B) Moduli spaces of pseudoholomorphic curves

(M,ω) symplectic, J compatible almost complex structure

MAIN EXAMPLE: $A \in H_2(M)$ fixed

$\bar\partial_J^{-1}(0) \subset \{u \in W^{1,p}(\mathbb{P}^1, M) \mid [u]=A\}$
$\shortparallel$ zero set of Fredholm section
but noncompact due to action of $\mathcal{J}$

$$\widetilde{\mathcal{M}} = \{u : \mathbb{P}^1 \to M \mid u_*[\mathbb{P}^1] = A,\ \bar\partial_J u = 0\}$$

$\mathbb{P}^1 \cong \mathbb{C} \cup \{\infty\}$

$$\mathcal{M} = \widetilde{\mathcal{M}}/\mathrm{Aut} \qquad \mathrm{Aut}(\mathbb{P}^1, \infty) = \left\{\varphi \circlearrowright \mathbb{P}^1,\ \varphi^* i = i,\ \varphi(\infty) = \infty \right\} = \{\varphi(z) = az + b \mid a \neq 0\}$$

"marked point" — noncompact 4-dim

$$\boxed{\overline{\mathcal{M}}_{0,1}(A,J)} = \overline{\mathcal{M}} = \widetilde{\mathcal{M}}/\mathrm{Aut} \underset{\text{gluing}}{\cup} \{\text{bubble trees with 1 marked point}\}$$

There is a continuous evaluation map $ev : \overline{\mathcal{M}} \to M$ given by $[u] \mapsto u(\infty)$ on $\widetilde{\mathcal{M}}/\mathrm{Aut}$ and we wish to define $[\overline{\mathcal{M}}] \in H_*(\overline{\mathcal{M}})$ or at least $ev_*[\overline{\mathcal{M}}] \in H_*(M)$.
We have no description $\overline{\mathcal{M}} = s^{-1}(0)$ as zero set of a section, just a subset $\bar\partial_J^{-1}(0)/\mathrm{Aut} \subset \overline{\mathcal{M}}$ that may not even be dense.

In some cases, perturbations of $J \in \mathcal{J}(M,\omega)$ provide regularization, corresponding to Aut-equivariant transverse perturbations $p = \bar\partial_{J'} - \bar\partial_J$ for $J' \in \mathcal{J}^{reg}(M,\omega)$ which moreover "preserve compactification type".

OTHER EXAMPLES of moduli spaces of pseudoholomorphic curves

are all essentially of the form

$$\overline{\mathcal{M}} = \widetilde{\mathcal{M}}/\mathrm{Aut} \underset{\text{gluing}}{\cup} \{\text{nodal/broken curves}\}$$

where $\widetilde{\mathcal{M}} = \bar{\partial}_J^{-1}(0)$ is a zero set of a Fredholm section $\begin{array}{c}\mathcal{E}\\ \downarrow \\ \mathcal{B}\end{array} \circlearrowleft \bar{\partial}_J$

over

$$\mathcal{B} = \Big\{ u:(\Sigma,j) \to (M,J) \;\Big|\; \underbrace{[u]=A}_{\substack{\text{fixed (relative)}\\ \text{homology/homotopy}\\ \text{class}}},\ \underbrace{u(\partial\Sigma)\subset L}_{\substack{\text{Lagrangian}\\ \text{(for each boundary}\\ \text{component)}}},\ \underbrace{(\Sigma,j)\in \mathcal{DM},\ (M,J)\in\mathcal{D}}_{\substack{\text{finite dimensional}\\ \text{smooth families}}} \Big\}$$

with

$\mathrm{Aut} = \{\text{biholomorphisms between } (\Sigma,j), (\Sigma',j') \in \mathcal{DM}\}$

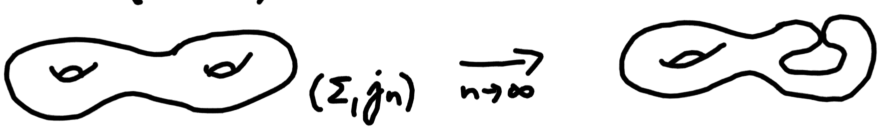
and nodal/broken curves arising from • bubbling

- compactification of $\mathcal{DM}/\mathrm{Aut}$
- compactification of $\mathcal{D}$

Examples of smooth families and compactifications

domain: Σ fixed, $j \in \{\text{complex structures on } \Sigma\}$

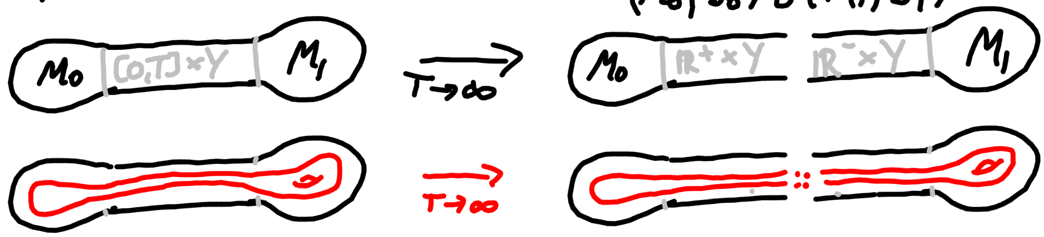
Deligne-Mumford compactification of $\{j \text{ on } \Sigma\} \big/ j \sim \varphi^* j \;\; \forall \varphi \in \mathrm{Diff}\,\Sigma$
contains (stable) nodal curves

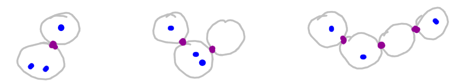
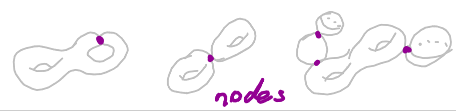
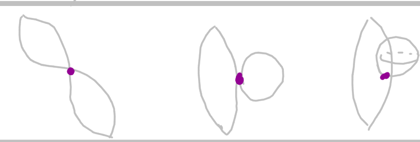
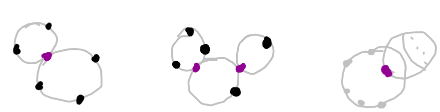
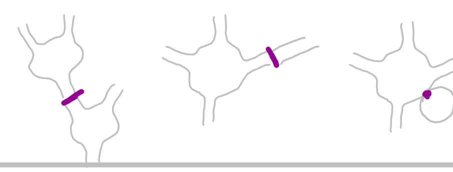
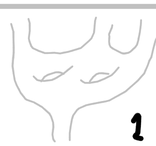
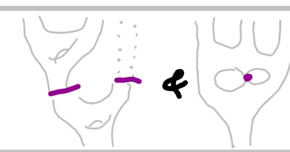
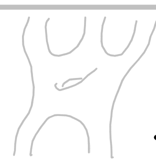
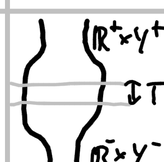


target (SFT-type splitting)

$\mathcal{D} = \{(M_0, J_0) \#_T (M_1, J_1) \mid T \geq 0\}$ with compactification $(M_0, J_0) \sqcup (M_1, J_1)$

this forces curves to break



	(Σ, j)	(M, J)	curves added in "compactification"
genus zero Gromov-Witten	$(\mathbb{P}^1, i)$ + marked points	fixed, compact	
Gromov-Witten	Σ fixed + marked points, j can vary	— " —	
Hamiltonian Floer	$(\mathbb{R} \times S^1, i)$ $\subset \mathbb{C}/\mathbb{Z}$	fixed	
Lagrangian Floer	$(\mathbb{R} \times [0,1], i)$ $\subset \mathbb{C}$	— " —	
Fukaya A_∞-algebra	(D, i) + marked points on ∂D; D = disk in $\mathbb{C}$	— " —	
Fukaya A_∞-category	$(D \setminus \{z_0, \ldots, z_k\}, i)$ $z_0, \ldots, z_k \in \partial D$ 	— " —	
contact homology	 k pos. punctures, 1 neg. puncture	$\mathbb{R} \times Y$	"buildings" & "nodes" 
Symplectic Field Theory	 punctured Riemann surfaces		buildings & nodes & sphere bubbles
relative SFT	punctured Riemann surfaces with boundary		buildings & interior/boundary nodes & sphere/disk bubbles