Polyfold - Fredholm theory

M-polyfold bundles and Fredholm sections

literature:
- Hofer-Wysocki-Zehnder
- Hofer - surveys
- Fabert-Fish-Golovko-Wehrheim: "Polyfolds - A first and second look"

Ex: $f:\mathbb{R}^n\to\mathbb{R}$ Morse function, $\operatorname{crit} f=\{0\}$

$\overline{\mathcal{M}}_{Morse}(\mathbb{R}^n,\mathbb{R}^n)=\sigma^{-1}(0)$ for M-polyfold Fredholm section $\begin{array}{c}\mathcal{E}\\ \downarrow\uparrow\sigma\\ \mathcal{X}\end{array}$

$$|\mathcal{X}|=\bigcup_{L>0} H^1([-L,L],\mathbb{R}^n)\ \cup_{\text{preglue}}\ H^1([0,\infty),\mathbb{R}^n)\times H^1((-\infty,0],\mathbb{R}^n)$$

second countable metric space

M-polyfold charts

• near $\gamma_0:[-L,L]\to\mathbb{R}^n$ $\quad \underbrace{\mathbb{R}\times \mathbb{H}^1([-L,L],\mathbb{R}^n)}_{\mathbb{E}}\supset\underbrace{(\tfrac12,2)\times\{\|\xi\|<\delta\}}_{\underset{\text{open}}{\supset}\ \mathcal{U}=\underset{r=\mathrm{id}}{r(\mathcal{U})}=\mathcal{O}}\xrightarrow{\ \sim\ }\operatorname{nbhd}(\gamma)\subset|\mathcal{X}|$

$(\theta,\xi)\longmapsto(\gamma_0+\xi)(\theta^{-1}\cdot):$ $[-\theta L,\theta L]\to\mathbb{R}^n$

• near (γ_-^0,γ_+^0)

$[0,v_0)\times\underbrace{\mathbb{H}^1([0,\infty),\mathbb{R}^n)\times\mathbb{H}^1((-\infty,0],\mathbb{R}^n)}_{\mathbb{E}}$ $\qquad \operatorname{nbhd}(\gamma_-^0,\gamma_+^0)\subset|\mathcal{X}|$

$r\,\hat{=}\,(\Pi_{R(v)})_{v\in[0,v_0)}$, $\ R(v)=e^{1/v}-e$ $\quad \big\downarrow$ retraction (splicing)

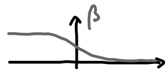
$\mathcal{O}\simeq\bigcup_{v\in[0,v_0)}\{v\}\times\ker\ominus_{R(v)}$

sc-retract (splicing core)

$\varphi:(v,\xi_-,\xi_+)\mapsto\oplus_{R(v)}(\gamma_-^0+\xi_-,\gamma_+^0+\xi_+)$ homeomorphism

$\oplus_{R(v)}(\ldots) = (\beta\ \ 1-\beta)\begin{pmatrix}\tau_R&0\\0&\tau_{-R}\end{pmatrix}\begin{pmatrix}\gamma_-^0+\xi_-\\ \gamma_+^0+\xi_+\end{pmatrix}$

$$\Pi_R=\begin{pmatrix}\tau_{-R}&0\\0&\tau_R\end{pmatrix}\begin{pmatrix}\beta&1-\beta\\ \beta-1&\beta\end{pmatrix}^{-1}\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}\beta&1-\beta\\ \beta-1&\beta\end{pmatrix}\begin{pmatrix}\tau_R&0\\0&\tau_{-R}\end{pmatrix}$$



$\tau_R(\gamma)=\gamma(\cdot+R)$

$$\boxed{\operatorname{Obj}\mathcal{X}=\bigsqcup_{\gamma_i}\mathcal{U}_i\ \cup\ \bigsqcup_{(\gamma_-^j,\gamma_+^j)}\mathcal{O}_j}$$

transition maps ($\hat{=}$ $t\circ s^{-1}$ for $(s\times t):\operatorname{Mor}\mathcal{X}\to\operatorname{Obj}\mathcal{X}$)

$$\mathcal{U}_j\supset\varphi_j^{-1}(\varphi_i(\mathcal{U}_i))\xrightarrow{sc^\infty}\mathcal{U}_i\qquad(\theta,\xi)\longmapsto\left(\tfrac{L_j}{L_i}\theta,(\gamma_j+\xi)(\tfrac{L_j}{L_i}\cdot)-\gamma_i\right)$$

is in fact $C^\infty(H^k)$ $\forall k$

$$\begin{array}{ccccc}
[0,v_0)\times\mathbb{E}_j & \supset & r_j^{-1}(\ldots) & \xrightarrow{sc^\infty} & \\
\downarrow r_j & & \downarrow r_j & & \\
\mathcal{O}_j & \supset & \varphi_j^{-1}(\varphi_i(\mathcal{U}_i)) & \longrightarrow & \mathcal{U}_i\subset\mathbb{E}_i
\end{array}$$

$$(v,\xi_-,\xi_+)\longmapsto\oplus_{R(v)}(\gamma_-^j+\xi_-,\gamma_+^j+\xi_+)\,(R(v)^{-1}\cdot)-\gamma_i$$

Note: $\varphi_i^{-1}\circ\varphi_j\circ r_j$ has same formula since r_j projects along $\ker\oplus$

$$\begin{array}{ccccc}
[0,v_0)\times\mathbb{E}_j & \supset & r_j^{-1}(\ldots) & \xrightarrow{sc^\infty} & [0,v_0')\times\mathbb{E}_i\\
\downarrow r_j & & \downarrow r_j & & \cup\\
\mathcal{O}_j & \supset & \varphi_j^{-1}(\varphi_i(\mathcal{U}_i)) & \longrightarrow & \mathcal{O}_i
\end{array}$$

$$(v,\xi_-,\xi_+)\longmapsto\varphi_i^{-1}\left(\oplus_{R(v)}(\gamma_-^j+\xi_-,\gamma_+^j+\xi_+)\right)$$

$$=\left(\underset{\underset{v}{\|}}{v'},\ \begin{pmatrix}\tau_{R'}&0\\0&\tau_{R'}\end{pmatrix}\begin{pmatrix}\beta_i&1-\beta_i\\ \beta_i-1&\beta_i\end{pmatrix}^{-1}\begin{pmatrix}\beta_j&1-\beta_j\\0&0\end{pmatrix}\begin{pmatrix}\tau_{R_j(v)}&0\\0&\tau_{-R(v)}\end{pmatrix}\begin{pmatrix}\gamma_-^j+\xi_-\\ \gamma_+^j+\xi_+\end{pmatrix}-\begin{pmatrix}\gamma_-^i\\ \gamma_+^i\end{pmatrix}\right),\quad R'=R$$

using fixed $R_i(v)=R_j(v)$

sc^∞ follows as for splicing when using exponential gluing profile $R_\cdot(v)=e^{1/v}-e$

Ex: $f:\mathbb{R}^n\to\mathbb{R}$ Morse function, $\mathrm{Crit} f=\{0\}$

$\overline{\mathcal{M}}_{Morse}(\mathbb{R}^n,\mathbb{R}^n)=\sigma^{-1}(0)$ for M-polyfold Fredholm section $\begin{array}{c}\mathcal{E}\\ \downarrow\uparrow\sigma\\ \mathcal{X}\end{array}$

$$|\mathcal{X}| = \bigcup_{L>0} H^1([-L,L],\mathbb{R}^n) \;\underset{\text{pregluing}}{\cup}\; H^1([0,\infty),\mathbb{R}^n)\times H^1((-\infty,0],\mathbb{R}^n)$$

$$\uparrow \downarrow |\sigma| \qquad \begin{array}{c}\gamma\\ \downarrow\\ \dot\gamma-\nabla f(\gamma)\end{array} \qquad \begin{array}{c}(\gamma_-,\gamma_+)\\ \downarrow\\ (\dot\gamma_--\nabla f(\gamma_-),\ \dot\gamma_+-\nabla f(\gamma_+))\end{array}$$

$|\mathcal{E}|$

M-poly bundle charts

- near $\gamma_0:[-L,L]\to\mathbb{R}^n$

$$\underbrace{\mathbb{R}\times H^1([-L,L],\mathbb{R}^n)}_{\mathbb{E}}\supset\mathcal{U}\xrightarrow{\sim}\mathrm{nbhd}(\gamma)\subset\mathcal{X}$$

$$(\theta,\xi)\mapsto(\gamma_0+\xi)(\theta^{-1}\cdot)$$

$$\mathrm{Obj}\,\mathcal{E}|_{\mathcal{U}\subset\mathrm{Obj}\,\mathcal{X}}=\underset{\mathbb{R}\times\mathbb{E}}{\mathcal{U}}\times\underbrace{H^0([-L,L],\mathbb{R}^n)}_{\mathbb{F}}$$

σ: $(\theta,\xi,(\theta^{-1}\frac{d}{dt}+\nabla f)(\gamma_0+\xi))$

$\downarrow \frac{d}{dt}+\nabla f$

$H^0([-\theta L,\theta L],\mathbb{R}^n)$

$\sim(\theta\cdot)$

- near (γ_-^0,γ_+^0)

$$[0,v_0)\times\underbrace{H^1([0,\infty),\mathbb{R}^n)\times H^1((-\infty,0],\mathbb{R}^n)}_{\mathbb{E}}=\mathcal{U}\quad \mathrm{nbhd}(\gamma_-^0,\gamma_+^0)\subset\mathcal{X}$$

$$\oplus_{R(v)}(\gamma_-^0+\xi_-,\gamma_+^0+\xi_+)$$

$$\downarrow (\pi_{R(v)})_{v\in[0,v_0)}$$

$$\mathcal{O}\simeq\bigcup_{v\in[0,v_0)}\{v\}\times\mathrm{im}\,\pi_{R(v)}\qquad \cancel{(v,\xi_-,\xi_+)}$$

$\ominus_{R(v)}(\xi_-,\xi_+)=0$

$\downarrow\sigma$

$$\mathrm{Obj}\,\mathcal{E}|_{\mathcal{O}\subset\mathrm{Obj}\,\mathcal{X}}=\bigcup_{v\in[0,v_0)}\{v\}\times\mathrm{im}\,\pi_{R(v)}\times\mathrm{im}\,\Pi_{R(v)}\qquad (v,\xi_-,\xi_+,\mathcal{D}_{R(v)}(\xi_-,\xi_+))$$

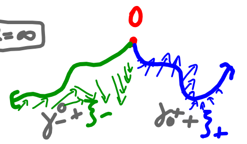
defined for all $(\xi_-,\xi_+)\in\mathbb{E}$

"filled section"

$$\uparrow (\pi_{R(v)}\times\Pi_{R(v)})_{v\in[0,v_0)}\qquad \Pi_R|_{F_1}=\pi_R$$

$$[0,v_0)\times\mathbb{E}\times\underbrace{H^0([0,\infty),\mathbb{R}^n)\times H^0((-\infty,0],\mathbb{R}^n)}_{\mathbb{F}}$$

$\boxed{R=\infty}$



$$\mathcal{D}_\infty(\xi_-,\xi_+)=\left((\tfrac{d}{dt}+\nabla f)(\gamma_-^0+\xi_-),\ (\tfrac{d}{dt}+\nabla f)(\gamma_+^0+\xi_+)\right)$$

$\boxed{R<\infty}$

$$\mathcal{D}_{R<\infty}(\xi_-,\xi_+)=\boxplus_R^{-1}\left((\tfrac{d}{dt}+\nabla f)(\oplus_{R(v)}(\gamma_-^0+\xi_-,\gamma_+^0+\xi_+)),\ (\tfrac{d}{dt}+\nabla^2 f(0))(\ominus_{R(v)}(\xi_-,\xi_+))\right)$$

linearization of $\frac{d}{dt}+\nabla f$ at $\gamma\equiv 0\in\mathrm{Crit} f$

$$\boxplus_R:\mathbb{F}\xrightarrow{\sim}H^0([-R,R],\mathbb{R}^n)\times H^0((-\infty,\infty),\mathbb{R}^n)$$

$$\uparrow\downarrow\ \tfrac{d}{dt}+\nabla^2 f(0)$$

Definition 6.1.4. *An* **M-polyfold bundle** *is an sc*$^\infty$ *surjection* $p : \overset{=\mathcal{E}}{\mathcal{Y}} \to \mathcal{X}$ *between two M-polyfolds together with a real vector space structure on each fiber* $\mathcal{Y}_x := p^{-1}(x) \subset \mathcal{Y}$ *over* $x \in \mathcal{X}$ *such that, for a sufficiently small neighbourhood* $U \subset \mathcal{X}$ *of any point in* $\mathcal{X}$ *there exists a* **local sc-trivialization** $\Phi : \mathcal{Y} \supset p^{-1}(U) \to \mathcal{R}$. *The latter is an sc*$^\infty$ *diffeomorphism to an sc-bundle retract* $\mathcal{R} = \bigcup_{p\in\mathcal{O}}\{p\} \times \mathcal{R}_p \subset \mathbb{E} \times \mathbb{F}$ *that covers an M-polyfold chart* $\phi : U \to \mathcal{O} \subset \mathbb{E}$ *in the sense that* $\mathrm{pr}_{\mathcal{O}} \circ \Phi = \phi \circ p$, *and preserves the linear structure in the sense that* $\Phi|_{\mathcal{Y}_x} : \mathcal{Y}_x \to \{\phi(x)\} \times \mathcal{R}_{\phi(x)}$ *is an isomorphism in every fiber over* $x \in U$.

$$\mathcal{E}_{(v,\xi)} = \begin{cases} H^0([-L,L],\mathbb{R}^n) & ; v>0 \\ H^0([0,\infty),\mathbb{R}^n) \times H^0((-\infty,0],\mathbb{R}^n) \end{cases} \underset{\oplus}{\simeq} \mathrm{im}\,\Pi_v \qquad \mathcal{R} = \bigcup_{(v,\xi)\in\mathcal{O}} \{(v,\xi)\} \times \mathrm{im}\,\Pi_v$$

Definition 6.1.1. *Let* $\mathcal{O} \subset [0,\infty)^k \times \mathbb{E}$ *be an sc-retract with corners in the sense of Definition 5.3.4, and let* $\mathbb{F}$ *be an sc-Banach space. Then a* **sc-bundle retract** *over* $\mathcal{O}$ *in* $\mathbb{F}$ *is a family of subspaces* $(\mathcal{R}_p \subset \mathbb{F})_{p\in\mathcal{O}}$ *that are scale smoothly parametrized by* $p \in \mathcal{O}$ *in the following sense: There exists a* **sc-retraction of bundle type**,

splicing

$$\mathcal{U} \times \mathbb{F} \longrightarrow [0,\infty)^k \times \mathbb{E} \times \mathbb{F}, \qquad (v,e,f) \longmapsto \big(\overset{(v,e)}{r(v,e)}, \overset{\Pi_v f}{\Pi_{(v,e)} f}\big), \tag{10}$$

given by a neat sc-retraction $r : \mathcal{U} \to [0,\infty)^k \times \mathbb{E}$ *with image* $r(\mathcal{U}) = \mathcal{O}$ *and a family of linear projections* $\Pi_{(v,e)} : \mathbb{F} \to \mathbb{F}$ *that are parametrized by* $(v,e) \in \mathcal{U}$, *and whose images for* $p = (v,e) \in \mathcal{O}$ *are the given subspaces* $\Pi_p(\mathbb{F}) = \mathcal{R}_p$.

Definition 6.2.8. *An sc$^\infty$ section $s : \mathcal{X} \to \mathcal{Y}$ of an M-polyfold bundle is a* **sc-Fredholm section** *if s is regularizing*⊛ *in the sense of Definition 6.1.8 and for each $x \in \mathcal{X}_\infty$ there is a local sc-trivialization $\Phi : p^{-1}(U) \to \mathcal{R}$ in the sense of Definition 6.1.4 over a neighbourhood $U \subset \mathcal{X}$ of x with $\Phi(x, 0) =$ 0, such that $\Phi_* s$ has a Fredholm filling in the sense of Definition 6.2.7.*

⊛ in trivializations:
$f(v,\xi) \in F_i \Rightarrow (v,\xi) \in \mathbb{R}^k \times E_i$

$(v,\xi) \mapsto (v,\xi,f(v,\xi))$

$$[0,v_0) \times \mathbb{H}^1([0,\infty)) \times \mathbb{H}^1((-\infty,0)) \xrightarrow{\bar{f}} \mathbb{H}^0([0,\infty)) \times \mathbb{H}^0((-\infty,0))$$

$$(v,\xi_-,\xi_+) \longmapsto \begin{cases} \left(\left(\frac{d}{dt}+\nabla f\right)(\gamma_-^0+\xi_-), \left(\frac{d}{dt}+\nabla f\right)(\gamma_+^0+\xi_+)\right) & ; v=0 \\ \boxplus_{R(v)}^{-1}\left(\left(\frac{d}{dt}+\nabla f\right)\left(\oplus_{R(v)}(\gamma_-^0+\xi_-,\gamma_+^0+\xi_+)\right), \left(\frac{d}{dt}+\nabla^2 f(0)\right)\left(\ominus_{R(v)}(\xi_-,\xi_+)\right)\right) & ; v>0 \end{cases}$$

Definition 6.2.7. *Let $s : \mathcal{O} \to \mathcal{R}$, $s(p) = (p, f(p))$ be an sc$^\infty$ section of an M-polyfold bundle model $\mathrm{pr}_{\mathcal{O}} : \mathcal{R} \to \mathcal{O}$ as in Definition 6.1.1, whose base is an sc-retract $\mathcal{O} \subset [0,\infty)^k \times \mathbb{E}$ containing $0 \in [0,\infty)^k \times \mathbb{E}$, and with fibers $\mathcal{R}_p \subset \mathbb{F}$ for $p \in \mathcal{O}$. Then a* **Fredholm filling at** 0 *for s over $\mathcal{O}$ consists of*

- *a [illegible] sc-retraction of bundle type $R : \mathcal{U} \times \mathbb{F} \to \mathcal{U} \times \mathbb{F}$, $R(p,h) = (r(p), \Pi_p h)$ on an open subset $\mathcal{U} \subset [0,\infty)^k \times \mathbb{E}$ such that $r(\mathcal{U}) = \mathcal{O}$ and $\Pi_p \mathbb{F} = \mathcal{R}_p$ for all $p \in \mathcal{O}$,*
- *an sc$^\infty$ map $\bar{f} : \mathcal{U} \to \mathbb{F}$ that is sc-Fredholm at 0 in the sense of Definition 6.2.4,*

with the following properties:

(i) $\bar{f}|_{\mathcal{O}} = f$;
(ii) if $p \in \mathcal{U}$ such that $\bar{f}(p) \in \mathcal{R}_{r(p)}$ then $p = r(p)$, that is $p \in \mathcal{O}$;

$\Rightarrow \boxed{\bar{f}^{-1}(0) = f^{-1}(0)}$

(iii) The linearisation of the map $[0,\infty)^k \times \mathbb{E} \to \mathbb{F}$, $p \mapsto (\mathrm{id}_{\mathbb{F}} - \Pi_{r(p)})\bar{f}(p)$ at each $p \in \mathcal{O}$ restricts to an isomorphism from $\ker \mathrm{D}_p r$ to $\ker \Pi_p$.

$(v,\xi) \mapsto (id - \Pi_v)\bar{f}(v,\xi)$

$\ker \mathrm{D}_p r = \ker D_{(v_0,\xi_0)}((v,\xi) \mapsto \Pi_v \xi) = T_p\mathcal{O}^\perp$

$\ker \Pi_p = \ker \Pi_{v_0} = \mathcal{E}_p^\perp$

$$\mathbb{R}^k \times \mathbb{E} = \underset{\substack{\| \\ \mathrm{im}\, D_p r \,\oplus\, \ker D_p r}}{T_p\mathcal{O} \oplus T_p\mathcal{O}^\perp} \supset T_p\mathcal{O}^\perp \xrightarrow[I_p]{\sim} \mathcal{E}_p^\perp \subset \underset{\substack{\| \\ \mathrm{im}\,\Pi_p \,\oplus\, \ker \Pi_p}}{\mathcal{E}_p \oplus \mathcal{E}_p^\perp} = \mathbb{F}$$

$$D\bar{f} = D(\Pi_\bullet \circ \bar{f}) + \overbrace{D\big(\underbrace{(id - \Pi_\bullet) \circ \bar{f}}_{\equiv 0 \text{ on } \mathcal{O}}\big)}^{I_p} = \Pi_{v_0} \circ D\bar{f} + (id - \Pi_{v_0}) \circ D\bar{f}$$

at $(v_0,\xi_0) \in \bar{f}^{-1}(0) \subset \mathcal{O}$

$$= \begin{pmatrix} \Pi_{v_0} \circ D\bar{f}|_{T_p\mathcal{O}} & \Pi_{v_0} \circ D\bar{f}|_{T_p\mathcal{O}^\perp} \\ 0 & I_p \end{pmatrix} : T_p\mathcal{O} \oplus T_p\mathcal{O}^\perp \longrightarrow \mathcal{E}_p \oplus \mathcal{E}_p^\perp$$

Note: I_p isomorphism $\Rightarrow \boxed{\left(D\bar{f} \text{ surjective iff } \Pi_{v_0} \circ D\bar{f}|_{T_p\mathcal{O}} \text{ surjective}\right)}$

Definition 6.3.1. *A scale smooth section $s : \mathcal{X} \to \mathcal{Y}$ is called* **transverse (to the zero section)** *if for every $x \in s^{-1}(0)$ the linearization $\mathrm{D}_x s : \mathrm{T}_x\mathcal{X} \to \mathcal{Y}_x$ is surjective. Here the* **linearization** $\mathrm{D}_x s$ *is represented by the differential $\boxed{\mathrm{D}_{\phi(x)}(\Pi \circ f \circ r)|_{\mathrm{T}_{\phi(x)}\mathcal{O}}} : \mathrm{T}_{\phi(x)}\mathcal{O} \to \Pi_{\phi(x)}(\mathbb{F})$ in any local sc-trivialization $p^{-1}(U) \xrightarrow{\sim} \bigcup_{p\in\mathcal{O}} \Pi_p(\mathbb{F})$ which covers $\phi : \mathcal{X} \supset U \xrightarrow{\sim} \mathcal{O} = r(\mathcal{U}) \subset \mathbb{E}$ and transforms s to $p \mapsto (p, f(p))$.*

(by Note on p.5) $\Downarrow$

$\mathbb{E}$

$\cup$ open

$\bar{f} : \mathcal{U} \to \mathbb{F} \quad \pitchfork 0$

$\Downarrow$ I.F.Thm. of scale calculus

M-polyfold Regularization

Theorem 6.3.2 ([HWZ2], Thm. 5.14)**.** *Let $s : \mathcal{X} \to \mathcal{Y}$ be a transverse sc-Fredholm section. Then the solution set $\mathcal{M} := s^{-1}(0)$ inherits from its ambient space $\mathcal{X}$ a smooth structure as finite dimensional manifold. Its dimension is given by the Fredholm index of s and the tangent bundle is given by the kernel of the linearized section, $\mathrm{T}_x\mathcal{M} = \ker \mathrm{D}_x s$.*

Theorem 6.3.7. ([HWZ2],*Theorem 5.22) Let $\mathrm{pr} : \mathcal{Y} \to \mathcal{X}$ be a strong* t.b.d *M-polyfold bundle modeled on sc-Hilbert spaces, and let $s : \mathcal{X} \to \mathcal{Y}$ be a proper Fredholm section.*

(i) For any auxiliary norm $N : \mathcal{Y}_1 \to [0, \infty)$ and neighbourhood $s^{-1}(0) \subset \mathcal{U} \subset \mathcal{X}$ controlling compactness, there exists an sc$^+$-section $\nu : \mathcal{X} \to \mathcal{Y}_1$ with $\operatorname{supp} \nu \subset \mathcal{U}$ and $\sup_{x\in\mathcal{X}} N(\nu(x)) < 1$, and such that $s + \nu$ is transverse to the zero section. In particular, $(s + \nu)^{-1}(0)$ carries the structure of a smooth compact manifold.

(ii) Given two transverse perturbations $\nu_i : \mathcal{X} \to \mathcal{Y}_1$ for $i = 0, 1$ as in (i), controlled by auxiliary norms and neighbourhoods $(N_i, \mathcal{U}_i)$ controlling compactness, there exists an sc$^+$-section $\widetilde{\nu} : \mathcal{X} \times [0, 1] \to \mathcal{Y}_1$ such that $\{(x, t) \in \mathcal{X} \times [0, 1] \,|\, s(x) + \widetilde{\nu}(x, t)\}$ is a smooth compact cobordism from $(s + \nu_0)^{-1}(0)$ to $(s + \nu_1)^{-1}(0)$.