Analytic description of moduli spaces

Geometric Regularization ~ equivariant transversality (Sard-Smale) & gluing (Newton iteration)

Abstract Regularization
- Why?
- Fredholm stabilization
- "stabilized gluing"

$\boxed{\text{Geometric Regularization}}$

0.) Global Fredholm description
1.) ~~Equivariant~~ Transversality
2.) Quotient Theorem
3.) Gromov Compactness & Gluing
4.) Cobordism / Continuation map

$\boxed{\text{Abstract Regularization}}$

0.) Gromov Compactness & Pregluing Topology
1.) local Fredholm descriptions
1'.) transition information
2.) regularization theorem

$$\overline{\mathcal{M}} = \bar{\partial}_J^{-1}(0) / \mathrm{Aut} \cup_{\text{glue}} \mathcal{M}^{\text{broken}}$$

$$\overline{\mathcal{M}} = s^{-1}(0)$$

Why abstract regularization?

* more general / quotable / transferrable

* regular J rarely exist
  $(\bar\partial_J \pitchfork 0)$
  ↳ when all curves are "somewhere injective"

typical obstructions to $\bar\partial_J \pitchfork 0$

if $\bar\partial_J \pitchfork 0$ then $\dim M_{0,1}(A,J) = 2n + 2c_1(A) - 4$

$2n-6 \geq 0$ $\qquad$ $2(n-k-2) < 0$ for $k \gg 1$

① multiply covered spheres

$$u_k : \mathbb{P}^1 \longrightarrow \mathbb{P}^1 \xhookrightarrow{u} C \subset M \qquad c_1(TM|_C) = -1 \Rightarrow c_1(u_k^* TM) = -k$$
$$z \longmapsto z^k$$

$$\bar\partial_J u = 0 \Rightarrow \bar\partial_J u_k = 0 \;\; \forall k \quad \text{but} \quad \operatorname{ind} D_{u_k} \bar\partial_J = 2n - 2k < 0 \quad \forall k \gg 1$$

①' nowhere injective disks

① occurs in almost all moduli spaces in almost all symplectic manifolds
by bubbling $\qquad$ except for special cases: - monotone
- exact

② branched covers of trivial cylinders

$$\mathbb{R} \times S^1 \xrightarrow{\psi} \mathbb{R} \times S^1 \longrightarrow (\mathbb{R} \times Y, J \text{ comp. w. } d(e^s \lambda)) \qquad (Y, \lambda) \text{ contact}$$
$$(s,t) \longmapsto (s, \gamma(t)) \qquad \gamma : S^1 \to Y \text{ Reeb orbit}$$

local Fredholm description

$$\begin{array}{c}\mathcal{E}\\ \downarrow \uparrow \sigma \\ \mathcal{B}\end{array}\quad \text{Fredholm section}, \quad \underbrace{\sigma^{-1}(0)}_{\text{finite symmetry group}} \xhookrightarrow[\text{local homeom}]{} \overline{\mathcal{M}}$$

obstruction "bundle" | **stabilization** | **finite dim. reduction**

$$\begin{array}{c} \bigcup_{b\in\sigma^{-1}(0)} \mathcal{E}_b/\operatorname{im} D_b\sigma \\ \downarrow \\ \sigma^{-1}(0)\end{array} \rightsquigarrow \quad E \subset \mathcal{E}|_{\text{nbhd}(\sigma^{-1}(0))}$$

$$E_b + \operatorname{im} D_b\sigma = \mathcal{E}_b$$

⚠ fiber dimension generally not constant

$$\sigma_E : \mathcal{B}\times E \to \mathcal{E} \qquad \pitchfork 0$$
$$(b,e) \mapsto \sigma(b)+e$$

$$\sigma^{-1}(0) \simeq \operatorname{zero}\begin{pmatrix}\sigma_E^{-1}(0) \to E\\ (b,e)\mapsto e\end{pmatrix}$$

$$\Rightarrow \quad \begin{array}{c} U\times E \\ \downarrow \uparrow s \\ U \end{array} \circlearrowleft \Gamma \text{ finite}$$

$$s^{-1}(0)/\Gamma \xhookrightarrow[\text{local homeom.}]{} \mathcal{M}$$

$$U = \sigma_E^{-1}(0)$$
$$s : (b,e) \mapsto e$$

$$\sigma_E^{-1}(0) \simeq \{ b\in\mathcal{B} \mid \sigma(b)\in E\} \quad \text{smooth manifold}$$

$$\begin{array}{c}(b,e)\\ \downarrow \\ e\end{array} \simeq \begin{array}{c} b \\ \downarrow \\ \sigma(b)\end{array}$$

<u>PHILOSOPHY</u>: "$[\sigma^{-1}(0)]$" $=$ Euler class of $\begin{array}{c}E\\ \downarrow \\ \sigma_E^{-1}(0)\end{array}$ $= [(s+\nu)^{-1}(0)]$

(Γ-equivariant)

$$\begin{array}{c}E\\ \uparrow \bar{0}\end{array},\ \simeq \{\bar{0},\ u\in E\}$$

$$\left((s+\nu)^{-1}(0)/\Gamma\right)$$

1.) **local Fredholm descriptions** (a) on $\bar\partial_J^{-1}(0)/\mathrm{Aut}$ $\quad \begin{matrix}\mathcal{E} \\ \downarrow \uparrow \bar\partial_J = \sigma \\ \text{local slice} \hookrightarrow \mathcal{B}/\mathrm{Aut}\end{matrix}$

(b) near $\mathcal{M}^{\text{broken}}$ $\xrightarrow{\text{polyfold approach}}$ "by interpreting pregluing as chart of generalized Banach manifold" $\to$ later

"by gluing after stabilization"

$$\mathcal{M}^{\text{broken}} \supset \mathrm{Nbhd}([u_-],[u_+]) \simeq \sigma_-^{-1}(0) \underset{\mathrm{Per}}{\times} \sigma_+^{-1}(0)$$

$$\mathcal{E}_\pm = \bigcup L^p(v^*TM) \quad \begin{matrix}\downarrow \uparrow \sigma_\pm = \bar\partial_{J,M}\end{matrix}$$

$$\mathcal{B}_\pm = \left\{ \begin{matrix} v \in W^{1,p}(\mathbb{R}\times S^1, M) \\ \lim_{s\to\pm\infty} v(s,\cdot) = \ldots\ldots \\ \exists!\, t \in S^1 : v(0,t) \in H_\pm \end{matrix} \right\}$$

- stabilize by $E_\pm \subset \mathcal{E}_\pm$

$$\sigma^\pm_{E_\pm} : \mathcal{B}_\pm \times E_\pm \to \mathcal{E}_\pm \quad \pitchfork 0$$
$$(v,e) \mapsto \sigma(v) - e$$

$$\simeq \mathrm{Zero}\left(\begin{matrix} E_- \times E_+ \quad (e_-,e_+) \\ \downarrow \qquad \uparrow \\ \left\{ \begin{matrix} ([v_-],[v_+],e_-,e_+) \\ \bar\partial_{J,M} v_\pm = e_\pm \end{matrix} \right\} \end{matrix} \right) = (\sigma^-_{E_-} \times \sigma^+_{E_+})^{-1}(0)$$

- glue $\quad (\sigma^-_{E_-} \times \sigma^+_{E_+})^{-1}(0) \times (R_1,\infty) \xrightarrow[\text{local homeo}]{} \left\{ \begin{pmatrix} v \approx v_- \#_R v_+ \\ e \approx e_- \#_R e_+ \end{pmatrix} \;\middle|\; \bar\partial_{J,M} v = e \right\}$

$$\Rightarrow \overline{\mathcal{M}} \supset \mathrm{Nbhd}([u_-],[u_+]) \simeq \mathrm{Zero}\left(\begin{matrix} E_- \times E_+ \\ \downarrow \\ (\sigma^-_{E_-} \times \sigma^+_{E_+})^{-1}(0) \times (R_1,\infty] \end{matrix} \right) \supset \{R<\infty\} \xrightarrow{g} \left\{ (v,e) \;\middle|\; \bar\partial_{J,M} v = \underbrace{e}_{e_- \#_R e_+} \right\} \xrightarrow{} (e_-,e_+)$$

⚠ need $E_\pm$ "somewhat geometric":

- $e_\pm$ "supported outside of pregluing region"
- $e_- \# e_+$ is a differentiable section (despite failure of $(v,R) \mapsto v(\cdot \pm R)$)

$$\begin{matrix} E_\pm \\ \downarrow \uparrow e_\pm \\ \mathcal{B}_\pm \end{matrix} \rightsquigarrow \quad e_- \# e_+ : \mathcal{B}_- \times \mathcal{B}_+ \times (R_0,\infty) \longrightarrow \mathcal{E}$$
$$(v_-,v_+,R) \longmapsto e_-(v_-(\cdot + R)) + e_+(v_+(\cdot - R))$$

1'.) transition information

for $\overline{\mathcal{M}}_{0,1}(A;J) = \{u : \mathbb{P}^1 \to M \mid u_*[\mathbb{P}^1] = A, \bar\partial_J u = 0\} / \{\varphi(\infty)=\infty\}$

$\downarrow ev \qquad \downarrow$

$M \qquad u(\infty)$

$\mathcal{E} \supset E \qquad\qquad \mathcal{E} \supset E'$

$\downarrow \circlearrowright \bar\partial_J \qquad\qquad \downarrow \circlearrowright \bar\partial_J$

$\left\{u \approx u_0 \,\middle|\, u_*[\mathbb{P}^1] = A, \begin{matrix} u(0)\in H_0 \\ u(1)\in H_1 \end{matrix}\right\} \supset \bar\partial_J^{-1}(0) \qquad \bar\partial_J^{-1}(0) \subset \left\{v \approx v_0 \,\middle|\, v_*[\mathbb{P}^1] = A \ \begin{matrix} v(0)\in H_0' \\ v(1)\in H_1' \end{matrix}\right\}$

$\searrow \overline{\mathcal{M}} \swarrow$

$\qquad = \mathcal{S} \qquad\qquad \cap \qquad\qquad = \mathcal{S}'$

$v\circ\varphi_v \leftarrow\!\!| \quad \longrightarrow \mathcal{B}/\mathrm{Aut} \longleftarrow \quad |\!\!\to v$

$\cup \qquad\qquad\qquad\qquad \cup$

$\{\bar\partial_J u \in E\}$ smooth submfd

polyfold approach: this is "smooth" → later

$\{\bar\partial_J v \in E'\}$ smooth submfd

$[\bar\partial_J u \in E] \cap [\bar\partial_J v \in E']$

$\cup$
$\overline{\mathcal{M}}$

⚠ transition map only defined over $\bar\partial_J^{-1}(0)$

overlap chart (naive version)

$\{\bar\partial_J w \in E + E'\}$

more precisely : $\{w \in \mathcal{S} \mid \bar\partial_J w \in E + \Phi^* E'\}$ smooth bundle?

$\Phi : v \mapsto v\circ\varphi_v$ not differentiable → special E (geometric)

$E \pitchfork \Phi^* E'$? → "stabilization"

Note: This overlap chart needs to be constructed analytically in the same way (Fredholm problem in local slice) as previous finite dimensional reduction charts