Polyfold-Fredholm theory

M-polyfold Regularization

literature: • Hofer-Wysocki-Zehnder

• Hofer - surveys

• Fabert-Fish-Golovko-Wehrheim: "Polyfolds - A first and second look"

M-polyfold Fredholm section in a chart - Gromov-Witten type

$\overline{\mathcal{M}}(A,J) =$ Gromov compactification of $\{u:\mathbb{P}^1\to M \mid \bar\partial_J u = 0,\ u_*[\mathbb{P}^1]=A\}/\mathrm{Aut}\,\mathbb{P}^1$

If all stabilizers $\{\varphi \mid u\circ\varphi = u\}$ are trivial, then there is a

Fredholm section of M-polyfold bundle s.t. $\sigma^{-1}(0)\simeq\overline{\mathcal{M}}(A,J)$.

$\mathcal{Y} \approx \bigcup_{(u)\in\mathcal{B}} \Omega^{0,1}(\underline{u}^*TM)$

$\downarrow \quad \circlearrowleft \sigma : [(u_\alpha)_{\alpha\in T}] \mapsto [(\bar\partial_J u_\alpha)_{\alpha\in T}]$

$\mathcal{X} = \{u:\mathbb{P}^1\to M \mid u_*[\mathbb{P}^1]=A\}/\mathrm{Aut}\,\mathbb{P}^1 \ \underset{\text{pregluing}}{\cup}\ \underbrace{\bigcup_{A=A^-+A^+} \{u^-:\mathbb{P}^1\to M \mid u^-_*[\mathbb{P}^1]=A^-\}/\mathrm{Aut}(\mathbb{P}^1,0) \underset{M}{\times} \{u^+:\mathbb{P}^1\to M \mid u^+_*[\mathbb{P}^1]=A^+\}/\mathrm{Aut}(\mathbb{P}^1,0)}_{\text{modeled by splicing cores in } \{a\in\mathbb{C} \mid |a|<\varepsilon\}\times \mathbb{H}^3(\mathbb{P}^1,\mathbb{C}^n)\underset{ev_0,ev_0}{\times}\mathbb{H}^3(\mathbb{P}^1,\mathbb{C}^n) = D\times\mathbb{E}} \ \underset{\text{pregluing}}{\cup} \cdots$

$\mathcal{Y}$ ——— '' ——— ——— '' ——— $\times\ \underbrace{\mathbb{H}^2(\mathbb{P}^1,\mathbb{C}^n)\times\mathbb{H}^2(\mathbb{P}^1,\mathbb{C}^n)}_{\mathbb{F}}$

splicings near (u_0^-,u_0^+) arise from anti-pre-gluing isomorphisms

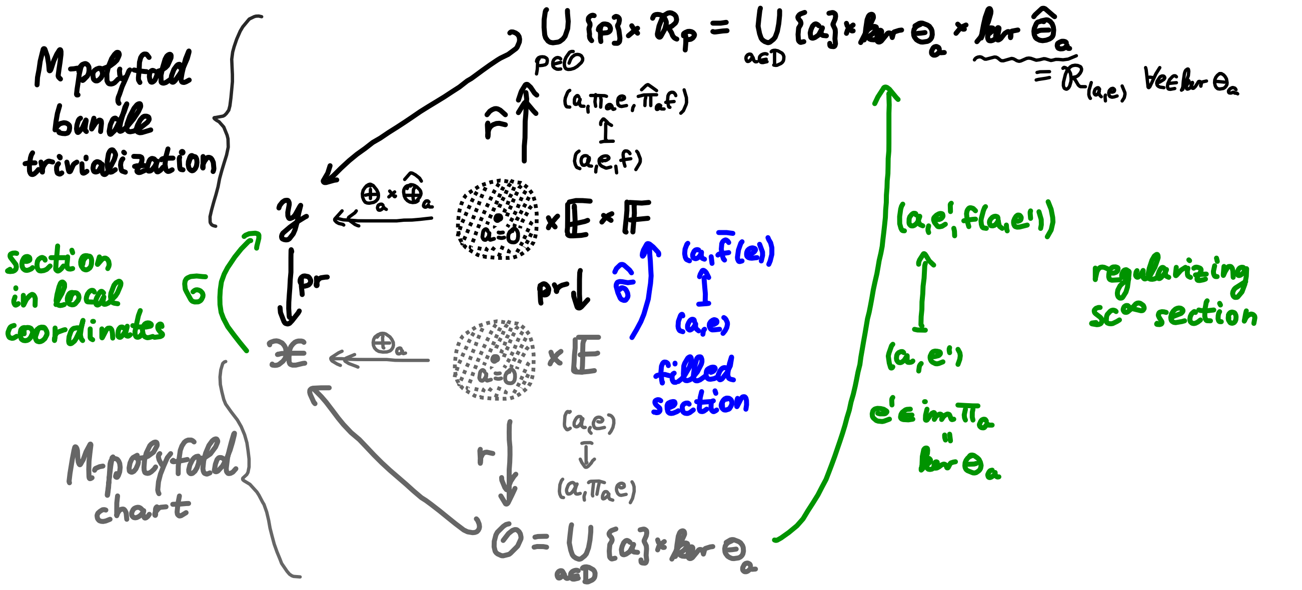
$$\boxplus_a = \oplus_a\times\ominus_a : \mathbb{H}^3(u_0^{-*}TM)\underset{\circledast}{\times}\mathbb{H}^3(u_0^{+*}TM) \xrightarrow{\simeq} \mathbb{H}^3(u_0^-\#_a u_0^{+*}TM)\times\mathbb{H}^3(\mathbb{R}\times S^1, T_{n_0}M)$$

$\int\!\downarrow D_{n_0}\bar\partial_J \qquad \underset{\approx\ \partial_s + J(n_0)\partial_t}{u_0^-(0)=u_0^+(0)}$

$$\widehat{\boxplus}_a = \widehat{\oplus}_a\times\widehat{\ominus}_a : \mathbb{H}^2(\Lambda^{0,1}u_0^{-*}TM)\times\mathbb{H}^2(\Lambda^{0,1}u_0^{+*}TM) \xrightarrow{\simeq} \mathbb{H}^2(\Lambda^{0,1}u_0^-\#_a u_0^{+*}TM)\times\mathbb{H}^2(\mathbb{R}\times S^1, T_{n_0}M)$$

"filler" arises from linearized operator at node

$\circledast$ fiber product and varying nodal values require some algebraic adjustments to this form



Definition 6.2.8. *An sc^∞ section $s : \mathcal{X} \to \mathcal{Y}$ of an M-polyfold bundle is a* **sc-Fredholm section** *if s is regularizing*$^{\circledast}$ *in the sense of Definition 6.1.8 and for each $x \in \mathcal{X}_\infty$ there is a local sc-trivialization $\Phi : p^{-1}(U) \to \mathcal{R}$ in the sense of Definition 6.1.4 over a neighbourhood $U \subset \mathcal{X}$ of x with $\Phi(x, 0) = 0$, such that $\underbrace{\Phi_* s}_{(v,\xi) \mapsto (v,\xi,f(v,\xi))}$ has a Fredholm filling in the sense of Definition 6.2.7.*

$\circledast$ in trivializations:
$f(v,\xi) \in F_i \Rightarrow (v,\xi) \in \mathbb{R}^k \times E_i$

$$D \times \mathbb{H}^3(\mathbb{P}^1) \times \mathbb{H}^3(\mathbb{P}^1) \xrightarrow{\bar{F}} \mathbb{H}^2(\mathbb{P}^1) \times \mathbb{H}^2(\mathbb{P}^1)$$

$$(a, \xi_-, \xi_+) \longmapsto \begin{cases} \left(\bar{\partial}_J(\exp_{u_0^-}\xi^-), \bar{\partial}_J(\exp_{u_0^+}\xi^+)\right) & ; a = 0 \\ \boxplus_a^{-1}\left(\bar{\partial}_J \exp_{u_0^- \#_a u_0^+} \oplus_a(\xi^-,\xi^+), D_{u=u_0^\pm(0)}\bar{\partial}_J \ominus_a(\xi^-,\xi^+)\right) & ; a \neq 0 \end{cases}$$

Definition 6.2.7. *Let $s : \mathcal{O} \to \mathcal{R}$, $s(p) = (p, f(p))$ be an sc^∞ section of an M-polyfold bundle model $\mathrm{pr}_{\mathcal{O}} : \mathcal{R} \to \mathcal{O}$ as in Definition 6.1.1, whose base is an sc-retract $\mathcal{O} \subset [0,\infty)^k \times \mathbb{E}$ containing $0 \in [0,\infty)^k \times \mathbb{E}$, and with fibers $\mathcal{R}_p \subset \mathbb{F}$ for $p \in \mathcal{O}$. Then a* **Fredholm filling at 0** *for s over $\mathcal{O}$ consists of*

- *a [illegible] sc-retraction of bundle type $R : \mathcal{U} \times \mathbb{F} \to \mathcal{U} \times \mathbb{F}$, $R(p,h) = (r(p), \Pi_p h)$ on an open subset $\mathcal{U} \subset [0,\infty)^k \times \mathbb{E}$ such that $r(\mathcal{U}) = \mathcal{O}$ and $\Pi_p \mathbb{F} = \mathcal{R}_p$ for all $p \in \mathcal{O}$,*
- *an sc^∞ map $\bar{f} : \mathcal{U} \to \mathbb{F}$ that is sc-Fredholm at 0 in the sense of Definition 6.2.4,*

with the following properties:

(i) $\bar{f}|_{\mathcal{O}} = f$;
(ii) *if $p \in \mathcal{U}$ such that $\bar{f}(p) \in \mathcal{R}_{r(p)}$ then $p = r(p)$, that is $p \in \mathcal{O}$;*

$\Rightarrow \boxed{\bar{F}^{-1}(0) = f^{-1}(0)}$

(iii) *The $\underbrace{\text{linearisation of the map } [0,\infty)^k \times \mathbb{E} \to \mathbb{F},\ p \mapsto (\mathrm{id}_{\mathbb{F}} - \Pi_{r(p)})\bar{f}(p)}_{= I_p}$ at each $p \in \mathcal{O}$ restricts to an isomorphism from $\ker \mathrm{D}_p r$ to $\ker \Pi_p$.*

$p = (v_0, \xi_0)$

ker resp. coker of $D_p\bar{F} = \begin{pmatrix} \Pi_{v_0} \circ D\bar{F}|_{T_p\mathcal{O}} & \Pi_{v_0} \circ D\bar{F}|_{T_p\mathcal{O}^\perp} \\ 0 & I_p \end{pmatrix}$

$\mathbb{R}^k \times \mathbb{E} = T_p\mathcal{O} \oplus T_p\mathcal{O}^\perp$
$\downarrow$
$\mathbb{F} = \mathcal{R}_p \oplus \mathcal{R}_p^\perp$

identified with

ker resp. coker of $D_p f := \Pi_{v_0} \circ D\bar{F}|_{T_p\mathcal{O}} : \underset{\mathbb{R}^k \times \mathbb{E}}{\underset{\cap}{T_p\mathcal{O}}} \longrightarrow \underset{\mathbb{F}}{\underset{\cap}{\mathcal{R}_p}}$

Definition 6.3.1. *A scale smooth section $s : \mathcal{X} \to \mathcal{Y}$ is called* **transverse** **(to the zero section)** *if for every $x \in s^{-1}(0)$ the linearization $\mathrm{D}_x s : \mathrm{T}_x\mathcal{X} \to \mathcal{Y}_x$ is surjective. Here the* **linearization** $\mathrm{D}_x s$ *is represented by the differential $\mathrm{D}_{\phi(x)}(\Pi \circ f \circ r)|_{\mathrm{T}_{\phi(x)}\mathcal{O}} : \mathrm{T}_{\phi(x)}\mathcal{O} \to \Pi_{\phi(x)}(\mathbb{F})$ in any local sc-trivialization $p^{-1}(U) \xrightarrow{\sim} \bigcup_{p\in\mathcal{O}} \Pi_p(\mathbb{F})$ which covers $\phi : \mathcal{X} \supset U \xrightarrow{\sim} \mathcal{O} = r(\mathcal{U}) \subset \mathbb{E}$ and transforms s to $p \mapsto (p, f(p))$.*

$$\text{(iii)} \Downarrow \qquad \mathbb{E} \supset_{\text{open}}$$

$$F : \mathcal{U} \to \mathbb{F} \quad \pitchfork 0$$

$$\Downarrow \text{ I.F.Thm. of scale calculus}$$

M-polyfold Regularization

Theorem 6.3.2 ([HWZ2], Thm. 5.14). *Let $s : \mathcal{X} \to \mathcal{Y}$ be a transverse sc-Fredholm section. Then the solution set $\mathcal{M} := s^{-1}(0)$ inherits from its ambient space $\mathcal{X}$ a smooth structure as finite dimensional manifold. Its dimension is given by the Fredholm index of s and the tangent bundle is given by the kernel of the linearized section, $\mathrm{T}_x\mathcal{M} = \ker \mathrm{D}_x s$.*

$\exists\, sc^\infty$ cutoff functions

$\exists$ compact perturbations : "sc^+"

Theorem 6.3.7. ([HWZ2],*Theorem 5.22) Let $\mathrm{pr} : \mathcal{Y} \to \mathcal{X}$ be a strong M-polyfold bundle modeled on sc-Hilbert spaces, and let $s : \mathcal{X} \to \mathcal{Y}$ be a proper Fredholm section.* $s^{-1}(0) \subset |\mathcal{X}|$ compact

(i) For any auxiliary norm $N : \mathcal{Y}_1 \to [0, \infty)$ and neighbourhood $s^{-1}(0) \subset \mathcal{U} \subset \mathcal{X}$ controlling compactness, there exists an sc^+-section $\nu : \mathcal{X} \to \mathcal{Y}_1$ with $\operatorname{supp}\nu \subset \mathcal{U}$ and $\sup_{x\in\mathcal{X}} N(\nu(x)) < 1$ and such that $s + \nu$ is transverse to the zero section. In particular, $(s+\nu)^{-1}(0)$ carries the structure of a smooth compact manifold.

(ii) Given two transverse perturbations $\nu_i : \mathcal{X} \to \mathcal{Y}_1$ for $i = 0, 1$ as in (i), controlled by auxiliary norms and neighbourhoods $(N_i, \mathcal{U}_i)$ controlling compactness, there exists an sc^+-section $\widetilde{\nu} : \mathcal{X} \times [0,1] \to \mathcal{Y}_1$ such that $\{(x,t) \in \mathcal{X} \times [0,1] \mid s(x) + \widetilde{\nu}(x,t)\}$ is a smooth compact cobordism from $(s+\nu_0)^{-1}(0)$ to $(s+\nu_1)^{-1}(0)$.

See addendum for notions of

- strong bundle, sc^+-section : formalizing "compact lower order perturbation"
- norm & nbhd controlling compactness : abstract version of "small lower order perturbations don't affect Gromov compactness"

Proof of

Theorem 6.3.7. ([HWZ2],*Theorem 5.22*) *Let* pr : $\mathcal{Y} \to \mathcal{X}$ *be a strong M-polyfold bundle modeled on sc-Hilbert spaces, and let* $s : \mathcal{X} \to \mathcal{Y}$ *be a proper Fredholm section.*

(i) For any auxiliary norm $N : \mathcal{Y}_1 \to [0, \infty)$ *and neighbourhood* $s^{-1}(0) \subset \mathcal{U} \subset \mathcal{X}$ *controlling compactness, there exists an sc*$^+$*-section* $\nu : \mathcal{X} \to \mathcal{Y}_1$ *with* $\operatorname{supp} \nu \subset \mathcal{U}$ *and* $\sup_{x \in \mathcal{X}} N(\nu(x)) < 1$ *and such that* $s + \nu$ *is transverse to the zero section. In particular,* $(s + \nu)^{-1}(0)$ *carries the structure of a smooth compact manifold.*

(ii) Given two transverse perturbations $\nu_i : \mathcal{X} \to \mathcal{Y}_1$ *for* $i = 0, 1$ *as in (i), controlled by auxiliary norms and neighbourhoods* $(N_i, \mathcal{U}_i)$ *controlling compactness, there exists an sc*$^+$*-section* $\widetilde{\nu} : \mathcal{X} \times [0, 1] \to \mathcal{Y}_1$ *such that* $\{(x, t) \in \mathcal{X} \times [0, 1] \mid s(x) + \widetilde{\nu}(x, t)\}$ *is a smooth compact cobordism from* $(s + \nu_0)^{-1}(0)$ *to* $(s + \nu_1)^{-1}(0)$.

uses Sard-Smale on universal moduli space

$$\left\{(x,\nu) \in \mathcal{X} \times \Gamma_{sc^+}(\mathcal{Y}) \;\middle|\; \operatorname{supp} \nu \subset \mathcal{U},\ \|\nu\|_{\infty,N} < 1,\ (s+\nu)(x) = 0\right\}$$

Rmk:

- can take $\mathcal{U}$ = nbhd of $\{x \in s^{-1}(0) \mid D_x s \text{ not onto}\} \subset \mathcal{X}$

  $\Rightarrow$ compact sets of transverse solutions remain unperturbed

- can find transverse $\|\nu_i\|_{\infty,N} \to 0$ with $\operatorname{supp} \nu_i \subset \mathcal{U}_i \xrightarrow[i\to\infty]{} s^{-1}(0)$

  $\Rightarrow$ perturbed solutions are near unperturbed solutions

- classical transversality should imply polyfold transversality
  ($D_u \bar\partial_J$ onto $\forall\, \bar\partial_J u = 0$) ($D_x s$ onto $\forall x \in s^{-1}(0)$)

Proof:

$$D_0 \overline{\mathcal{F}} = D\left((a,\xi_-,\xi_+) \longmapsto \begin{cases} \left(\bar\partial_J(\exp_{u_0^-}\xi^-),\ \bar\partial_J(\exp_{u_0^+}\xi^+)\right) \\ \boxplus_a^{-1}\left(\bar\partial_J \exp_{u_0^- \#_a u_0^+} \oplus_a(\xi^-,\xi^+),\ D_{n_0 = u_0^\pm(0)} \bar\partial_J \ominus_a(\xi^-,\xi^+)\right) \end{cases} \right)$$

$$: (A, Y_-, Y_+) \longmapsto \left(D_{u_0^-}\bar\partial_J Y_-,\ D_{u_0^+}\bar\partial_J Y_+\right) \longleftarrow \text{onto}$$
$$+ \partial_s\big|_{s=0} \boxplus_{sA}^{-1}\left(\bar\partial_J\, u_0^- \#_{sA} u_0^+,\ 0\right)$$

remains to check that $\overline{\mathcal{F}}$ is sc-Fredholm: requires comparison between $D_0\overline{\mathcal{F}}$ and

$$D_{(a,0,0)}\overline{\mathcal{F}} : (A, Y_-, Y_+) \longmapsto \boxplus_a^{-1}\left(D_{u_0^- \#_a u_0^+}\bar\partial_J \oplus_a(Y_-,Y_+),\ D_{n_0}\bar\partial_J \ominus_a(Y_-,Y_+)\right)$$
$$+ \partial_s\big|_{s=0}\left(\bar\partial_J\, u_0^- \#_{a+sA} u_0^+,\ 0\right)$$

__Lemma__: $f : \mathbb{R}^k \times \mathbb{E} \to \mathbb{F}$, $(a,e) \mapsto f_a(e)$ is sc-Fredholm if

- regularizing: $f_a(e) \in F_k \Rightarrow e \in E_k$ ✓ elliptic regularity
- uniformly $\mathcal{C}^1$ up to finite dimensions:
  - → $\forall k, a$ small: $f_a : E_k \to F_k$ classically $\mathcal{C}^1$ ✓ $\mathcal{C}^1$ for $\bar\partial_J$
  - → continuity of $Df_a : E_k \to L(E_k, F_k)$ uniform

    $\forall \delta > 0,\ k \in \mathbb{N}_0\ \exists \varepsilon > 0 :\ |a|, \|e\|_k, \|e - e'\|_k < \varepsilon \Rightarrow \|D_e f_a - D_{e'} f_a\|_{L(E_k, F_k)} < \delta$
  - → $a_i \to 0$, $\|e_i\|_{E_k} \leq 1$, $\|D_0 f_{a_i}(e_i)\|_{F_k} \to 0 \Rightarrow \|D_0 f_0(e_i)\|_{F_k} \to 0$

  classical gluing estimates
- uniformly linearized Fredholm:

  $D_0 f_a : \mathbb{E} \to \mathbb{F}$ sc-Fredholm operator $\forall a$ small with index independent of a

  $\|$

$$D_{(a,0,0)} \bar{\mathcal{F}} : (\cancel{X}, \underbrace{Y_-, Y_+}_{e}) \longmapsto \boxplus_a^{-1}\left(D_{u_0^- \#_a u_0^+} \bar\partial_J \oplus_a (Y_-, Y_+),\ D_{n_0} \bar\partial_J \ominus_a (Y_-, Y_+) \right)$$

$$\cancel{+ \partial_s|_{s=0} \left(\bar\partial_J\, u_0^- \#_{a+sA} u_0^+, 0 \right)}$$

Addendum: Nonlinear sc-Fredholm theory

Definition 6.1.5. *An M-polyfold bundle $p : \mathcal{Y} \to \mathcal{X}$ is called* **strong** *if it has trivializations in strong M-polyfold bundle models that are strongly compatible in the following sense.*

$D \times \mathbb{H}^3 \times \mathbb{H}^2$

(i) A **strong sc-retraction of bundle type** *is a retraction $R : \mathcal{U} \times \mathbb{F} \to \cancel{[0,\infty)}^k \times \mathbb{E} \times \mathbb{F}$, $(v, e, f) \mapsto \big(r(v, e), \Pi_{(v,e)} f\big)$ as in (10) that restricts to an sc$^\infty$ map $\mathcal{U} \times \mathbb{F}_1 \to \cancel{[0,\infty)}^k \times \mathbb{E} \times \mathbb{F}_1$, i.e. a retraction in the sc-Banach space $\big(\mathbb{R}^k \times E_m \times F_{m+1}\big)_{m \in \mathbb{N}_0}$.* GW case: $D \times \mathbb{H}^3 \times \mathbb{H}^3$

(ii) A **strong M-polyfold bundle model** *is the projection $\mathrm{pr}_{\mathcal{O}} : \mathcal{R} = \bigcup_{p \in \mathcal{O}} \{p\} \times \mathcal{R}_p \to \mathcal{O}$ from the total space of a* **strong sc-bundle retract** *$(\mathcal{R}_p \subset \mathbb{F})_{p \in \mathcal{O}}$ to its base retract $\mathcal{O}$ as in Definition 6.1.1, where $\mathcal{R}$ is the image of a strong retraction of bundle type.*

(iii) Two local sc-trivializations $\Phi : p^{-1}(U) \to \mathcal{R} \subset [0,\infty)^k \times \mathbb{E} \times \mathbb{F}$, and $\Phi' : p^{-1}(U') \to \mathcal{R}' \subset [0,\infty)^{k'} \times \mathbb{E}' \times \mathbb{F}'$ to strong M-polyfold bundle models $\mathcal{R} \to \mathcal{O}$ and $\mathcal{R}' \to \mathcal{O}'$ are **strongly compatible** *if their transition map restricts to a scale smooth map with respect to the ambient sc-sectors $[0,\infty)^k \times \mathbb{E} \times \mathbb{F}_1$ and $[0,\infty)^{k'} \times \mathbb{E}' \times \mathbb{F}'_1$. That is, we require sc$^\infty$ regularity of the map between these sectors in sc-Banach spaces of*

$$\iota_{\mathcal{R}'} \circ \Phi' \circ \Phi^{-1} \circ R : \; R^{-1}\big(\Phi\big(p^{-1}(U \cap U')\big)\big) \cap [0,\infty)^k \times \mathbb{E} \times \mathbb{F}_1 \;\longrightarrow\; [0,\infty)^{k'} \times \mathbb{E}' \times \mathbb{F}'_1$$

for any strong sc-retraction of bundle type with $R(\mathcal{U} \times \mathbb{F}) = \mathcal{R}$ (and hence $R(\mathcal{U} \times F_1) = \mathcal{R} \cap (\mathcal{U} \times F_1)$), and the inclusion $\iota_{\mathcal{R}'} : \mathcal{R}' \cap (\mathcal{U}' \times F'_1) \hookrightarrow [0,\infty)^{k'} \times E'_0 \times F'_1$.

For a strong M-polyfold bundle $p : \mathcal{Y} \to \mathcal{X}$ we denote by $p|_{\mathcal{Y}^1} : \mathcal{Y}^1 \to \mathcal{X}$ the subbundle of vectors $Y \in \mathcal{Y}$ such that for some (and hence any) trivialization $\Phi : p^{-1}(U) \to \mathcal{R} \subset \cancel{[0,\infty)}^k \times \mathbb{E} \times \mathbb{F}$ to a strong M-polyfold bundle model we have $\Phi(Y) \in \cancel{[0,\infty)}^k \times E_0 \times F_1$. $D \times \mathbb{H}^3 \times \mathbb{H}^2$

$D \times H^3 \times H^3$

Ex: $\mathcal{Y}' = \{(\underline{u}, \eta) \mid \eta \in H^3(\Lambda^{0,1} \underline{u}^* TM)\}$
is a well defined set because $H^3(u^*TM)$ is well defined for $u \in H^3$
(not just H^2, as required for $\mathcal{Y}$, but $H^k(u^*TM)$ isn't well defined if $u \notin H^k$)

Definition 6.3.6. *An* **auxiliary norm** *N for the strong M-polyfold bundle $\mathrm{pr} : \mathcal{Y} \to \mathcal{X}$ is a continuous map $N : \mathcal{Y}_1 \to [0,\infty)$ such that the restriction to each fiber $\mathrm{pr}^{-1}(x) \cap \mathcal{Y}_1$ for $x \in \mathcal{X}$ is a complete norm.* $\xrightarrow{\text{locally}} \mathbb{R}^k \times \mathbb{E} \times \mathbb{F}' \ni (v, I, \eta) \longmapsto \|\eta\|_{F_1}$

Moreover, if $s : \mathcal{X} \to \mathcal{Y}$ is a proper section, then a pair of an auxiliary norm N and an open neighbourhood $\mathcal{U} \subset \mathcal{X}$ of $s^{-1}(0)$ is said to **control compactness** *if for any sc$^+$-section $\nu : \mathcal{X} \to \mathcal{Y}_1$ with $\mathrm{supp}\,\nu \subset \mathcal{U}$ and $\sup_{x \in \mathcal{X}} N(\nu(x)) \leq 1$ the perturbed solution set $(s + \nu)^{-1}(0) \subset \mathcal{X}$ is compact.*

Lemma: There exist and are unique up to ...

Definition 6.1.8. *Let $p : \mathcal{Y} \to \mathcal{X}$ be a strong M-polyfold bundle. We denote the space of sc^∞ sections by*

$$\Gamma(p) := \{\boldsymbol{\gamma} : \mathcal{Y} \to \mathcal{X} \ sc^\infty \,|\, \boldsymbol{\gamma} \circ s = \mathrm{Id}_{\mathcal{X}}\}.$$

The subset of **sc$^+$ sections** $\Gamma^+(p) \subset \Gamma(p)$ *is the subset of those sections $s \in \Gamma(p)$ with values in $\mathcal{Y}_1$, or equivalently $\Gamma^+(p) \cong \Gamma(p|_{\mathcal{Y}_1})$.*

Moreover, we call a section $s \in \Gamma(p)$ **regularizing** *if the following implication holds:*

$$m \in \mathbb{N}_0, x \in \mathcal{X}_m, s(x) \in \mathcal{Y}^1_m \implies x \in \mathcal{X}_{m+1}.$$

"elliptic regularity"

The space of regularizing sections is equivalently defined and denoted by

$$\Gamma^{reg}(p) := \{p \in \Gamma(p) \,|\, \forall m \in \mathbb{N}_0 : s^{-1}(\mathcal{Y}^1_m) \subset \mathcal{X}_{m+1}\}.$$

<u>Rmk</u>: $sc^+ \mathrel{\hat{=}}$ lower order compact perturbation

$s \in \Gamma^{reg}, \nu \in \Gamma^+ \Rightarrow s + \nu \in \Gamma^{reg}$

stability

Theorem 6.2.10 ([HWZ2], Thm. 3.9). *Let $p : \mathcal{Y} \to \mathcal{X}$ be a strong M-polyfold bundle. Then for any sc-Fredholm section $s : \mathcal{X} \to \mathcal{Y}$ and sc^+ section $\nu : \mathcal{X} \to \mathcal{Y}^1$ the section $s + \nu : \mathcal{X} \to \mathcal{Y}$ is again sc-Fredholm.*

<u>further resources for polyfold theory:</u>

survey (as advertised throughout - and feedback most welcome):

http://arxiv.org/abs/1210.6670

and the references therein

lecture videos:

* 2009, for entertainment value: http://www.msri.org/workshops/479/schedules/3779

* 2012 lecture series at IAS: http://video.ias.edu/intropolyfolds/wehrheim

http://video.ias.edu/intropolyfolds/albers

http://video.ias.edu/polyfoldsminicourse/hofer1

reading courses / working groups at UC Berkeley: possible almost anytime - contact: wehrheim@berkeley.edu

July 5-19, 2015 there will be a summer school "Moduli Problems in Symplectic Geometry" The scientific committee is Hofer, Hutchings, McDuff and organizing committee Cristofaro-Gardiner, Fish, Nelson.

Organized as part of the IHÉS Lectures, this Summer School aims to provide PhD students, post-docs, and young researchers with an overview of recent developments in moduli spaces of pseudoholomorphic curves in symplectic and contact geometry.

Pseudoholomorphic curves arise as the zero set of a Fredholm section of a suitable bundle. Provided the section can be appropriately perturbed, these moduli spaces yield powerful contact and symplectic invariants such as Gromov-Witten theory, Hamiltonian Floer homology, contact homology, symplectic homology, and Symplectic Field Theory, which will be addressed in detail during the workshop. There are two main perturbative techniques, geometric and functional analytic.

The geometric perturbation methods are powerful for applications and practical from a computational point of view but typically require many restrictive assumptions and fail to generalize broadly. We will introduce researchers to the polyfold machinery of Hofer, Wysocki, and Zehnder, a new analytic framework designed to resolve the issue of transversality systematically. As computations are integral in applications of the aforementioned invariants, we will also explore how geometric perturbation schemes can be incorporated into the polyfold package.

We will supplement 7 mini courses with moderated discussions and related talks by senior faculty on current and future directions for the field.