 #178)
- [GA09] Cristian D. González-Avilés, *Arithmetic duality theorems for 1-motives over function fields*, J. reine angew. Math. **632** (2009), 203–231, DOI 10.1515/CRELLE.2009.055. MR2544149 (2010i:11169)
- [Gab81] Ofer Gabber, *Some theorems on Azumaya algebras*, The Brauer group (Sem., Les Plans-sur-Bex, 1980), Lecture Notes in Math., vol. 844, Springer, Berlin-New York, 1981, pp. 129–209. MR611868 (83d:13004)
- [Gir71] Jean Giraud, *Cohomologie non abélienne*, Springer-Verlag, Berlin, 1971 (French). Die Grundlehren der mathematischen Wissenschaften, Band 179. MR0344253 (49 #8992)
- [Gro68] Alexander Grothendieck, *Le groupe de Brauer. III. Exemples et compléments*, Dix Exposés sur la Cohomologie des Schémas, North-Holland, Amsterdam, 1968, pp. 88–188 (French). MR0244271 (39 #5586c)
- [HR79] Edwin Hewitt and Kenneth A. Ross, *Abstract harmonic analysis. Vol. I*, 2nd ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 115, Springer-Verlag, Berlin, 1979. Structure of topological groups, integration theory, group representations. MR551496 (81k:43001)
- [HS05] David Harari and Tamás Szamuely, *Arithmetic duality theorems for 1-motives*, J. reine angew. Math. **578** (2005), 93–128, DOI 10.1515/crll.2005.2005.578.93. MR2113891 (2006f:14053)
- [Lan56] Serge Lang, *Algebraic groups over finite fields*, Amer. J. Math. **78** (1956), 555–563. MR0086367 (19,174a)
- [Mil06] J. S. Milne, *Arithmetic duality theorems*, 2nd ed., BookSurge, LLC, Charleston, SC, 2006. MR2261462 (2007e:14029)
- [Ryd15] David Rydh, *Noetherian approximation of algebraic spaces and stacks*, J. Algebra **422** (2015), 105–147, DOI 10.1016/j.jalgebra.2014.09.012. MR3272071
- [Ser02] Jean-Pierre Serre, *Galois cohomology*, Corrected reprint of the 1997 English edition, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2002. Translated from the French by Patrick Ion and revised by the author. MR1867431 (2002i:12004)
- [SGA 3_{I new}] Philippe Gille and Patrick Polo (eds.), *Schémas en groupes (SGA 3). Tome I. Propriétés générales des schémas en groupes*, Documents Mathématiques (Paris) [Mathematical Documents (Paris)], 7, Société Mathématique de France, Paris, 2011 (French). Séminaire de Géométrie Algébrique du Bois Marie 1962–64. [Algebraic Geometry Seminar of Bois Marie 1962–64]; A seminar directed by M. Demazure and A. Grothendieck with the collaboration of M. Artin, J.-E. Bertin, P. Gabriel, M. Raynaud and J.-P. Serre; Revised and annotated edition of the 1970 French original. MR2867621
- [SGA 4_{II}] *Théorie des topos et cohomologie étale des schémas. Tome 2*, Lecture Notes in Mathematics, Vol. 270, Springer-Verlag, Berlin, 1972 (French). Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4); Dirigé par M. Artin, A. Grothendieck et J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat. MR0354653 (50 #7131)

- [SP] *The Stacks Project*. <http://stacks.math.columbia.edu>.
- [Tat50] J. T. Tate, *Fourier analysis in number fields, and Hecke's zeta-functions*, Algebraic Number Theory (Proc. Instructional Conf., Brighton, 1965), Thompson, Washington, D.C., 1967, pp. 305–347. MR0217026 (36 #121)
- [TO70] John Tate and Frans Oort, *Group schemes of prime order*, Ann. Sci. École Norm. Sup. (4) **3** (1970), 1–21. MR0265368 (42 #278)
- [Toë11] Bertrand Toën, *Descente fidèlement plate pour les n -champs d'Artin*, Compos. Math. **147** (2011), no. 5, 1382–1412, DOI 10.1112/S0010437X10005245 (French, with English and French summaries). MR2834725

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CA 94720-3840, USA

E-mail address: kestutis@berkeley.edu

URL: <http://math.berkeley.edu/~kestutis/>