Application of abstract regularization techniques

at the example of

a "non-equivariant" proof of Arnold Conjecture

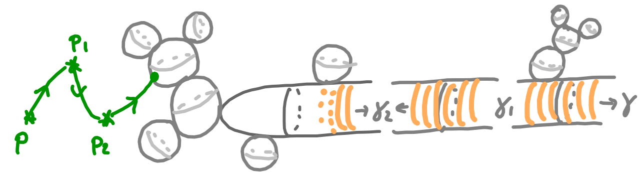
- using SFT moduli spaces

and

- introducing polyfold language

compact Piunikhin-Salamon-Schwarz moduli spaces via SFT

$$CM(f) \xrightarrow{PSS} CF(H) \xrightarrow{SSP} CM(f), \qquad \Downarrow h, \qquad I$$



PSS: $\overline{\mathcal{M}}(p,\gamma) := \bigcup_{k=0}^{\infty} \bigcup_{\gamma_1 \ldots \gamma_k \in \mathcal{P}_H} \overline{\mathcal{M}}(p,M) \times_M \overline{\mathcal{M}}_+^{S^2}(\gamma_k) \times \overline{\mathcal{M}}^{S^2}(\gamma_k,\gamma_{k-1}) \ldots \times \overline{\mathcal{M}}^{S^2}(\gamma_1,\gamma)$

$$\simeq \overline{\mathcal{M}}(p,M) \underset{ev \searrow M \swarrow ev}{\times} \overline{\mathcal{N}}_+(\gamma)$$ ← SFT compact-cation of

$$\mathcal{N}_\pm(\gamma) := \{ \hat{u} : \mathbb{C} \to \mathbb{C}^\pm \times M \mid \bar{\partial}_{\hat{J}_\pm} \hat{u} = 0,\ [\hat{u}] = [id] \times [u],\ E(u) < \infty,\ \hat{u}|_{nbhd(\infty)} \sim id_{\mathbb{C}} \times \gamma \} / Aut(\mathbb{C})$$

(pos/neg puncture)

$\mathcal{N}_\pm(\gamma) \to M$, $[\hat{u}] \mapsto im\, \hat{u} \cap \{0\} \times M$

$$\hat{J}_\pm = \begin{pmatrix} j_{\mathbb{C}} & 0 \\ *_\pm & J \end{pmatrix} \text{ on } T\mathbb{C} \times TM$$

$*_\pm$ domain dependent Hamiltonian perturbation

SSP: $\overline{\mathcal{M}}(\gamma,p) := \overline{\mathcal{N}}_-(\gamma) \underset{ev \searrow M \swarrow ev}{\times} \overline{\mathcal{M}}(M,p)$

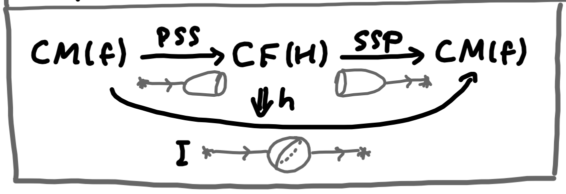
SFT [Hofer et al]: $\mathbb{C}^\pm \times M$ symplectic cobordism $\emptyset \rightleftarrows S^1 \times M$

with pos/neg end $\mathbb{R}^\pm \times S^1 \times M$

compact-cation: "buildings of nodal holomorphic curves"

first/last level $\subset \mathbb{C}^\pm \times M$, other levels $\subset \mathbb{R} \times S^1 \times M$

compact Piunikhin-Salamon-Schwarz moduli spaces via SFT



PSS: $\overline{\mathcal{M}}(p,\gamma) := \overline{\mathcal{M}}(p,M) \underset{ev \searrow M \swarrow ev}{\times} \overline{\mathcal{N}}_+(\gamma)$ ← SFT compact-cation of

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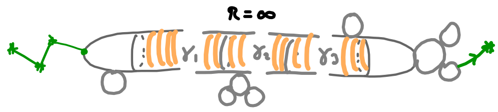
$\downarrow$ M ; $[\hat{u}] \downarrow \operatorname{im}\hat{u} \cap \{0\}\times M$

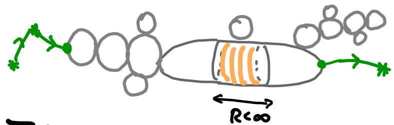
$\hat{J}_\pm = \begin{pmatrix} j_{\mathbb{C}} & 0 \\ *_\pm & J \end{pmatrix}$ on $T\mathbb{C} \times TM$; $*_\pm$ domain dependent Hamiltonian perturbation

SSP: $\overline{\mathcal{M}}(\gamma,p) := \overline{\mathcal{N}}_-(\gamma) \underset{ev \searrow M \swarrow ev}{\times} \overline{\mathcal{M}}(M,p)$

h: $\overline{\mathcal{M}}(p_-,p_+) = \bigcup_{0\le R<\infty} \overline{\mathcal{M}}(p_-,M) \underset{M}{\times} \overline{\mathcal{M}}^{S^2}_R \underset{M}{\times} \overline{\mathcal{M}}(M,p_+)$



$\cup \bigcup_{k=0}^{\infty} \bigcup_{\gamma_1 \ldots \gamma_k \in \mathcal{P}_H} \overline{\mathcal{M}}(p_-,M) \underset{M}{\times} \overline{\mathcal{M}}^{S^2}_+(\gamma_1) \times \overline{\mathcal{M}}^{S^2}(\gamma_1,\gamma_2) \ldots$



$\times \overline{\mathcal{M}}^{S^2}(\gamma_{k-1},\gamma_k) \times \overline{\mathcal{M}}^{S^2}_-(\gamma_k) \underset{M}{\times} \overline{\mathcal{M}}(M,p_+)$

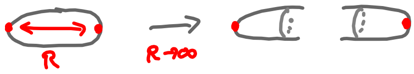
$= \overline{\mathcal{M}}(p_-,M) \underset{ev \searrow M \swarrow ev_0}{\times} \overline{\mathcal{N}}_{stretch} \underset{ev_\infty \searrow M \swarrow ev}{\times} \overline{\mathcal{M}}(M,p_+)$

$\overline{\mathcal{N}}_{stretch}$: SFT-compact-cation under "neck stretching" $R \to \infty$ of

$$\mathcal{N}_{stretch} := \bigcup_{R\ge 0} \{\hat{u}: \mathbb{P}^1 \to \mathbb{P}^1 \times M \mid \bar\partial_{\hat{J}_R}\hat{u} = 0,\ [\hat{u}] = [id]\times[u],\ E(u) < \infty\} / Aut(\mathbb{P}^1)$$

$ev_0 \swarrow M$, $\searrow ev_\infty M$; $\downarrow \operatorname{im}\hat{u} \cap \{{}^0_\infty\}\times M$

$(\mathbb{P}^1\times M, \hat{J}_R) \xrightarrow[R\to\infty]{} (\mathbb{C}^+\times M, \hat{J}_+) \cup (\mathbb{C}^-\times M, \hat{J}_-)$

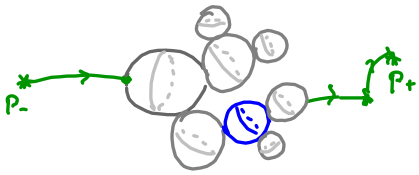


$\hat{J}_R$ compatible with $\omega_{\mathbb{P}^1}\times\omega_M$ → metrics g_R on $\mathbb{P}^1\times M$ with $dist_{g_R}((0,p),(\infty,q)) \xrightarrow[R\to\infty]{} \infty \quad \forall p,q \in M$

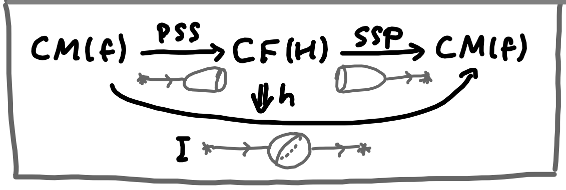
SFT-stretching-limit: buildings of nodal holomorphic curves in levels $\mathbb{C}^+\times M$, … $\mathbb{R}\times S^1\times M$ …, $\mathbb{C}^-\times M$

I: $\overline{\mathcal{M}}^I(p_-,p_+) = \overline{\mathcal{M}}(p_-,p_+) \cap \{R=0\}$

$= \overline{\mathcal{M}}(p_-,M) \underset{M}{\times} \overline{\mathcal{N}}_{R=0} \underset{M}{\times} \overline{\mathcal{M}}(M,p_+)$



compact Piunikhin-Salamon-Schwarz moduli spaces



Fredholm index of section cutting out $\overline{\mathcal{M}}$
//

		"vdim" = dimension if regular	expected boundary phenomena
PSS	$\overline{\mathcal{M}}(p_-,\gamma) := \overline{\mathcal{M}}(p_-,M) \times_M \overline{\mathcal{N}}_+(\gamma)$	$\operatorname{ind}\bar\partial_J - \dim \operatorname{Aut} - \dim W^u_{p_-}$	Morse breaking building levels
SSP	$\overline{\mathcal{M}}(\gamma,p_+) := \overline{\mathcal{N}}_-(\gamma) \times_M \overline{\mathcal{M}}(M,p_+)$	$\operatorname{ind}\bar\partial_J - \dim \operatorname{Aut} - \dim W^s_{p_+}$	Morse breaking building levels
h	$\overline{\mathcal{M}}(p_-,p_+) := \overline{\mathcal{M}}(p_-,M) \times_M \overline{\mathcal{N}}_{stretch} \times_M \overline{\mathcal{M}}(M,p_+)$	$\operatorname{ind}\bar\partial_J + 1 - \dim \operatorname{Aut} - \dim W^u_{p_-} - \dim W^s_{p_+}$	Morse breaking building levels
I	$\overline{\mathcal{M}}^I(p_-,p_+) := \overline{\mathcal{M}}(p_-,M) \times_M \overline{\mathcal{N}}_{R=0} \times_M \overline{\mathcal{M}}(M,p_+)$	$\operatorname{ind}\bar\partial_J - \dim \operatorname{Aut} - \dim W^u_{p_-} - \dim W^s_{p_+}$	Morse breaking

+ nodal curves expected in in codimension ≥ 2

energy $E : \overline{\mathcal{M}}(\dots) \to \mathbb{R}$ given by $E : \overline{\mathcal{N}}(\dots) \to \mathbb{R}$

$(x_-, \hat u, x_+) \longmapsto E(\hat u)$

index $I : \overline{\mathcal{M}}(\dots) \to \mathbb{Z}$

$(x_-, \hat u, x_+) \longmapsto \operatorname{ind} D_{\hat u}\bar\partial_J \ (+1) - \dim \operatorname{Aut} - |p_-| + |p_+| - (2n-1)$ $\qquad |p| := \dim W^u_p$

$\overline{\mathcal{M}}(\dots)_{k,E_0} := \{ I = k,\ E = E_0 \} \subset \overline{\mathcal{M}}(\dots) \qquad \left(\subset \overline{\mathcal{M}}(p_-,M) \times_M \overline{\mathcal{N}}_{E=E_0} \times_M \overline{\mathcal{M}}(M,p_+) \right)$

(after regularization) expect

④ $\partial \overline{\mathcal{M}}^I(p_-,p_+)_{E_0} \setminus \text{corners} = \partial \overline{\mathcal{M}}(p_-,M) \times_M \overline{\mathcal{N}}_{R=0, E=E_0} \times_M \mathcal{M}(M,p_+) \ \cup\ \mathcal{M}(p_-,M) \times_M \overline{\mathcal{N}}_{R=0, E=E_0} \times_M \partial \overline{\mathcal{M}}(M,p_+)$

③ $\partial \overline{\mathcal{N}}_{stretch} \setminus \text{corners} = \mathcal{N}_{R=0} \cup \bigcup_\gamma \mathcal{N}_+(\gamma) \times \mathcal{N}_-(\gamma)$ $\ (R=\infty)$

$$\Rightarrow \partial \overline{\mathcal{M}}(p_-,p_+)_{E_0} \setminus \text{corners} = \partial \overline{\mathcal{M}}(p_-,M) \times_M \operatorname{int} \overline{\mathcal{N}}_{stretch, E=E_0} \times_M \mathcal{M}(M,p_+) \ \cup\ \mathcal{M}(p_-,M) \times_M \operatorname{int} \overline{\mathcal{N}}_{stretch, E=E_0} \times_M \partial \overline{\mathcal{M}}(M,p_+)$$
$$\cup\ \operatorname{int} \overline{\mathcal{M}}^I_{E_0}(p_-,p_+) \ \cup \bigcup_{\gamma,\ E_+ + E_- = E_0} \operatorname{int} \overline{\mathcal{M}}(p_-,\gamma)_{E_+} \times \operatorname{int} \overline{\mathcal{M}}(\gamma,p_+)_{E_-}$$

"non-equivariant" proof of Arnold Conjecture

① Morse complex with Novikov coefficients

② $\overline{\mathcal{M}}(\dots)_{0,E_i} = \sigma^{-1}(0)$ for "proper" index 0 [redacted]-sections

Claim: PSS, SSP, h, I : $\langle * \rangle \longmapsto \sum_{\cdot, i} \langle [\overline{\mathcal{M}}(*,\cdot)_{0,E_i}], 1 \rangle \, q^{E_i} \langle \cdot \rangle$ well defined

Proof:

- $\overline{\mathcal{M}}.(*,\cdot)_{0,E_i} = \sigma^{-1}(0)$ is the (compact) zero set of a [redacted]-section of index 0
- [redacted]-sections of index 0 with compact zero set have regularizations

as
- Čech homology class $\in \check{H}_0(\sigma^{-1}(0), \mathbb{Q})$
- branched weighted 0-manifold $(\sigma + \underline{\nu})^{-1}(0)$ unique up to weighted cobordism

[redacted] : $\overline{\mathcal{M}} = |\sigma^{-1}(0)| = \sigma^{-1}(0) / \text{morphisms}$ $\mathcal{E} \downarrow \circlearrowleft \sigma$ $\mathcal{B}$ functor between categories

	Siebert	Hofer-Wysocki-Zehnder	"Kuranishi"
$Obj_{\mathcal{E}}$	union of open subsets in Banach space	M-polyfold with boundary & corners	union of finite dimensional manifolds with boundary & corners
$Mor_{\mathcal{B}}$	homeomorphisms ⊛	scale-diffeomorphisms	(stratified) [MW in progress] smooth embeddings
$\sigma\|_{Obj}$	$\mathcal{C}^1$ up to finite dimensions	scale-smooth	(stratified) smooth

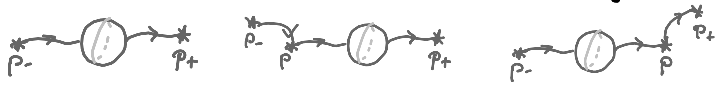
Rmk: index = index $D_u\sigma$ locally constant $\forall u \in \mathcal{B}$, $E : \mathcal{B} \to \mathbb{R}$ locally constant

⊛ boundary & corner notion requires differentiability since  $\underset{\text{homeo}}{\simeq}$ 

"non-equivariant" proof of Arnold Conjecture

④ $\overline{\mathcal{M}}^I(p_-,p_+)_{1,E_i} = \sigma^{-1}(0)$ for proper index 1 [redacted]-sections $\begin{matrix} \mathcal{E}_i \\ \downarrow \uparrow \sigma_i \\ \mathcal{B}_i \end{matrix}$

$\overline{\mathcal{M}}(p_-,M) \times_M \overline{\mathcal{N}}_{R=0, E=E_i} \times_M \overline{\mathcal{M}}(M,p_+)$



over $\mathcal{B}$ with "boundary stratification induced from Morse trajectory spaces"

polyfold: $\mathcal{B} = \overline{\mathcal{M}}(p_-,M) \times \mathcal{B}_N \times \overline{\mathcal{M}}(M,p_+)$ $\mathcal{B}_N$ polyfold without boundary

$\sigma(x_-,\hat{u},x_+) = \left(\bar{\partial}_J \hat{u}, \text{"}ev(x_-) - ev_0(\hat{u})\text{"}, \text{"}ev(x_+) - ev_\infty(\hat{u})\text{"} \right)$ $\bar{\partial}_J : \mathcal{B}_N \to \mathcal{E}$ polyfold Fredholm

$\Rightarrow \partial'\mathcal{B} = \partial'\overline{\mathcal{M}}(p_-,M) \times \mathcal{B}_N \times \mathrm{int}\,\overline{\mathcal{M}}(M,p_+) \cup \mathrm{int}\,\overline{\mathcal{M}}(p_-,M) \times \mathcal{B}_N \times \partial'\overline{\mathcal{M}}(M,p_+)$

Notation : ∂' = boundary ∖ corners

$\Rightarrow |\sigma^{-1}(0) \cap \partial'\mathcal{B}| = \underbrace{\partial'\overline{\mathcal{M}}(p_-,M)}_{\bigcup_p \mathcal{M}(p_-,p)\times\overline{\mathcal{M}}(p,M)} \times_M \overline{\mathcal{N}}_{R=0, E=E_i} \times_M \mathcal{M}(M,p_+) \cup \mathcal{M}(p_-,M) \times_M \overline{\mathcal{N}}_{R=0, E=E_i} \times_M \underbrace{\partial'\overline{\mathcal{M}}(M,p_+)}_{\bigcup_p \overline{\mathcal{M}}(M,p)\times\mathcal{M}(p,p_+)}$

$= \bigcup_p \mathcal{M}(p_-,p) \times \mathrm{int}\,\overline{\mathcal{M}}^I(p,p_+)_{E_i} \cup \bigcup_p \mathrm{int}\,\overline{\mathcal{M}}^I(p_-,p)_{E_i} \times \mathcal{M}(p,p_+)$

from index formula (⇒ additivity in breaking): $1 = \underbrace{|p|-|p_-|}_{\geq 1 \text{ or } \emptyset} + \underbrace{\mathrm{vdim}\,\overline{\mathcal{M}}^I(p,p_+)_{E_i}}$ resp. $\underbrace{\mathrm{vdim}\,\overline{\mathcal{M}}^I(p_-,p)_{E_i}} + \underbrace{|p_+|-|p|}_{\geq 1 \text{ or } \emptyset}$

expected ≥ 0 or $\emptyset$

⇝ If σ is regular then $|\sigma^{-1}(0)|$ is a 1-dimensional manifold/orbifold

with $\partial|\sigma^{-1}(0)| = |\sigma^{-1}(0) \cap \partial\mathcal{B}| = \bigcup_{|p|=|p_-|+1} \mathcal{M}(p_-,p) \times \overline{\mathcal{M}}^I(p,p_+)_{E_i,0} \cup \bigcup_{|p|=|p_+|-1} \overline{\mathcal{M}}^I(p_-,p)_{E_i,0} \times \mathcal{M}(p,p_+)$

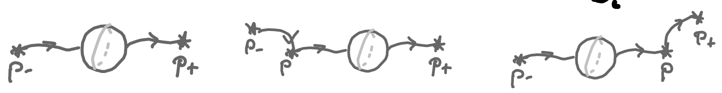
≙ contributions to $I \circ \partial$ and $\partial \circ I$

$\Rightarrow 0 = I \circ \partial + \partial \circ I$

"non-equivariant" proof of Arnold Conjecture

④ $\overline{\mathcal{M}}^I(p_-,p_+)_{1,E_i} = \sigma^{-1}(0)$ for proper index 1 [redacted]-sections $\begin{matrix} \mathcal{E}_i \\ \downarrow \uparrow \sigma_i \\ \mathcal{B}_i \end{matrix}$

$\overline{\mathcal{M}}(p_-,M) \underset{M}{\times} \overline{\mathcal{N}}_{\substack{R=0 \\ E=E_i}} \underset{M}{\times} \overline{\mathcal{M}}(M,p_+)$



over $\mathcal{B}$ with "boundary stratification induced from Morse trajectory spaces"

polyfold: $\mathcal{B} = \overline{\mathcal{M}}(p_-,M) \times \mathcal{B}_N \times \overline{\mathcal{M}}(M,p_+)$ $\quad \mathcal{B}_N$ polyfold without boundary

$\sigma(x_-,\hat{u},x_+) = \left(\bar\partial_J \hat{u}, \text{"}ev(x_-) - ev_0(\hat{u})\text{"}, \text{"}ev(x_+) - ev_\infty(\hat{u})\text{"}\right)$ $\quad \bar\partial_J : \mathcal{B}_N \to \mathcal{E}$ polyfold Fredholm

$\Rightarrow \partial'\mathcal{B} = \partial'\overline{\mathcal{M}}(p_-,M) \times \mathcal{B}_N \times \mathrm{int}\,\overline{\mathcal{M}}(M,p_+) \cup \mathrm{int}\,\overline{\mathcal{M}}(p_-,M) \times \mathcal{B}_N \times \partial'\overline{\mathcal{M}}(M,p_+)$

In general, abstract regularization for [redacted]-sections provides a multisection $\nu : \mathcal{B} \to \text{finite subsets}(\mathcal{E})$ so that all branches of $\sigma+\nu$ are $\pitchfork 0_{\mathcal{E}}$ and hence (by [redacted] implicit function theorem) $\overline{\mathcal{M}}^I(p_-,p_+)_{1,E_i}$ is "regularized to" $\overline{\mathcal{M}}^I(p_-,p_+)^{reg}_{1,E_i} := |(\sigma+\nu)^{-1}(0)|$ compact weighted branched 1-manifold with boundary $\partial|(\sigma+\nu)^{-1}(0)| = |(\sigma+\nu)^{-1}(0) \cap \partial'\mathcal{B}|$

(in polyfold setting) $= \bigcup_p \left| (\sigma+\nu)^{-1}(0) \cap \left(\mathcal{M}(p_-,p) \times \mathrm{int}\,\mathcal{B}(p,p_+)_{E_i} \cup \mathcal{B}(p_-,p)_{E_i} \times \mathcal{M}(p,p_+)\right)\right|$

by above argument with abstract regularity $= \bigcup_{|p|=|p_-|+1} \mathcal{M}(p_-,p) \times \overline{\mathcal{M}}^I(p,p_+)^{reg}_{0,E_i} \cup \bigcup_{|p|=|p_+|-1} \overline{\mathcal{M}}^I(p_-,p)^{reg}_{0,E_i} \times \mathcal{M}(p,p_+)$

|| index 0 component of $|(\sigma+\nu)^{-1}(0) \cap \mathcal{B}(p,p_+)_{E_i}|$ || index 0 component of $|(\sigma+\nu)^{-1}(0) \cap \mathcal{B}(p_-,p)_{E_i}|$

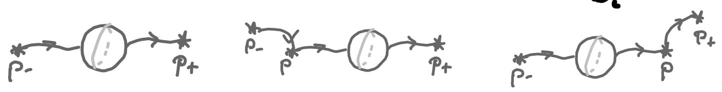
$\Rightarrow 0 = I \circ \partial + \partial \circ I$ $\quad$ for I defined from $(\sigma+\nu)^{-1}(0)$ &

⊛ need to be able to prescribe $\nu|_{\partial'\mathcal{B}}$ by choice made in ② when constructing I

"non-equivariant" proof of Arnold Conjecture

④ $\overline{\mathcal{M}}^I(p_-,p_+)_{1,E_i} = \sigma^{-1}(0)$ for proper index 1 [redacted]-sections $\mathcal{E}_i \downarrow \uparrow \sigma_i$, $\mathcal{B}_i$

$$\overline{\mathcal{M}}^I(p_-,p_+)_{1,E_i} = \overline{\mathcal{M}}(p_-,M) \times_M \overline{\mathcal{N}}_{R=0,\,E=E_i} \times_M \overline{\mathcal{M}}(M,p_+)$$



over $\mathcal{B} = \overline{\mathcal{M}}(p_-,M) \times \mathcal{B}_N \times \overline{\mathcal{M}}(M,p_+)$ with boundary stratification induced by

$$\partial \overline{\mathcal{M}}(p_-,M) = \bigcup_p \mathcal{M}(p_-,p) \times \overline{\mathcal{M}}(p,M)$$
$$\partial \overline{\mathcal{M}}(M,p_+) = \bigcup_p \overline{\mathcal{M}}(M,p) \times \mathcal{M}(p,p_+)$$

$$\Rightarrow 0 = I \circ \partial + \partial \circ I$$

③ $\overline{\mathcal{M}}^h(p_-,p_+)_{1,E_i} = \sigma^{-1}(0)$ for proper index 1 [redacted]-sections

over $\mathcal{B} = \overline{\mathcal{M}}(p,M) \times \mathcal{B}_{N_{stretch}} \times \overline{\mathcal{M}}(M,p)$ with boundary stratification induced by $\partial \overline{\mathcal{M}}(p,M)$, $\partial \overline{\mathcal{M}}(M,p)$, $\partial \mathcal{B}_{N_{stretch}} = \mathcal{B}_{N_{R=0}} \cup \mathcal{B}_{N_+} \times \mathcal{B}_{N_-}$

($\partial \overline{\mathcal{M}}(p,M) \rightsquigarrow h \circ \partial$, $\partial \overline{\mathcal{M}}(M,p) \rightsquigarrow \partial \circ h$, $\mathcal{B}_{N_{R=0}} \rightsquigarrow I$, $\mathcal{B}_{N_+} \rightsquigarrow PSS$, $\mathcal{B}_{N_-} \rightsquigarrow SSP$)

$$\Rightarrow I - SSP \circ PSS = \partial \circ h - h \circ \partial$$

"non-equivariant" proof of Arnold Conjecture

① $CM(f) := \bigoplus_{p \in \mathrm{Crit} f} \Lambda \langle p \rangle \circlearrowleft \partial \quad \rightsquigarrow \ker\partial / \mathrm{im}\,\partial = HM(f) \simeq H_*(M;\mathbb{Q}) \otimes_{\mathbb{Q}} \Lambda$

②-④ $CM(f) \xrightarrow{PSS} CF(H) \xrightarrow{SSP} CM(f)$ $\quad SSP \circ PSS = I : HM(f) \to HM(f)$

$\Downarrow h$

⑤ $I : HM_* \to HM_*$ isomorphism because

$\overline{\mathcal{M}}^I(p_-, p_+)_{0,E} = \emptyset$ for $E < 0$

$E(u) = \int u^*\omega \qquad = \{u \equiv p_- = p_+\}$ for $E = 0$

are regular and don't need to be perturbed for coherence with $E > 0$

$\Rightarrow I = id_{CM(f)} + \sum_{i=0}^{\infty} q^{E_i} I_i \qquad 0 < E_0 < E_1 < \dots E_i \xrightarrow[i\to\infty]{} \infty$

$\Rightarrow \exists I^{-1} = id_{CM(f)} +$ (determined iteratively)