Application of abstract regularization techniques

Proof of Arnold Conjecture via Floer Theory

- geometric : as before + ...
- via Kuranishi regularization
- via polyfold regularization

$\boxed{\text{Arnold Conjecture}}$ (M,ω) symplectic, closed

<u>autonomous Hamiltonian system:</u>

$H : M \to \mathbb{R} \rightsquigarrow$ Hamiltonian vector field $X_H : \omega(\cdot, X_H) = dH$

$X_H = J\nabla H \Leftarrow \omega(\cdot, X_H) = g(\cdot, -JX_H) \quad dH = g(\cdot, \nabla H)$

$\{p \mid X_H(p) = 0\}$ = rest points of Hamiltonian system

$= \operatorname{crit} H$ = generators of Morse complex – with homology $H_*(M)$

$\Rightarrow$ # rest points $\geq \sum_{i=0}^{\dim M} \operatorname{rank} H_i(M)$ if H "nondegenerate"

$\forall p \in \operatorname{crit} H \ \nabla^2 H(p)$ nondegenerate

<u>periodic Hamiltonian system:</u>

$H : S^1 \times M \to \mathbb{R} \rightsquigarrow$ periodic vector field $(X_{H_t})_{t \in S^1}$

$\rightsquigarrow$ periodic orbits $\mathcal{P}_H = \{\gamma : S^1 \to M \mid \dot\gamma(t) = X_{H_t}(\gamma(t))\}$

Arnold Conjecture

(M,ω) symplectic, closed

autonomous Hamiltonian system:

$H: M \to \mathbb{R} \rightsquigarrow$ Hamiltonian vector field X_H : $\omega(\cdot, X_H) = dH$

$\#$ rest points $\geq \sum_{i=0}^{\dim M} \operatorname{rank} H_i(M)$ if H "nondegenerate"

periodic Hamiltonian system:

$H: S^1 \times M \to \mathbb{R} \rightsquigarrow$ periodic orbits $\mathcal{P}_H = \{\gamma : S^1 \to M \mid \dot\gamma(t) = X_{H_t}(\gamma(t))\}$

Conj. [Arnold] (a) $\# \mathcal{P}_H \geq \min \#$ of rest points of autonomous Hamiltonian

(b) $\# \mathcal{P}_H \geq \sum_{i=1}^{\dim M} \operatorname{rk} H_i(M)$ if H is "nondegenerate"

$\mathcal{P}_H$ "cut out transversely"

$\left(\forall \gamma \in \mathcal{P}_H \quad \nabla_t + \nabla X(\gamma) : W^{1,p}(S^1, \gamma^* TM) \to L^p(S^1, \gamma^* TM) \text{ surjective}\right)$

Plan of proof of (b) [Floer]

→ construct a chain complex

(CF, ∂) generated by $\mathcal{P}_H$

→ show $HF = \ker\partial / \operatorname{im}\partial \simeq H_*(M)$

autonomous case

(CM, ∂) generated by Crit H

$HM = \ker\partial / \operatorname{im}\partial \simeq H_*(M)$

Arnold Conjecture (nondegenerate, homology case)

Claim: $\# \mathcal{P}_H = \{\gamma : S^1 \to M \mid \dot\gamma(t) = X_{H_t}(\gamma(t))\} \geq \sum_{i=1}^{\dim M} \operatorname{rk} H_i(M)$

① Floer homology $HF(H)$

② $HF(H) \simeq H_*(M)$

Floer:
- invariance $HF(H) \simeq HF(G)$ $\forall H, G$
- $G : M \to \mathbb{R}$ C^2 small $\Rightarrow$

Floer complex $\equiv$ Morse complex
(CF, ∂_F) $\quad$ (CM, ∂_M)

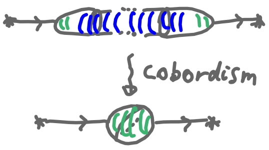
Piunikhin-Salamon-Schwarz:

$PSS : HM(f : M \to \mathbb{R}) \xrightarrow{\sim} HF(H)$ for any H, f $\quad$ $\boxed{⚠ \text{ need } S^1\text{-equivariant transversality}}$

[AFW]: $CM(f) \xrightarrow{PSS} CF(H) \xrightarrow{SSP} CM(f)$ induces isomorphism on $HM(f)$

$\bar\partial_J u = JX(u)$

$\bar\partial_J u = 0$



no S^1-symmetry (since H depends on S^1)

PSS $\to$ $J \in \mathcal{J}(M, \omega)$ S^1-symmetry, ?transversality
$\rightsquigarrow$ $PSS \circ SSP = id$

AFW $\to$ $J : S^1 \to \mathcal{J}(M, \omega)$ no — " —, ✓ transversality
$\rightsquigarrow$ $PSS \circ SSP = id +$ "upper triangular wrt energy filtration"

Note: This doesn't even require the Floer differential.

$\boxed{\text{"minimized" PSS proof of Arnold Conjecture}}$ & corrections of lecture

① Morse complex with Novikov coefficients

$f : M \to \mathbb{R}$ Morse (nondegenerate) $\Rightarrow$ $\mathrm{Crit} f = \{p \mid \nabla f(p) = 0\}$ finite

$CM(f) := \bigoplus_{p \in \mathrm{Crit} f} \Lambda \langle p \rangle$ $\qquad \Lambda = \{ \sum_{i=0}^{\infty} a_i q^{\lambda_i} \mid 0 \le \lambda_0 < \lambda_1 < \dots, \lambda_i \xrightarrow[i\to\infty]{} \infty, a_i \in \mathbb{Q} \}$

Novikov ring in formal variable q

field when dropping $\lambda_0 \geq 0$

$\circlearrowright$

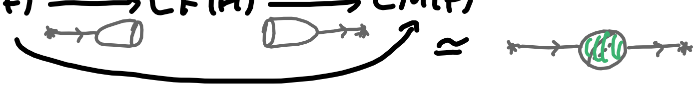
∂ Λ linear

$\langle p_- \rangle \mapsto \sum_{p_+ \in \mathrm{Crit} f} \# \underbrace{\mathcal{M}_1^{Morse}(p_-, p_+)}\langle p_+ \rangle$

$= \{ x : \mathbb{R} \to M \mid \dot{x} + \nabla f(x) = 0 \quad x(s) \xrightarrow[s \to \pm\infty]{} p_\pm \} / \mathbb{R}$

$\simeq W^u_{p_-} \cap W^s_{p_+} \cap f^{-1}\left(\tfrac{1}{2}(f(p_-) + f(p_+))\right)$ $\qquad \pitchfork$ iff g "Morse-Smale"

$CM(f) \xrightarrow{PSS} CF(H) \xrightarrow{SSP} CM(f)$ $\simeq$ $\simeq$ on HM



② PSS : $CM(f) \to CF(H) := \bigoplus_{\gamma \in \mathcal{P}_H} \Lambda \langle \gamma \rangle$

$\langle p_- \rangle \mapsto \sum_{\gamma \in \mathcal{P}_H} \sum_{u \in \mathcal{M}(p_-, \gamma)} a_u \, q^{E(u)} \langle \gamma \rangle$

$a_u \in \mathbb{Q}$ "weight" of solution u - determined by orientation and local "multiplicity"

SSP : $CF(H) \to CM(f)$, $\langle \gamma \rangle \mapsto \sum_{p_+ \in \mathrm{Crit} f} \sum_{u \in \mathcal{M}(\gamma, p_+)} a_u \, q^{E(u)} \langle p_+ \rangle$

I : $CM(f) \to CM(f)$, $\langle p_- \rangle \mapsto \sum_{p_+ \in \mathrm{Crit} f} \sum_{u \in \mathcal{M}(p_-, p_+)} a_u \, q^{E(u)} \langle p_+ \rangle$

are well defined because $\mathcal{M}(p_-, \gamma)$, $\mathcal{M}(\gamma, p_+)$, $\mathcal{M}(p_-, p_+)$ are oriented $\underline{0\text{-manifolds}}$ with $\{[u] \in \mathcal{M}(..) \mid E(u) < \lambda\}$ finite $\forall \lambda > 0$.

yielding $a_u \in \mathbb{Z}$

or $\underline{\text{weighted branched } 0\text{-manifolds}}$ yielding $a_u \in \mathbb{Q}$

"minimized" PSS proof of Arnold Conjecture

① $CM(f) := \bigoplus_{p\in \mathrm{Crit} f} \Lambda\langle p\rangle \quad \circlearrowleft \quad \partial : \langle p_-\rangle \mapsto \sum_{p_+\in \mathrm{Crit} f} \#\mathcal{M}_1^{Morse}(p_-,p_+)\langle p_+\rangle$

$$\begin{tikzcd} CM(f) \arrow[r, "PSS"] \arrow[rr, bend right, "I"'] & CF(H) \arrow[r, "SSP"] & CM(f) \end{tikzcd} \simeq \quad \simeq \text{ on } HM$$

② $PSS,\ SSP,\ I,\ h \ : \langle * \rangle \mapsto \sum_{\bullet} \sum_{u\in \mathcal{M}(*,\bullet)} a_u\, q^{E(u)} \langle \bullet \rangle$

③ $\boxed{SSP\circ PSS - I = \partial h - h\partial}$

because there are compact oriented 1-manifolds $\tilde{\mathcal{M}}^h(p_-,p_+)$ with

$$\partial\tilde{\mathcal{M}}^h(p_-,p_+) = \coprod_{\gamma\in P_H} \mathcal{M}^{PSS}(p_-,\gamma)\times\mathcal{M}^{SSP}(\gamma,p_+) \sqcup \mathcal{M}^{I}(p_-,p_+)$$
$$\sqcup \coprod_{p\in \mathrm{Crit} f} \mathcal{M}^{h}(p_-,p)\times\mathcal{M}^{Morse}(p,p_+) \sqcup \coprod_{p\in \mathrm{Crit} f} \mathcal{M}^{Morse}(p_-,p)\times\mathcal{M}^{h}(p,p_+)$$

④ $\boxed{I\partial = \partial I}$

$$\rightsquigarrow \partial\tilde{\mathcal{M}}^I(p_-,p_+) = \coprod_{p} \mathcal{M}^{Morse}(p_-,p)\times\mathcal{M}^{I}(p,p_+) \sqcup \coprod_{p} \mathcal{M}^{I}(p_-,p)\times\mathcal{M}^{Morse}(p,p_+)$$

⑤ I maps onto $H_*(M) \simeq \ker\partial / \mathrm{im}\,\partial$

"minimized" PSS proof of Arnold Conjecture

① $CM(f) := \bigoplus_{p \in \mathrm{Crit} f} \Lambda \langle p \rangle \circlearrowleft \partial : \langle p_- \rangle \mapsto \sum_{p_+ \in \mathrm{Crit} f} \# \mathcal{M}_1^{Morse}(p_-, p_+) \langle p_+ \rangle$

$$\boxed{\begin{array}{c} CM(f) \xrightarrow{PSS} CF(H) \xrightarrow{SSP} CM(f) \\ \Downarrow h \qquad\qquad \curvearrowright I \end{array} \simeq}$$

$\simeq$ on HM

② $PSS,\ SSP,\ I,\ h \ : \langle * \rangle \mapsto \sum_{\bullet} \sum_{u \in \mathcal{M}(*, \bullet)} q^{E(u)} \langle \bullet \rangle$

③ $SSP \circ PSS - I = \partial h - h \partial$

④ $I \partial = \partial I$

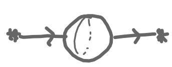
⑤ $\boxed{I : HM_* \to HM_* \text{ isomorphism}}$ on $HM_*(f) \simeq H_*(M; \Lambda)$

because $\mathcal{M}^I(p_-, p_+) = \{u = \text{const}, E(u) = 0\} \cup \{u \text{ nonconst.}, E(u) \geq \lambda_0\}$

$\Rightarrow \quad I = id_{CM} + q^{\lambda_0} I'$ for some $I' : CM \to CM$ with $\lambda_0 > 0$

$\Rightarrow \quad \exists\, I^{-1}$ determined iteratively from $\{E(u) = \lambda_n\}$ $\quad \lambda_0 < \lambda_1 < \lambda_2 < \ldots$

$\mathcal{M}^I(p_-, p_+) = \{u : \mathbb{C}P^1 \to M \mid \bar{\partial}_J u = 0,\ u(0) \in W^u_{p_-},\ u(\infty) \in W^s_{p_+}\}$



$\to$ $u = u_0$ const $\rightsquigarrow u_0 \in W^u_{p_-} \cap W^s_{p_+}$ $\quad \emptyset$ or $\dim \geq 1$ unless $p_- = p_+ = u_0$

$\rightsquigarrow id_{CM}$

$\to$ u nonconst $\overset{?}{\Longrightarrow}$ comes in S^1 family if J regular and S^1-invariant

$\Downarrow$ $\quad E(u) \geq \lambda_0 > 0 \rightsquigarrow q^{\lambda_0} I'$ $\quad \left(\overset{\Downarrow}{I = id_{CM}} \right)$