Polyfold - Fredholm theory

Scale Calculus

- relation to classical calculus
- use in polyfold description
- relation to elliptic operators
- implicit function theorem & Fredholm notions

literature:
* Hofer-Wysocki-Zehnder
* Hofer - surveys
* Fabert-Fish-Golovko-Wehrheim: "Polyfolds - A first and second look"
* Wehrheim: "Fredholm notions in scale calculus and Hamiltonian Floer Theory"

SCALE CALCULUS

Guiding Example: $\tau : S^1 \times \mathcal{C}^\infty(S^1) \longrightarrow \mathcal{C}^\infty(S^1)$ $\qquad S^1 = \mathbb{R}/\mathbb{Z}$

$(s, \gamma) \longmapsto \gamma(s + \cdot)$

is scale-smooth on sc-Banach space $\mathbb{E} := (\mathcal{C}^k(S^1))_{k\in\mathbb{N}_0}$

- $\tau : S^1 \times \mathcal{C}^{k+\ell}(S^1) \longrightarrow \mathcal{C}^k(S^1)$ is classically $\mathcal{C}^\ell$ for $k, \ell \geq 0$
- norms $\|\cdot\|_1, \|\cdot\|_2$ on finite dim. v-spaces are equivalent $\frac{1}{C}\|\cdot\|_1 \leq \|\cdot\|_2 \leq C\|\cdot\|_1$

$\Rightarrow$ $\tau : S^1 \times N \to \mathcal{C}^\infty(S^1)$ is smooth for $N \subset \mathcal{C}^\infty(S^1)$ finite dim. submfd

General Facts:

$\mathbb{E} = (E_k)_{k\in\mathbb{N}_0}$ scale-Banach space

(i) E_∞ finite dim. $\Rightarrow$ $E_k = E_\infty \ \forall k$ (i.e. $\|\cdot\|_k \sim \|\cdot\|_j \ \forall k,j$)

(ii) E_∞ finite dim. $\Leftarrow$ $E_k = E_j$ for $k \neq j$ (i.e. $\|\cdot\|_k \sim \|\cdot\|_j$)

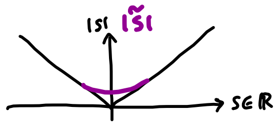
- $(E_k, \|\cdot\|_k)$ Banach space $\forall k$
- $E_k \hookrightarrow E_j$ continuous, compact $\forall k > j$ & $id : E_k \to E_k$ compact $\Leftrightarrow$ $\dim E_k < \infty$
- $E_\infty := \bigcap_{j\in\mathbb{N}_0} E_j \subset E_k$ dense $\forall k$ & $\dim E_\infty < \infty \Rightarrow \overline{E_\infty}^{\|\cdot\|} = E_\infty$

Sobolev spaces as scale-Banach spaces

Ω compact $\rightsquigarrow \mathbb{W}^{\ell,p}(\Omega) := \left(W^{\ell+k,p}(\Omega)\right)_{k\geq 0}$ $\quad \longleftrightarrow E_\infty = \mathcal{C}^\infty(\Omega)$

$\hookrightarrow \mathbb{W}^{\ell,p}_{\underline{\delta}}(\mathbb{R}\times\Omega) := \left(W^{\ell+k,p}_{\delta_k}(\mathbb{R}\times\Omega)\right)_{k\geq 0} \quad 0\leq\delta_0<\delta_1<\dots$

$$W^{\ell+k,p}_{\delta_k}(\mathbb{R}\times\Omega) = e^{-\delta_k \widetilde{|\cdot|}}\, W^{\ell+k,p}(\mathbb{R}\times\Omega), \quad \mathbb{R} \ni s$$



- $(E_k, \|\cdot\|_k)$ Banach space $\forall k$
- $E_k \hookrightarrow E_j$ continuous, compact $\forall k > j$
- $E_\infty := \bigcap_{j\in\mathbb{N}_0} E_j \subset E_k$ dense $\forall k$

$E_\infty \supset \mathcal{C}^\infty_c$ ← compact support

General Facts for $\tau : \mathbb{F} \to \mathbb{E}$

(0) scale-continuous $\left(\tau|_{F_k} : F_k \to E_k \ \ \mathcal{C}^0 \ \forall k\right) \Rightarrow \tau : F_\infty \to E_\infty$ determines τ uniquely

(i) scale-differentiable $\Leftrightarrow \forall k \geq 0$ $\left(\begin{array}{l} \tau|_{F_{k+1}} : F_{k+1} \to E_k \text{ differentiable } \forall k \\ D\tau : \underbrace{\mathbb{F}^1 \mathbb{F}}_{(F_{k+1} \times F_k)_{k \in \mathbb{N}_0}} \to \mathbb{E},\ (f,e) \mapsto D_f \tau \cdot e \end{array}\right)$

$$\frac{\| \tau(x+h) - \tau(x) - D_x\tau(h) \|_{E_k}}{\|h\|_{F_{k+1}}} \xrightarrow[\|h\|_{F_{k+1}} \to 0]{} 0$$

and $\forall x \in F_{k+1}$ $\sup\limits_{h \neq 0} \dfrac{\| D_x \tau(h) \|_{E_k}}{\|h\|_{F_k}} < \infty$

(ii) τ sc^1 $\Leftrightarrow$ (i) for $k=0$, $D\tau$ sc^0

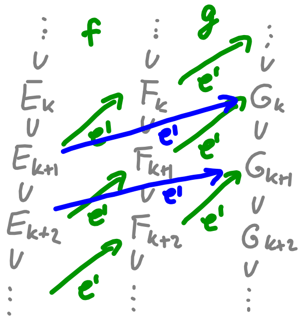
τ sc^ℓ $\Leftrightarrow$ (i) for $k=0$, $D\tau$ $sc^{\ell-1}$ $\underset{\cancel{\Leftarrow}}{\Rightarrow}$ $\tau : F^{k+\ell} \to E^k$ $\mathcal{C}^\ell$ $\forall k$ $\circledast$

(iii) τ sc^∞, $N \subset F_\infty$ finite dim. $\Rightarrow$ $\tau|_N : N \to E_\infty$ $\mathcal{C}^\infty$ wrt $\|\cdot\|_{E_k}$ $\forall k$

Chain Rule: $f : \mathbb{E} \to \mathbb{F}$, $g : \mathbb{F} \to \mathbb{G}$ sc^1

$\Rightarrow$ $g \circ f : \mathbb{E} \to \mathbb{G}$ sc^1, $T(g \circ f) = Tg \circ Tf$: $\underset{\underset{(E_{k+1} \times E_k)_{k \geq 0} =: \mathbb{E}^1 \times \mathbb{E}}{\parallel}}{T\mathbb{E}} \longrightarrow \underset{\underset{\mathbb{G}^1 \times \mathbb{G}}{\parallel}}{T\mathbb{G}}$

$(e, X) \mapsto \left((g \circ f)(e), D_e(g \circ f) X\right)$



fog

Note here how chain rule for "$\mathcal{C}^1$ after shift" does not hold.

Thm [HWZ]: $\exists$ polyfold $\mathcal{B}$

p.Fredholm section $\sigma_J : \mathcal{B} \to \mathcal{E}^J \qquad \forall J \in \mathcal{J}(M,\omega)$

s.t. $|\sigma_J^{-1}(0)| \simeq \bigcup_{A \neq 0} \underbrace{\overline{\mathcal{M}}(A,J)}_{\text{Gromov compactcation of}}$ $\quad \{u : \mathbb{P}^1 \to M \mid \bar\partial_J u = 0,\ u_*[\mathbb{P}^1] = A\}/_{\operatorname{Aut}\mathbb{P}^1}$

- **object level** near smooth curve $[u] \in \overline{\mathcal{M}}$ $\qquad \sigma^{-1}(0)/_{\Gamma = \operatorname{Stab}(u)} \hookrightarrow F_u \underset{\text{open}}{\subset} \overline{\mathcal{M}}$ homeomorphism

scale Banach bundle $\quad \mathcal{E}|_{\mathcal{U}} = \bigcup_{v \in \mathcal{U}} \overline{\Omega^{0,1}(\mathbb{P}^1, v^*TM)}^{H^2}$

$\pi \downarrow \qquad \downarrow \qquad \uparrow \bar\partial_J = \sigma_{\mathcal{U}}$ $\qquad$ scale-smooth Fredholm section

scale Banach manifold $\quad \mathcal{U} = \{v \in H^3(\mathbb{P}^1, M) \mid d(v,u) < \delta,\ v(z) \in \tilde{H}_z \text{ for } z = 0,1,\infty\}$

Hausdorff space
locally homeomorphic to sc-Banach space
with sc^∞ transition maps

π sc^∞ with linear structure on fibers

$\uparrow v = \exp_u(\xi)$ $\qquad$ totally geodesic wrt exp

$\left(\{\xi \in \underset{= W^{k,2}}{H^k}(\mathbb{P}^1, u^*TM) \mid \xi(z) \in T_{u(z)} H_z \text{ for } k = 0,1,\infty\}\right)_{k \geq 3}$

- **morphism level**:

$$\operatorname{Mor}\mathcal{B} \supset (s \times t)^{-1}(\mathcal{U}, \mathcal{U}') = \{(v,\varphi) \in \mathcal{U} \times \operatorname{Aut}\mathbb{P}^1 \mid v \circ \varphi \in \mathcal{U}'\}$$

scale Banach manifold

e.g. $\mathcal{U} \times \operatorname{Stab}(u) \simeq (s \times t)^{-1}(\mathcal{U}, \mathcal{U})$

$(v, g) \longmapsto (v, \varphi) \quad \underbrace{\varphi \approx g \text{ s.t. } v \circ \varphi \in \mathcal{U}}$

$\Leftrightarrow \varphi(z) = v^{-1}(H_z) \cap g(\text{nbhd}(z)) \quad$ for $z = 0,1,\infty$

This determines $\varphi \in \operatorname{Aut}(\mathbb{P}^1)$ uniquely.

- **structure maps**:

$$\left.\begin{array}{l} id : \operatorname{Obj}\mathcal{B} \to \operatorname{Mor}\mathcal{B}, \quad v \longmapsto (v, id) \\ s : \operatorname{Mor}\mathcal{B} \to \operatorname{Obj}\mathcal{B}, \quad (v,\varphi) \longmapsto v \\ t : \operatorname{Mor}\mathcal{B} \to \operatorname{Obj}\mathcal{B}, \quad (v,\varphi) \longmapsto v \circ \varphi \quad (*) \\ \circ : \operatorname{Mor}\mathcal{B} \times \operatorname{Mor}\mathcal{B} \to \operatorname{Mor}\mathcal{B}, \quad ((v,\varphi),(v\circ\varphi,\psi)) \longmapsto (v, \psi\circ\varphi) \end{array}\right\} \text{scale smooth}$$

$(*)$

$$\begin{tikzcd} \operatorname{Mor}\mathcal{B} \arrow[rr] & & \operatorname{Obj}\mathcal{B} \\ \mathcal{V}_{\mathcal{U}'} \subset \mathcal{U} \arrow[u, phantom, "\cup_{s \text{ locally}}"] \arrow[d] & & \mathcal{U}' \arrow[u, phantom, "\cup"] \\ \mathcal{V}_{\mathcal{U}'} \times (\mathbb{P}^1)^3 \arrow[rr] & & \mathcal{V}_{\mathcal{U}'} \times \operatorname{Aut}\mathbb{P}^1 \arrow[u] \end{tikzcd}$$

$v \longmapsto \left(v, \left(v^{-1}(H_z) \cap \text{nbhd}(\varphi_0(z))\right)_{z=0,1,\infty}\right) \overset{e^{\infty}}{\longmapsto} (v,\varphi) \longmapsto v \circ \varphi$

$= \varphi(z)$ for $z = 0,1,\infty$

$sc^\infty \Leftarrow \begin{cases} e^{\ell} \text{ on} \\ \mathcal{U} \cap H^{2+\ell} \curvearrowright e^{\ell} \end{cases}$

sc^∞ analogous to $\tau : \text{Maps} \times S^1 \to \text{Maps}$

TODO: objets & morphisms near nodal curves

Elliptic Operators and Scale Calculus

<u>Ex</u>: $D := \partial_s + J_0 \partial_t : \mathcal{C}^\infty(T^2, \mathbb{C}^n) \to \mathcal{C}^\infty(T^2, \mathbb{C}^n)$ $\qquad T^2 = \mathbb{R}^2/\mathbb{Z}^2$ compact domain

- "first order" : $D : \mathbb{E} \to \mathbb{F}$ sc^0 $\qquad \mathbb{E} = (W^{k+1,p})_{k \geq 0}$, $\mathbb{F} = (W^{k,p})_{k \geq 0}$

  $\Updownarrow$

  $D|_{E_k} : E_k \to F_k$ bounded linear operator $\forall k$

- "elliptic regularity": $\underset{W^{k,p}}{Du \in F_k} \Rightarrow \underset{W^{k+1,p}}{u \in E_k}$ $\qquad$ ("<u>regularizing</u>")

  $\Downarrow$

- Fredholm : $D|_{E_0} : \underset{W^{1,p}}{E_0} \to \underset{L^p}{F_0}$ $\qquad$ has kernel $= \ker D \subset E_\infty$ $\quad$ finite dim.

  cokernel $= F_0 / DE_0$ $\quad$ finite dim.

<u>Lemma</u>: $D : \mathbb{E} \to \mathbb{F}$ linear
- sc^0
- regularizing
- $D : E_0 \to F_0$ Fredholm

$\Rightarrow$ <u>sc-Fredholm</u>

$\mathbb{E} = \ker D \oplus_{sc} \mathbb{E}^c$, $\mathbb{F} = \operatorname{Im} D \oplus_{sc} C$

i.e. $\forall k$: $E_k = \underset{E_\infty}{\underset{\cap}{-''-}} \oplus E_k^c$ $\qquad F_k = DE_k \oplus \underset{F_\infty}{\underset{\cap}{-''-}}$

$D|_{\mathbb{E}^c} : \mathbb{E}^c \to \operatorname{Im} D$ sc-isomorphism

i.e. $\forall k$ $\quad E_k^c \to DE_k$ $\quad$ isomorphism

$\Downarrow$ $\qquad \Leftarrow$

$\forall k$: $D|_{E_k} : E_k \to F_k$ Fredholm

$$\left.\begin{aligned} \ker D|_{E_k} &= \ker D|_{E_0} \\ F_k/DE_k &\simeq F_0/DE_0 \end{aligned}\right\} \Rightarrow \operatorname{ind} D|_{E_k} = \operatorname{ind} D|_{E_0}$$

Implicit Function Theorem:

E, F Banach spaces

$f: E \to F$ $\mathcal{C}^\infty$

Fredholm & transverse

$\left(\forall e \in f^{-1}(0) : D_e f : E \to F \text{ Fredholm surjective} \right)$

$\Rightarrow f^{-1}(0) \subset E$ submanifold of (local) dimension $\operatorname{ind} Df$

DREAM

$\mathbb{E}, \mathbb{F}$ sc-Banach spaces

$f: \mathbb{E} \to \mathbb{F}$ sc^∞, regularizing

sc-Fredholm & transverse

$\left(\forall e \in \underset{E_\infty}{f^{-1}(0)} : D_e f : \mathbb{E} \to \mathbb{F} \text{ sc-Fredholm surjective} \right)$

$\Rightarrow f^{-1}(0) \subset E_\infty$ submanifold of (local) dimension $\operatorname{ind} Df$

DREAM PROOF local chart near $e \in f^{-1}(0)$ from Newton iteration $\subset E_\infty$

$$E_\infty \supset \ker D_e f \longrightarrow f^{-1}(0) \qquad \xi \in E^c_\infty$$

$$X \longmapsto e + X + \xi \qquad f(e+X+\xi) = 0$$

$$\begin{cases} \xi_0 = 0 \\ \xi_{n+1} = \xi_n - Q\, \underbrace{f(e+X+\xi_n)}_{E_\infty} \in E_\infty \end{cases} \qquad D_e f : \mathbb{E}^c \underset{Q}{\overset{sc}{\rightleftarrows}} \operatorname{Im} D_e f$$

$$\|\xi_{n+1} - \xi_n\|_{E_k} = \|Q\, f(e+X+\xi_n)\|_{E_k} \le \|Q\|_{L(F_k, E_k)} \|f(e+X+\xi_n)\|_{F_k}$$

$$\wedge$$

$$\|f(e+X+\xi_{n-1})\|_{F_k} + \underbrace{\|D_{e+X+\xi_{n-1}} f(\xi_n - \xi_{n-1})\|_{F_k}}_{\le C \|\xi_n - \xi_{n-1}\|_{E_{k+1}} \text{ if } \|X+\xi_{n-1}\|_{E_{k+1}} \text{ small}} + \underbrace{\frac{\|f(e+X+\xi_n) - f(e+X+\xi_{n-1}) - D_{e+X+\xi_{n-1}} f(\xi_n - \xi_{n-1})\|_{F_k}}{\|\xi_n - \xi_{n-1}\|_{E_{k+1}}}}_{\downarrow\, 0 \text{ if } \|\xi_n - \xi_{n-1}\|_{E_{k+1}} \to 0} \|\xi_n - \xi_{n-1}\|_{E_{k+1}}$$

$\Rightarrow$ proof needs more differentiability for contraction property

Implicit Function Theorem:

$\mathbb{E}, \mathbb{F}$ sc-Banach spaces

$f: \mathbb{E} \to \mathbb{F}$ sc^∞, sc-Fredholm

$f \pitchfork 0$ ($\forall e \in f^{-1}(0)$: $D_e f : \mathbb{E} \to \mathbb{F}$ surjective)

$\Rightarrow f^{-1}(0) \subset E_\infty$ submanifold of (local) dimension ind Df

Def^{n}: $f: \mathbb{E} \to \mathbb{F}$ is sc-Fredholm

- regularizing: $f(e) \in F_k \Rightarrow e \in E_k$
- contraction in local coordinates near each $e \in f^{-1}(0)$ is $(0,0)$

$$\mathbb{R}^n \times \mathbb{E}^c \longrightarrow \mathbb{R}^m \times \mathbb{F}^c$$

$$(v, w) \longmapsto (A(v,w), w - B(v,w))$$

$$\forall k \in \mathbb{N}_0, \theta > 0\ \exists \varepsilon > 0 : \forall \|v\|, \|w_1\|_{E_k}, \|w_2\|_{E_k} < \varepsilon :$$

$$\|B(v,w_1) - B(v,w_2)\|_{F_k} \leq \theta \|w_1 - w_2\|_{E_k}$$

Lemma: $f: \mathbb{E} \to \mathbb{F}$ sc^∞, regularizing

$$\left.\begin{array}{l} \bullet \text{ uniformly } C^1 \text{ up to finite dimensions} \\ \bullet \text{ uniformly linearized Fredholm near } e=0 \end{array}\right\} \Rightarrow \text{contraction in local coordinates near } e=0$$