

Analytic description of moduli spaces,

Geometric Regularization ~ equivariant transversality
(Sard-Smale)
& gluing (Newton iteration)

Abstract Regularization

- Why?
- Fredholm stabilization
- "stabilized gluing"

Geometric Regularization

- 0.) Global Fredholm description
- 1.) ~~Equivariant~~ Transversality
- 2.) Quotient Theorem
- 3.) Gromov Compactness & Gluing
- 4.) Cobordism / Continuation map

Abstract Regularization

- 0.) Gromov Compactness & Pre-gluing
Topology
- 1.) local Fredholm descriptions
- 1') transition information
- 2.) regularization theorem

$$\bar{M} = \frac{\bar{\partial}_j^{-1}(0)}{\text{Aut}} \cup M^{\text{broken}}$$

$$\bar{M} = s^{-1}(0)$$

Why abstract regularization?

* more general / quotable / transferrable

* regular J rarely exist

$(\bar{\omega}, \neq 0)$

↳ when all curves are "somewhat injective"

typical obstructions to $\bar{\omega}, \neq 0$

if $\bar{\omega}, \neq 0$ then $\dim M_{0,1}(A, J) = 2n + 2c_1(A) - 4$

$2n - 6 \geq 0$

$2(n - k - 2) < 0$ for $k \gg 1$

① multiply covered spheres

$$u_k: \mathbb{P}^1 \xrightarrow{\quad} \mathbb{P}^1 \xleftarrow{u} C \subset M$$

$$z \mapsto z^k$$

$$c_1(TM|_C) = -1 \Rightarrow c_1(u_k^*TM) = -k$$

$$\bar{\omega}, u = 0 \Rightarrow \bar{\omega}, u_k = 0 \quad \forall k \quad \text{but} \quad \text{ind } D_{u_k} \bar{\omega} = 2n - 2k < 0 \quad \forall k \gg 1$$

① nowhere injective disks

① occurs in almost all moduli spaces in almost all symplectic manifolds
by bubbling

except for special cases: - monotone
- exact

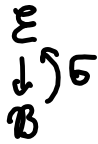
② branched covers of trivial cylinders

$$\mathbb{R} \times S^1 \xrightarrow{\Psi} \mathbb{R} \times S^1 \longrightarrow (\mathbb{R} \times Y, J \text{ comp. w. } d(e^f \lambda)) \quad (Y, \lambda) \text{ contact}$$

$$(s, t) \longmapsto (s, \gamma(t))$$

$$\gamma: S^1 \rightarrow Y \text{ Reeb orbit}$$

local Fredholm description

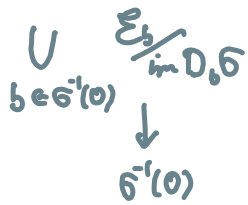


Fredholm section

$\sigma^{-1}(0)$
finite symmetry group

$\hookrightarrow \mathcal{M}$
local homeom.

obstruction "bundle"



\triangle fiber dimension generally not constant

stabilization

$$E \subset \mathcal{E}|_{\text{nbhd}(\sigma^{-1}(0))}$$

$$E_b + \text{im } \mathcal{D}_b \sigma = \mathcal{E}_b$$

$$\sigma_E: \mathcal{B} \times E \rightarrow \mathcal{E} \neq 0$$

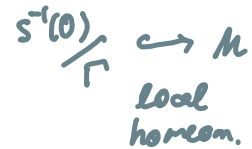
$$(b, e) \mapsto \sigma(b) + e$$

$$\sigma^{-1}(0) \simeq \text{zero} \left(\begin{array}{c} \sigma_E^{-1}(0) \rightarrow E \\ (b, e) \mapsto e \end{array} \right)$$

$$\sigma_E^{-1}(0) \simeq \{b \in \mathcal{B} \mid \sigma(b) \in E\} \text{ smooth manifold}$$

$$\begin{array}{c} (b, e) \\ \downarrow \\ e \end{array} \simeq \begin{array}{c} b \\ \downarrow \\ \sigma(b) \end{array}$$

finite dim. reduction

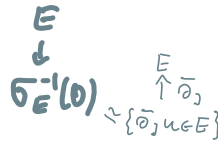


$$U = \sigma_E^{-1}(0)$$

$$s: (b, e) \mapsto e$$

PHILOSOPHY: "[$\sigma^{-1}(0)$]"

= Euler class of Γ -equivariant



1.) local Fredholm descriptions

(a) on $\bar{\partial}_2^{-1}(0) / \text{Aut}$

$\mathcal{E} \uparrow \bar{\partial}_2 = \mathcal{G}$
 \downarrow Local slice $\hookrightarrow \mathcal{B} / \text{Aut}$

(b) near $\mathcal{M}^{\text{broken}}$

polyfold approach

"by gluing after stabilization"

"by interpreting pregluing as chart of generalized Banach manifold" \rightarrow later

$$\mathcal{M}^{\text{broken}} \supset \text{Nbhd}([u_-], [u_+]) \simeq \mathcal{G}_-^{-1}(0) \times_{\text{Per}} \mathcal{G}_+^{-1}(0)$$

• stabilize by $E_{\pm} \subset \mathcal{E}_{\pm}$

$$\mathcal{G}_{E_{\pm}}^{\pm}: \mathcal{F}_{\pm} \times E_{\pm} \rightarrow \mathcal{E}_{\pm} \quad \neq 0$$

$$(v, e) \mapsto \mathcal{G}(v) - e$$

$$\simeq \text{Zero} \left(\begin{array}{c} E_- \times E_+ \quad (e_-, e_+) \\ \downarrow \\ \{([v_-], [v_+], e_-, e_+)\} \\ \bar{\partial}_{2, \mathcal{H}} v_{\pm} = e_{\pm} \\ \uparrow \\ \text{"} \\ (\mathcal{G}_-^{-1} \times \mathcal{G}_+^{-1})^{-1}(0) \end{array} \right)$$

$$\mathcal{E}_{\pm} = \bigcup L^p(v^*TM)$$

$$\downarrow \bar{\mathcal{G}}_{\pm} = \bar{\partial}_{2, \mathcal{H}}$$

$$\mathcal{F}_{\pm} = \left\{ v \in W^{1,p}(R \times S^1, M) \right.$$

$$\left. \begin{array}{l} \lim_{s \rightarrow \pm\infty} v(s, \cdot) = \dots \\ \exists! \text{ } \mathcal{G} \in S^1: v(0, \mathcal{G}) \in H_{\pm} \end{array} \right\}$$

• glue $(\mathcal{G}_-^{-1} \times \mathcal{G}_+^{-1})^{-1}(0) \times (R, \infty) \xrightarrow[\text{local homeo}]{} \left\{ \begin{array}{l} v \approx v_- \#_R v_+ \\ e \approx e_- \#_R e_+ \end{array} \middle| \bar{\partial}_{2, \mathcal{H}} v = e \right\}$

$$\Rightarrow \bar{\mathcal{M}} \supset \text{Nbhd}([u_-], [u_+]) \simeq \text{Zero} \left(\begin{array}{c} E_- \times E_+ \\ \downarrow \\ (\mathcal{G}_-^{-1} \times \mathcal{G}_+^{-1})^{-1}(0) \times (R, \infty) \end{array} \right) \xrightarrow{S} \left\{ (v, e) \mid \bar{\partial}_{2, \mathcal{H}} v = e \#_R e_+ \right\}$$

Δ need E_{\pm} "somewhat geometric":

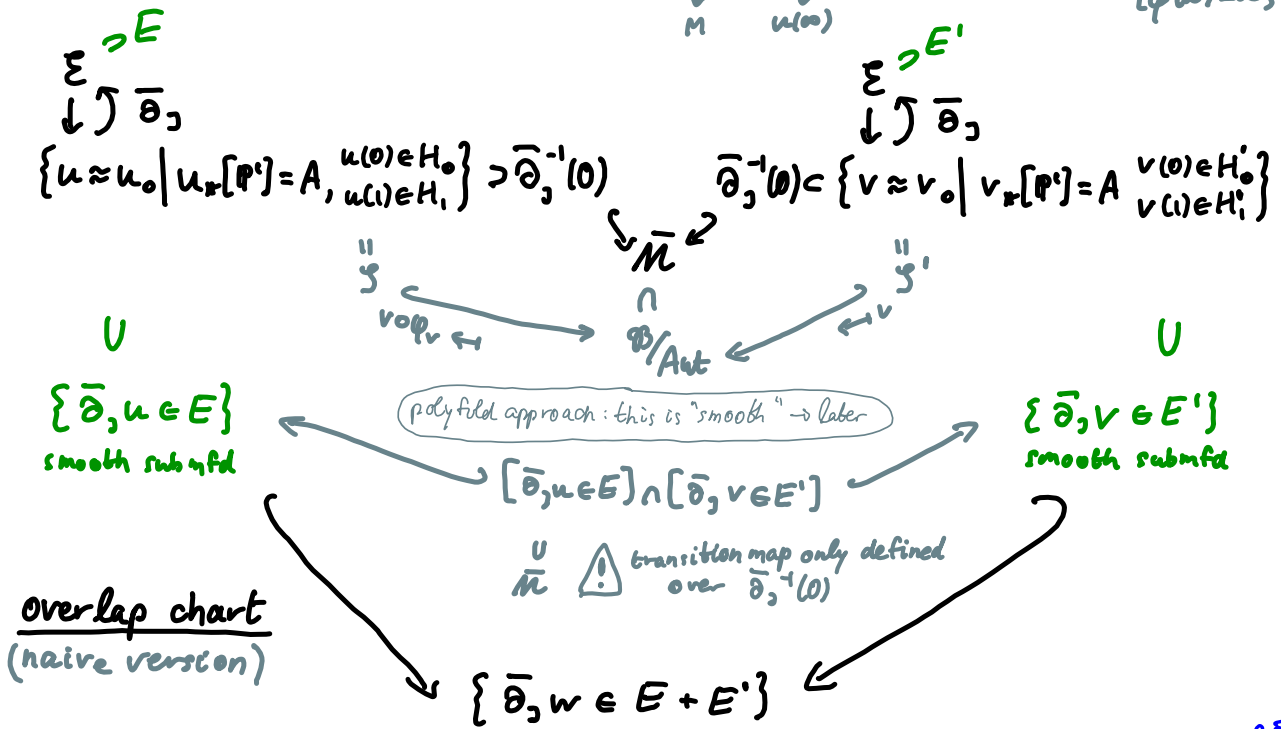
* e_{\pm} "supported outside of pregluing region"

* $e_- \# e_+$ is a differentiable section (despite failure of $(v, R) \mapsto v(\cdot \pm R)$)

$$\begin{array}{c} E_{\pm} \\ \downarrow \mathcal{F}_{\pm} \end{array} \uparrow e_{\pm} \quad \sigma \rightarrow \quad e_- \# e_+: \mathcal{F}_- \times \mathcal{F}_+ \times (R_0, \infty) \rightarrow \mathcal{E}$$

$$(v_-, v_+, R) \mapsto e_-(v_-(\cdot + R)) + e_+(v_+(\cdot - R))$$

1.) transition information for $\bar{M}_{0,1}(A_1) = \{u: P^1 \rightarrow M \mid u_*[P^1] = A, \bar{\partial}_2 u = 0\}$
 $\downarrow \text{ev} \quad \downarrow u(\infty)$
 $M \quad u(\infty)$
 $\{\varphi(\infty) = \infty\}$



more precisely: $\{w \in E \mid \bar{\partial}_2 w \in E + \Phi^* E'\}$ $\Phi: v \mapsto v \circ \varphi_v$ not differentiable \rightarrow special E (geometric)
 smooth bundle? $E \cap \Phi^* E' ? \rightarrow$ 'stabilization'

Note: This overlap chart needs to be constructed analytically in the same way (Fredholm problem in local slice) as previous finite dimensional reduction charts