Abstract approaches to regularizing moduli spaces of pseudoholomorphic curves

Approach #1 : "Euler class on Banach orbifolds"
[Siebert]

* Gromov-Witten invariant
* Kuranishi section for
  - nontrivial Deligne-Mumford space
  - neighbourhood of nodal curve

stabilized gluing

Approach #2 : finite dimensional reductions

Fukaya-Ono-Oh-Ohta (1999) Kuranishi structure (of germs)

Joyce — " — ⇝ d-orbifold

McDuff-W. Kuranishi atlas

Theorem 0.1 *Let (M, ω) be a closed symplectic manifold with a tame almost complex structure J. Then the space $\mathcal{C}(M;p)$ of stable parametrized marked complex curves in M of Sobolev class L^p_1 (Definition 3.1) is a Banach orbifold. Moreover, there is a Banach orbibundle E over $\mathcal{C}(M;p)$ with fiber $\bar{L}^p(C; \varphi^* T_M \otimes \bar{\Omega}_C)$ at $(C, \mathbf{x} = (x_1, \ldots, x_k), \varphi : C \to M)$ with an oriented Kuranishi section s (Definition 1.15) with $\hat{s}(C, \mathbf{x}, \varphi) = \bar{\partial}_J \varphi$. The zero locus of s is the set $\mathcal{C}^{\mathrm{hol}}(M, J)$ of stable pseudo holomorphic curves in (M, J) (Definition 3.5), which is a locally finite dimensional Hausdorff space with compact components.*

Let $\mathcal{M}_{g,k}$ be the moduli space of Deligne-Mumford stable k-marked algebraic curves of genus g, with the convention $\mathcal{M}_{g,k} = \{\mathrm{pt}\}$ whenever $2g + k < 3$. The localized Euler class $\boxed{GW^{M,J}_{g,k} \in H_*(\mathcal{C}^{\mathrm{hol}}(M, J))}$ associated to (E, s) (Theorem 1.21) gives rise to GW-correspondences (Definition 7.2)

$$GW^{M,J}_{g,k} : H^*(M)^{\otimes k} \longrightarrow H_*(\mathcal{M}_{g,k}), \quad \overset{\simeq \mathrm{pt}}{} \simeq \mathbb{Q}$$

that are invariants of the symplectic deformation type of (M, ω). They coincide with the ones defined in [RuTi2] in case (M, ω) is semi-positive. (by geometric regularization – eg [McDuff-Salamon])

basic example: $[\overline{\mathcal{M}}] = \mathrm{Euler}\left(\begin{matrix} \mathcal{E} \\ \downarrow \\ W^{1,p}(\mathbb{P}^1, M)/\mathrm{Aut} \end{matrix} \curvearrowright s = \bar{\partial}_J \right) \in H_*(\overline{\mathcal{M}} = s^{-1}(0); \mathbb{Q})$

$$\overline{\mathcal{M}} = \{ u \in W^{1,p}(\mathbb{P}^1, M) \mid \bar{\partial}_J u = 0,\ u_*[\mathbb{P}^1] = A \} / \mathrm{Aut}(\mathbb{P}^1; i, \infty) \xrightarrow{ev} M$$
$$[u] \longmapsto u(\infty)$$

$$\rightsquigarrow GW^{M,J}_{0,1} : H^*(M) \longrightarrow \mathbb{Q}$$
$$\alpha \longmapsto \langle \underbrace{ev^* \alpha}_{\in H^*(\overline{\mathcal{M}})}, [\overline{\mathcal{M}}] \rangle = \langle \alpha, \underbrace{ev_*[\overline{\mathcal{M}}]}_{\in H_*(M)} \rangle$$

Rmk: semi-positive GW is constructed the same way with

- ideally $[\overline{\mathcal{M}}]$ = classical fundamental class from $\overline{\mathcal{M}} = \bar{\partial}_J^{-1}(0)/\mathrm{Aut} \cup_{\text{glue}} \{\text{nodal curves}\}$ for regular J "being" a compact manifold
- generally, — " — give "pseudocycle"

Then Euler class axiom ① says $\mathrm{Euler}(s) = [s^{-1}(0) \simeq \overline{\mathcal{M}}]$ since "$\pitchfork\ \bar{\partial}_J \curvearrowright \begin{matrix} \mathcal{E} \\ \mathcal{C}(M;p) \end{matrix}$"

<u>Guiding Questions</u> for studying regularization approaches via abstract perturbations / "virtual" fundamental class

$\overline{\mathcal{M}} = s^{-1}(0)$ $\quad [(s+\nu)^{-1}(0)]$ / Euler(s) $= [\overline{\mathcal{M}}]$

→ what is the abstract form of section s?

topological Banach bundle

$$\begin{tikzcd} \mathcal{E} \arrow[d, shift right] & E_0 \arrow[l, "\sim"'] & s_R(\zeta) \\ \mathcal{B} \arrow[u, shift right, "s"'] & \mathbb{R}^b \times V \arrow[l, "\sim"] \arrow[u] & \ni (R,\zeta) \arrow[u, mapsto] \end{tikzcd}$$

locally

continuous (wrt $R \in \mathbb{R}^b$) family of Fredholm sections $s_R \in \mathcal{C}^1(V, E_0)$ "differentiable relative $\mathbb{R}^b$"

→ how is s constructed for pseudoholomorphic curve moduli spaces? from local Fredholm descriptions:

① near smooth curves as before, where nondifferentiability of reparametrization is no issue since transition maps are only required to be continuous

$$\begin{tikzcd} \bigcup \Omega^{0,1} \arrow[d, shift right] & \mathcal{E} \arrow[d, shift right] & \bigcup_{\mathfrak{s}'} \Omega^{0,1} \arrow[d, shift left] \\ \{v(0)\in H_0, v(1)\in H_1\} = \mathfrak{S} \arrow[u, shift right, "\bar\partial_J"'] \arrow[r, hook] \arrow[rr, bend right, "\text{continuous transition map}"'] & \mathcal{B} = W^{1,p}(\mathbb{P}^1, M)/\mathrm{Aut} \arrow[u, shift right, "s"'] & \mathfrak{S}' = \{v'(0)\in H_0', v'(1)\in H_1'\} \arrow[l, hook'] \arrow[u, shift left, "\bar\partial_J"] \end{tikzcd}$$

$v \longmapsto v \circ \varphi_v$

② for $\mathcal{M}_{g,k} \neq pt$ $\qquad$ ③ for nodal curves

(1') analytic issues appear in construction of Kuranishi structure

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\tau"] \arrow[d] & \mathcal{E} \arrow[d, shift right] \\
\mathcal{B}' \arrow[r, hook] \arrow[d, phantom, "\cup" rotate=-90] & \mathcal{B} = W^{1,p}(\mathbb{P}^1; M)/\mathrm{Aut} \arrow[u, shift right, "s"'] \\
s^{-1}(0) &
\end{tikzcd}$$

since τ is required to be "differentiable relative $\mathbb{R}^b$" in local differentiable structures

$$\begin{tikzcd}
\mathcal{E} \arrow[d] & \mathbb{R}^b \times V \times E_0 \arrow[l, hook'] \arrow[d] \\
\mathcal{B} & \mathbb{R}^b \times V \arrow[l, hook']
\end{tikzcd}$$

(part of "Fredholm structure" of s)

This requires construction of stabilizations in local slices that "transform as if the transition map was differentiable":

$$\begin{tikzcd}
 & U\Omega^{0,1} \arrow[dd, shift right] & F \arrow[l, "\tau"'] \arrow[ddl] & \cong_{loc} & \mathcal{F} \arrow[d, "\tau"] & \cong_{loc} & F' \arrow[r, "\tau'"] \arrow[ddr] & U'\Omega^{0,1} \arrow[dd, shift left] & \\
 & & & & \mathcal{E} \arrow[d] & & & & \\
\{v(0)\in H_0, v(1)\in H_1\} = & \mathcal{S} \arrow[uu, shift right, "\bar\partial_J"'] \arrow[rrr, "\sim loc"] \arrow[rrrrrr, bend right, "\Phi: v \mapsto v\circ\varphi_v"'] & & & \mathcal{B}' & & & \mathcal{S}' \arrow[lll, "\sim loc"'] \arrow[uu, shift left, "\bar\partial_J"] & = \{v'(0)\in H'_0, v'(1)\in H'_1\}
\end{tikzcd}$$

$F' = \Phi^* F$ should be a differentiable finite rank bundle

i.e. both τ and $\tau \circ \hat{\Phi}$ should be differentiable

although $\hat{\Phi}$ covers nondifferentiable map $\Phi : \mathcal{S} \to \mathcal{S}'$.

(This can be achieved by "geometric construction" of $\mathcal{F}$ – from $\Omega^{0,1}$ on "universal curve" – which can be reinterpreted as generalized perturbation of J.)

② local Fredholm description for $\overline{\mathcal{M}}_{g,k} \neq pt$ "Deligne-Mumford space"

Ex: $\overline{\mathcal{M}}_{0,\substack{1\\2\\3}} = \{pt\} = \{(\mathbb{P}^1, i, \substack{0\\(0,1)\\(0,1,\infty)})\}$ "fixed marked points"

general definition for fixed (closed, oriented) surface – of genus g

$$\mathcal{M}_{g,k} = \{(\Sigma, j, (z_0,\dots,z_k)) \mid j \text{ complex structure},\ z_0..z_k \in \Sigma \text{ distinct}\}/\sim$$

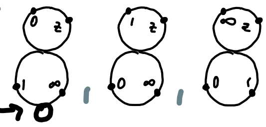
$\wr$ $(\Sigma, \varphi^* j, (\varphi^{-1}(z_0)\dots\varphi^{-1}(z_k)))$ $\forall \varphi \circlearrowleft$ diffeomorphism

$\overline{\mathcal{M}}_{g,k} = \mathcal{M}_{g,k} \cup \{$nodal Riemann surfaces arising from degeneration of j or coincidence of marked points$\}$

Ex: For $g=0, k=1$ can use uniformization theorem to fix representatives

- with $j=i$ → remaining symmetry $Aut(\mathbb{P}^1)$
- with $z_0=0, z_1=1, z_2=\infty$ → no remaining $\sim$ relation

$\Rightarrow \mathcal{M}_{0,4} \simeq \{(\mathbb{P}^1, i, (0, 1, \infty, z)) \mid z \in \mathbb{P}^1 \setminus \{0,1,\infty\}\}$

$\overline{\mathcal{M}}_{0,4} = \mathcal{M}_{0,4} \cup \{$  $\}$ nodal domains $\simeq \mathbb{P}^1$

compactification for $z \to 0$

<u>Fredholm description for $\bar\partial_J$ on</u> $\mathcal{C}(M;p) = \{(\Sigma, j, u: \Sigma \xrightarrow{W^{1,p}} M, \underline{z}) \mid j \text{ cx. str} \dots\}/\sim$

(J almost cx str on M fixed)

$\downarrow$ $\overline{\mathcal{M}}_{g,k} \neq \{pt\}$ $\quad \wr$ $(\Sigma, \varphi^* j, u\circ\varphi, \varphi^{-1}(\underline{z}))$ $\forall \varphi \in Diff(\Sigma)$

$$\bar\partial_J : (\Sigma, j, u, \underline{z}) \longmapsto \bar\partial_{J,j} u = \tfrac{1}{2}(du + J\circ du\circ j)$$

$\wr$ locally on Σ or for $\Sigma \simeq \mathbb{P}^1$

$(\Sigma, j_0, u\circ\varphi_j, \varphi_j^{-1}(\underline{z})) \longmapsto \bar\partial_J(u\circ\varphi_j)$

⚠ differentiability fails for $\overline{\mathcal{M}}_{g,k} \times W^{1,p} \to L^p$

but we have

<u>"differentiability relative $\overline{\mathcal{M}}_{g,k}$"</u>

$$\mathcal{C}(M;p) \underset{locally}{\simeq} \mathcal{M}_{g,k} \times W^{1,p}(\Sigma, M) \dots \xrightarrow{\bar\partial_J} \mathcal{E} = L^p\text{-}(0,1)\text{-forms}$$

$$(j, u) \longmapsto s_j(u) = \bar\partial_{J,j} u$$

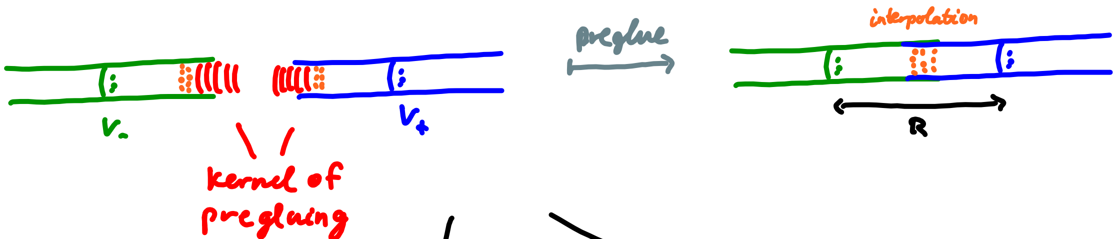
- each s_j is $\mathcal{C}^1$, Fredholm
- $j \mapsto ds_j$ is continuous wrt j

③ local Fredholm description near nodal curves — naive

$$\begin{array}{ccccccc} \cup\,\Omega^{0,1} & & \mathcal{E} & & (L\times V)\times E_0 & & \bar\partial_J(v_- \#_R v_+) \\ \downarrow \uparrow \bar\partial_J & & \downarrow \uparrow s & & \downarrow \uparrow & & \uparrow \\ \{v(0)\in H_0,\ v(1)\in H_1\} = \mathcal{S} & \longrightarrow & \mathcal{B} \cup \mathcal{B}^{nodal} & \overset{\triangle}{\longleftarrow} & L\times V & & (R, v_-, v_+) \end{array}$$

continuous transition map

$\triangle$ pregluing $(R, v_-, v_+) \longmapsto v_- \#_R v_+$ is not injective



Siebert → algebraic geometry ②

HWZ polyfold theory → pregluing <u>is</u> a local chart of polyfold $\mathcal{B} \cup \mathcal{B}^{nodal}$

"gluing after stabilization"

$$\begin{array}{ccc} F_\pm & \xrightarrow{\tau_\pm} & \mathcal{E}_\pm \\ & \searrow & \downarrow \uparrow \bar\partial_J = \sigma_\pm \\ & & \mathcal{B}_\pm \hookrightarrow \overline{\mathcal{B}} \supset \sigma_\pm^{-1}(0) \end{array}$$

$\bar\partial_J + \tau_\pm : F_\pm \to \mathcal{E}_\pm \quad \pitchfork 0$

$$\Rightarrow \quad \begin{array}{c} F_\pm|_{U_\pm} \quad f \\ \downarrow \quad \uparrow s_\pm \\ U_\pm = \{(u,f) \mid \bar\partial_J u = \tau_\pm(f)\} \end{array}$$

$\overline{\mathcal{M}} \hookleftarrow s_\pm^{-1}(0) \subset U_\pm$

local stabilizations

near $(v_-)(v_+)$

(for simplicity forgetting to encode matching at node $v_-(0) = v_+(\infty)$)

local finite dimensional reductions

$\Downarrow$

"glued finite dimensional reduction"

$$\begin{array}{ccc} \tilde F & & F_-|_{U_-} \times F_+|_{U_+} \qquad (f_-, f_+) \\ \downarrow \uparrow \tilde s & \text{natural transition map} & \downarrow \uparrow s \qquad\qquad \uparrow \\ \tilde U \supset \tilde s^{-1}(0) & \hookrightarrow \overline{\mathcal{M}} \hookleftarrow & s^{-1}(0) \subset D \times U_- \times U_+ \ni (e^{-(R+i\theta)}, v_-, f_-, v_+, f_+) \end{array}$$

or $0 \mathrel{\hat=} R = \infty$

$\overline{\mathcal{M}} \ni ([v_-, v_+]) \leftarrow (0, v_-, 0, v_+, 0)$

$\tilde U \ni (\exp_{v_- \# v_+}(\zeta),\ f_- \# f_+ + \eta) \leftarrow$

extended transition map

constructed by stabilized gluing s.t. $f_\pm = 0$ $\Leftarrow$ $f_- \# f_+ + \eta = 0$

via Newton iteration & choices

$\tilde U \dashrightarrow \overline{\mathcal{B}} \dashleftarrow D \times U_- \times U_+$

⚠ extended transition map doesn't naturally arise from an ambient space of "not necc. hol. curves"

#2 Regularization via finite dimensional reductions

Executive summary:

$\overline{\mathcal{M}} = |s^{-1}(0)|$

$= \coprod s_i^{-1}(0) / \mathrm{Mor}$

$$\begin{array}{c} \mathcal{E} \\ \downarrow \uparrow s \\ \mathcal{U} \end{array}$$

section of etale category bundle

Obj: $\begin{array}{c} U_i \times F_i \\ \downarrow \uparrow s_i \\ \coprod_i U_i \end{array} \supset s_i^{-1}(0) \hookrightarrow \overline{\mathcal{M}}$

finite dimensional reductions

Mor:
- $\Gamma_i \circlearrowright U_i \times F_i$ isotropy
- transition

regularization theorem:

$\exists\, \mathcal{P} \subset \{\text{sections } \nu : \mathcal{U} \to \mathcal{E}\}$:

$\forall \nu \in \mathcal{P}$: $|(s+\nu)^{-1}(0)|$ compact manifold

$\forall \nu_0 \neq \nu_1 \in \mathcal{P}$ $\exists$ cobordism $|(s+\nu_0)^{-1}(0)| \sim |(s+\nu_1)^{-1}(0)|$

$\Rightarrow [\overline{\mathcal{M}}] := [\,|(s+\nu)^{-1}(0)|\,]$

Rmk: This trades some analytic issues in approach #1 (more ∞ dimensional) for topological issues.

E.g. can make compatible (also nontrivial) perturbations $s_i + \nu_i \pitchfork 0$

so that $\bigsqcup_i (s+\nu_i)^{-1}(0) / \mathrm{Mor}$ is locally homeomorphic to $\mathbb{R}^n$

but need to ensure Hausdorff & compactness property – which is nontrivial for quotient topologies.