Application of abstract regularization techniques

- wish lists for general form & properties
- application to Arnold conjecture

General form & properties of abstract regularization

$\overline{\mathcal{M}}$ compact (metrizable) moduli space

$\cong$

$|\sigma^{-1}(0)|$ topological realization of "zero set of section"

$\mathcal{B}, \mathcal{E}$ categories, $\forall b \in \mathrm{Obj}\,\mathcal{B} : \pi^{-1}(b) \subset \mathrm{Obj}\,\mathcal{E}$ has vector space structure
$\forall \varphi \in \mathrm{Mor}_{\mathcal{B}}(b_1,b_2) : \mathrm{Mor}\,\mathcal{E} \supset \pi^{-1}(\varphi) =$ graph of a linear map $\pi^{-1}(b_1) \to \pi^{-1}(b_2)$

$\pi : \mathcal{E} \to \mathcal{B}$, $0, \sigma : \mathcal{B} \to \mathcal{E}$ functors, $0 \circ \pi = id_{\mathcal{B}} = \sigma \circ \pi$

⊕ "σ induced by moduli problem"

$\sigma : \mathrm{Obj}_{\mathcal{B}} \to \mathrm{Obj}_{\mathcal{E}}$ given by local Fredholm descriptions
(+ stabilizations
+ finite dimensional reduction)

$\mathcal{E} \underset{\bar\partial_J}{\downarrow\uparrow} \circlearrowleft \Gamma$, $\mathcal{U}$, $\bar\partial_J^{-1}(0)/\Gamma \hookrightarrow \overline{\mathcal{M}}$

⊕ locally constant index

	Siebert	Hofer-Wysocki-Zehnder	"Kuranishi"
$\mathrm{Obj}_{\mathcal{B}, \mathcal{E}}$	union of open subsets in Banach space	M-polyfold with boundary & corners	union of finite dimensional manifolds with boundary & corners
$\mathrm{Mor}_{\mathcal{B}, \mathcal{E}}$	homeomorphisms	scale-diffeomorphisms	smooth embeddings
$\sigma\vert_{\mathrm{Obj}}$	$\mathcal{C}^1$ up to finite dimensions	scale-smooth	smooth
Fredholm property	global stabilization $\mathcal{F} \xrightarrow{\iota} \mathcal{E}$, $\iota(\mathrm{Obj}_{\mathcal{F}}) \subset \mathrm{Obj}_{\mathcal{E}}$ $pr \searrow \mathcal{B} \swarrow$ covers "coker $D\sigma$"	local $\sigma : \mathrm{Obj}_{\mathcal{B}} \to \mathrm{Obj}_{\mathcal{E}}$ "regularizing" (elliptic regularity) & $D_b\sigma$ Fredholm $\forall b \in \sigma^{-1}(0)$	local finite dimensional reductions $\mathcal{E} = pr^*\mathcal{F}\vert_{\iota^{-1}(\dots)}$ $\downarrow\uparrow pr^* id_{\mathcal{F}}$ $\mathcal{B} = \iota^{-1}(\tilde\sigma(\tilde{\mathcal{B}}))$ induced by (local) $\mathcal{F} \xrightarrow{\iota} \tilde{\mathcal{E}}$ $pr \searrow \tilde{\mathcal{B}} \swarrow$
index	index "$D\sigma$" + nondiff. dim. $= \dim \underset{\iota^{-1}(\sigma(\mathcal{B}))}{\mathrm{nbhd}(f)} - \dim \text{fiber-}\mathcal{F}$	$\dim \ker D_b\sigma - \mathrm{codim}\, \mathrm{im}\, D_b\sigma$	$\dim \underset{\mathrm{Obj}\,\mathcal{B}}{\mathrm{nbhd}(b)} - \dim \underset{\mathrm{Obj}\,\mathcal{E}}{\pi^{-1}(b)} = \mathrm{index}\, D\sigma$

Rmk: Expect HWZ polyfold-Fredholm sections to have a global stabilization (and hence global finite dimensional reduction) only if $\forall b \in \sigma^{-1}(0)$
$\exists F \subset \pi^{-1}(b) \subset \mathrm{Obj}\,\mathcal{E}$ finite dimensional subspace covering $\pi^{-1}(b) / \mathrm{im}\, D_b\sigma$
invariant under $\mathrm{Aut}\, \pi^{-1}(b)$
(i.e. $\mathrm{Mor}_{\mathcal{E}}(e_1,e_2) \neq \emptyset$ for $e_1, e_2 \in \pi^{-1}(b)$
$\Rightarrow e_1, e_2 \in F$ or $e_1, e_2 \notin F$)

General form & properties of abstract regularization,

$\overline{\mathcal{M}}$ compact (metrizable) moduli space

$\wr\!|$

$|\sigma^{-1}(0)|$ topological realization of "zero set of section"

$\mathcal{B}, \mathcal{E}$ categories, "$\forall b \in \mathcal{B} : \pi^{-1}(b) \in \mathrm{Vect}$"

$\pi : \mathcal{E} \to \mathcal{B}$, $0, \sigma : \mathcal{B} \to \mathcal{E}$ functors, $0 \circ \pi = id_{\mathcal{B}} = \sigma \circ \pi$

⊛ "σ induced by moduli problem"

$\sigma : \mathrm{Obj}_{\mathcal{B}} \to \mathrm{Obj}_{\mathcal{E}}$ given by local Fredholm descriptions

(+ stabilizations
+ finite dimensional reduction)

$$\begin{tikzcd} \mathcal{E} \arrow[d, shift right] \\ \mathcal{U} \arrow[u, shift right, "\bar\partial_J"'] \end{tikzcd} \circlearrowleft \Gamma \qquad \bar\partial_J^{-1}(0)/\Gamma \hookrightarrow \overline{\mathcal{M}}$$

⊛ morphisms are transition data

$\mathrm{Mor}_{\mathcal{B}} |_{\sigma^{-1}(0)}$ determined by $\sigma^{-1}(0) /_{\text{morphisms}} \simeq \overline{\mathcal{M}}$

but outside of $\sigma^{-1}(0)$ this is additional structure

⊛ locally constant index $\simeq$ "virtual dimension of $(\overline{\mathcal{M}}, \sigma)$"

⊛ uniqueness: $(\mathcal{B}, \mathcal{E}, \sigma)$ unique up to "cobordism" e.g.

$$\begin{tikzcd}
 & & \tilde{\mathcal{E}} \arrow[d, shift right] & & \\
\mathcal{E} \arrow[urr] \arrow[d, shift right] & & & & \mathcal{E}' \arrow[ull] \arrow[d, shift right] \\
\mathcal{B} \arrow[u, shift right, "\sigma"'] \arrow[r, hook] & & \tilde{\mathcal{B}} \arrow[uu, shift right, "\tilde\sigma"'] & & \mathcal{B}' \arrow[u, shift right, "\sigma'"'] \arrow[ll, hook'] \\
\sigma^{-1}(0) \arrow[u, phantom, "\cup"] \arrow[d] & & \tilde\sigma^{-1}(0) \arrow[u, phantom, "\cup"] \arrow[d] & & \sigma'^{-1}(0) \arrow[u, phantom, "\cup"] \arrow[d] \\
\overline{\mathcal{M}} \arrow[rr, hook, "\times\{0\}"'] & & \overline{\mathcal{M}} \times [0,1] & & \overline{\mathcal{M}} \arrow[ll, hook', "\times\{1\}"]
\end{tikzcd}$$

General form & properties of abstract regularization

$\overline{\mathcal{M}}$ compact (metrizable) moduli space

$$\overset{\text{is}}{|\sigma^{-1}(0)|} \qquad \begin{array}{c}\mathcal{E}\\ 0\!\left(\uparrow\downarrow^{\pi}\right)\!\uparrow\sigma \\ \mathcal{B}\end{array}$$ "Fredholm section of categories" of index d

WISH LIST

$\partial\mathcal{B} = \emptyset$:

$\updownarrow$

index 0 for Arnold Conj. (minus cobordism)

$\exists\, \mathcal{P} \subset$ {multi-sections of $\mathcal{E} \to \mathcal{B}$} :

- $\forall \nu \in \mathcal{P}$: $|(\sigma+\nu)^{-1}(0)|$ weighted branched manifold of dimension d
- $\forall \nu \pitchfork \mu \in \mathcal{P}$: $|(\sigma+\mu)^{-1}(0)|$ cobordant
- if $\sigma|_{\mathcal{U}} \pitchfork 0$ for $\mathcal{U} \subset \mathrm{Obj}_{\mathcal{B}}$ open, $|\sigma^{-1}(0) \cap \mathcal{U}|$ compact then $\exists\, \nu \in \mathcal{P}$: $\nu|_{\mathcal{U}} \equiv 0$

Arnold Proof

(5) $I: HM_* \to HM_*$ isomorphism because

$$\overline{\mathcal{M}}^I(p_-, p_+)_{0,E} = \emptyset \quad \text{for } E<0$$
$$= \{u \equiv p_- = p_+\} \quad \text{for } E = 0$$

$E(u) = \int u^*\omega$

are regular connected components so don't need to be perturbed

$$\Rightarrow I = id_{CM(f)} + \sum_{i=0}^{\infty} q^{E_i} I_i \quad \Rightarrow \exists\, I^{-1}$$

General form & properties of abstract regularization

$\overline{\mathcal{M}}$ compact (metrizable) moduli space with "$\partial\overline{\mathcal{M}} \simeq \overline{\mathcal{M}} \times_{\cdots} \overline{\mathcal{M}}$" $\cdots$ discrete set

$\overline{\mathcal{M}} \cong |\sigma^{-1}(0)|$, $\quad \partial\overline{\mathcal{M}} = |\sigma|_{\partial\mathcal{B}}^{-1}(0)|$, $\quad \overline{\mathcal{M}} \times_{\cdots} \overline{\mathcal{M}} = |\sigma^{-1}(0) \times_{\cdots} \sigma^{-1}(0)|$

$$\begin{tikzcd} \mathcal{E} \arrow[d, "\pi"] \\ \mathcal{B} \arrow[u, bend left, "0"] \arrow[u, bend right, "\sigma"'] \end{tikzcd}$$

(*) with "gluing functor"

$$\begin{tikzcd} \mathcal{E} \times_{\cdots} \mathcal{E} \arrow[r, "\simeq"] \arrow[d] & \mathcal{E}|_{\partial\mathcal{B}} \arrow[r, hook] \arrow[d] & \mathcal{E} \arrow[d] \\ \mathcal{B} \times_{\cdots} \mathcal{B} \arrow[r, "\simeq"] & \partial\mathcal{B} \arrow[r, hook] & \mathcal{B} \end{tikzcd}$$

s.t. $\sigma|_{\partial\mathcal{B}} \simeq \sigma \times \sigma$

$\mathrm{ind} = \mathrm{ind} + \mathrm{ind} + 1$

(*) with extensions of maps

$ev : \overline{\mathcal{M}} \to M$, $lim : \overline{\mathcal{M}} \to \mathcal{P}_H$, $E : \overline{\mathcal{M}} \to \mathbb{R}$ induced by functors $ev, lim, E : \mathcal{B} \to \ldots$

Arnold Example: via SFT polyfolds

$$\partial\overline{\mathcal{M}}^h(p_-,p_+)_{1,E_i} \simeq \overline{\mathcal{M}}^I(p_-,p_+)_{0,E_i} \cup \overline{\mathcal{M}}^{PSS}(p_-,\cdot)_0 \times \overline{\mathcal{M}}^{SSP}(\cdot,p_+)$$

$$\overline{\mathcal{M}}^{PSS}(p_-,\cdot)_0 \xrightarrow{\lim_{+\infty} \times E} \mathcal{P}_H \times \mathbb{R} \xleftarrow{\lim_{-\infty} \times (E_i - E)} \overline{\mathcal{M}}^{SSP}(\cdot,p_+)$$

* nondiscrete $\mathbb{R}$ can be replaced by discrete set of possible energy differences

$$\cup\ \overline{\mathcal{M}}^{Morse}(p_-,\cdot) \times_{crit f} \overline{\mathcal{M}}^h(\cdot,p_+)_{0,E_i} \cup \overline{\mathcal{M}}^h(p_-,\cdot)_{0,E_i} \times_{crit f} \overline{\mathcal{M}}^{Morse}(\cdot,p_+)$$

$$\mathcal{B}^h(p_-,p_+) = \overline{\mathcal{M}}(p_-,M) \times \mathcal{B}^{SFT}_{stretch} \times \overline{\mathcal{M}}(M,p_+) \ni (x_-,R,\hat{u},x_+)$$

$$\mathcal{B}^{SFT}_{stretch} \overset{dense}{\supset} [0,\infty) \times \mathcal{C}^\infty(\mathbb{CP}^1, \mathbb{CP}^1 \times M)$$

$$(x_-,R,\hat{u},x_+) \overset{\sigma|_{\mathcal{C}^\infty}}{\longmapsto} \left(\bar{\partial}_{\hat{J}_R}\hat{u},\ ev(x_-) - ev_0(\hat{u}),\ ev(x_+) - ev_\infty(\hat{u})\right)$$

$$\partial\mathcal{B}^h(p_-,p_+) = \partial\overline{\mathcal{M}}(p_-,M) \times \mathcal{B}^{SFT}_{stretch} \times \overline{\mathcal{M}}(M,p_+)$$
$$\cup\ \overline{\mathcal{M}}(p_-,M) \times \mathcal{B}^{SFT}_{stretch} \times \partial\overline{\mathcal{M}}(M,p_+) \cup \overline{\mathcal{M}}(p_-,M) \times \partial\mathcal{B}^{SFT}_{stretch} \times \overline{\mathcal{M}}(M,p_+)$$

$$\partial\overline{\mathcal{M}}(M,p_+) = \bigcup_p \overline{\mathcal{M}}(M,p) \times \mathcal{M}(p,p_+)$$

$$\partial\mathcal{B}^{SFT}_{stretch} = \mathcal{B}^{SFT}_{R=0} \cup \mathcal{B}^{SFT}_+ \times_{\mathcal{P}_H} \mathcal{B}^{SFT}_-$$

$$\mathcal{B}^{SFT}_\pm \overset{dense}{\supset} \left\{ \hat{u} \in \mathcal{C}^\infty(\mathbb{C}, \mathbb{C} \times M) \;\middle|\; E(\hat{u}) < \infty,\ \hat{u}|_{nbhd(\infty)} \sim id_{\mathbb{C}} \times \mathcal{P}_H \right\} \xrightarrow{lim} \mathcal{P}_H$$

$$\simeq \mathcal{M}(p_-,\cdot) \times_{crit f} \mathcal{B}^h(\cdot,p_+)$$
$$\cup\ \mathcal{B}^h(p_-,\cdot) \times_{crit f} \mathcal{M}(\cdot,p_+)$$
$$\cup\ \mathcal{B}^I(p_-,p_+) \cup \mathcal{B}^{PSS}(p_-,\cdot) \times_{\mathcal{P}_H} \mathcal{B}^{SSP}(\cdot,p_+)$$

General form & properties of abstract regularization

$\overline{\mathcal{M}}$ compact (metrizable) moduli space with "$\partial\overline{\mathcal{M}} = \overline{\mathcal{M}} \times \overline{\mathcal{M}}$"

$$\overline{\mathcal{M}} \cong |\mathfrak{S}^{-1}(0)| \qquad \partial\overline{\mathcal{M}} = |\mathfrak{S}|_{\partial\mathcal{B}}^{-1}(0)| \qquad \overline{\mathcal{M}}\times\overline{\mathcal{M}} = |\mathfrak{S}^{-1}(0) \times \mathfrak{S}^{-1}(0)|$$

$$\begin{tikzcd} \mathcal{E} \arrow[d, "\pi"'] \\ \mathcal{B} \arrow[u, bend left, "\mathfrak{S}"] \arrow[u, bend right, "0"'] \end{tikzcd}$$

with "gluing functor"

$$\begin{tikzcd} \mathcal{E} \times \mathcal{E} \arrow[r, "\sim"] \arrow[d] & \mathcal{E}|_{\partial\mathcal{B}} \arrow[r, hook] \arrow[d] & \mathcal{E} \arrow[d] \\ \mathcal{B} \times \mathcal{B} \arrow[r, "\sim"] & \partial\mathcal{B} \arrow[r, hook] & \mathcal{B} \end{tikzcd}$$

s.t. $\mathfrak{S}|_{\partial\mathcal{B}} \simeq \mathfrak{S} \times \mathfrak{S}$

WISH LIST

$\partial\mathcal{B} \neq \emptyset$:

$\Updownarrow$

index 1 for Arnold Conj.

$\exists\ \mathcal{P} \subset \{$multi-sections of $\mathcal{E} \to \mathcal{B}\}$:

- $\forall \nu \in \mathcal{P}$: $|(\mathfrak{S}+\nu)^{-1}(0)|$ weighted branched manifold of dimension d
  with boundary $\partial|(\mathfrak{S}+\nu)^{-1}(0)| = |(\mathfrak{S}+\nu)^{-1}(0) \cap \partial\mathcal{B}|$
  corner degeneracy = corner degeneracy

- if $\mathfrak{S}|_{\partial\mathcal{B}} + \nu^{\partial} \pitchfork 0$ $\quad (\nu^{\partial} \in \mathcal{P}$ for $\mathfrak{S}|_{\partial\mathcal{B}})$
  then $\exists \nu \in \mathcal{P}$: $\nu|_{\partial\mathcal{B}} = \nu^{\partial}$

- $\mathfrak{S}_1 + \nu_1 \pitchfork 0,\ \mathfrak{S}_2 + \nu_2 \pitchfork 0 \Rightarrow \mathfrak{S}_1 \times \mathfrak{S}_2 + \nu_1 \times \nu_2 \pitchfork 0$
  $(\nu_1 \in \mathcal{P}_{\mathfrak{S}_1},\ \nu_2 \in \mathcal{P}_{\mathfrak{S}_2} \Rightarrow \nu_1 \times \nu_2 \in \mathcal{P}_{\mathfrak{S}_1 \times \mathfrak{S}_2})$

Arnold application: boundary strata with energy and index considerations

$$\partial\mathcal{B}^h(p_-,p_+)_{E,1} \simeq \mathcal{M}(p_-,\cdot) \underset{crit}{\times} \mathcal{B}^h(\cdot,p_+)_{E,\le 0}$$

$$\cup\ \mathcal{B}^h(p_-,\cdot)_{E,\le 0} \underset{crit}{\times} \mathcal{M}(\cdot,p_+) \ \cup\ \mathcal{B}^I(p_-,p_+)_{E,0} \ \cup\ \mathcal{B}^{PSS}(p_-,\cdot) \underset{p_H \times \mathbb{R}}{\times} \mathcal{B}^{SSP}(\cdot,p_+)$$

$\mathfrak{S}|_{\partial\mathcal{B}^h} = \qquad \mathfrak{S} \qquad\qquad \mathfrak{S} \qquad\qquad \mathfrak{S} \times \mathfrak{S}$

$E^{PSS} + E^{SSP} = E$

$\text{ind}^{PSS} + \text{ind}^{SSP} = 0$

$\Downarrow \exists \nu \in \mathcal{P}$:

$\nu|_{\partial\mathcal{B}^h} = \quad \nu|_{\text{ind} \le 0}$ resp. $\nu \times \nu|_{\text{ind} \le 0}$

[by first choosing perturbations for ind ≤ 0 connected components, then applying above regularization]

$\Downarrow$

$\partial|(\mathfrak{S}+\nu)^{-1}(0)|$

$\|$

$$|(\mathfrak{S}+\nu)^{-1}(0) \cap \partial\mathcal{B}^h| \simeq \mathcal{M}(p_-,\cdot) \underset{crit}{\times} |(\mathfrak{S}+\nu)^{-1}(0)|$$

$\cap\, \mathcal{B}(p_-,p_+) \qquad\qquad \cap\, \mathcal{B}(\cdot,p_+)$

$$\cup\ |(\mathfrak{S}+\nu)^{-1}(0)| \underset{crit}{\times} \mathcal{M}(\cdot,p_+) \ \cup\ |(\mathfrak{S}+\nu)^{-1}(0)| \ \cup\ |(\mathfrak{S}+\nu)^{-1}(0)| \underset{p_H \times \mathbb{R}}{\times} |(\mathfrak{S}+\nu)^{-1}(0)|$$

$\cap\, \mathcal{B}(p_-,\cdot) \qquad \cap\, \{R=0\} \qquad \cap\, \mathcal{B}^{PSS}(p_-,\cdot) \qquad \cap\, \mathcal{B}^{SSP}(\cdot,p_+)$

$\Downarrow$ [$\emptyset$ solution sets for ind < 0]

$$\partial\overline{\mathcal{M}}^h(p_-,p_+)^{reg}_{E,1} \simeq \mathcal{M}(p_-,\cdot) \underset{crit}{\times} \overline{\mathcal{M}}^h(\cdot,p_+)^{reg}_{E,0}$$

$$\cup\ \overline{\mathcal{M}}^h(p_-,\cdot)^{reg}_{E,0} \underset{crit}{\times} \mathcal{M}(\cdot,p_+) \ \cup\ \overline{\mathcal{M}}^I(p_-,p_+)^{reg}_{E,0} \ \cup\ \overline{\mathcal{M}}^{PSS}(p_-,\cdot)^{reg}_{0} \underset{p_H \times \mathbb{R}}{\times} \overline{\mathcal{M}}^{SSP}(\cdot,p_+)^{reg}_{0}$$

$\Downarrow$

$$0 = h \circ \partial \pm \partial \circ h \pm I \pm SSP \circ PSS$$