Abstract approaches to regularizing moduli spaces of pseudoholomorphic curves

manifolds

Approach #1 : "Euler class on Banach ~~or~~bifolds"
[Siebert]

L11

Gromov-Witten invariants for general symplectic manifolds

Bernd Siebert

basic example

$C = \mathbb{P}^1 \xrightarrow{\varphi} M$

$\Downarrow$

$x = \infty$

$\mathcal{C}^{hol}(..) = \overline{\mathcal{M}}$

$g = 0, k = 1$

Theorem 0.1 *Let (M, ω) be a closed symplectic manifold with a tame almost complex structure J. Then the space $\mathcal{C}(M; p)$ of stable parametrized marked complex curves in M of Sobolev class L^p_1 (Definition 3.1) is a Banach orbifold. Moreover, there is a Banach orbibundle E over $\mathcal{C}(M; p)$ with fiber $\tilde{L}^p(C; \varphi^* T_M \otimes \bar{\Omega}_C)$ at $(C, \mathbf{x} = (x_1, \ldots, x_k), \varphi : C \to M)$ with an oriented Kuranishi section s (Definition 1.15) with $\boxed{\tilde{s}(C, \mathbf{x}, \varphi) = \bar{\partial}_J \varphi.}$ The zero locus of s is the set $\mathcal{C}^{\mathrm{hol}}(M, J)$ of stable pseudo holomorphic curves in (M, J) (Definition 3.5), which is a locally finite dimensional Hausdorff space with compact components.*

Let $\mathcal{M}_{g,k}$ be the moduli space of Deligne-Mumford stable k-marked algebraic curves of genus g, with the convention $\boxed{\mathcal{M}_{g,k} = \{\mathrm{pt}\}}$ whenever $2g + k < 3$. The localized Euler class $\boxed{GW^{M,J}_{g,k} \in H_*(\mathcal{C}^{\mathrm{hol}}(M, J))}$ associated to (E, s) (Theorem 1.21) gives rise to GW-correspondences (Definition 7.2)

$$GW^{M,J}_{g,k} : H^*(M)^{\otimes k} \longrightarrow H_*(\underbrace{\mathcal{M}_{g,k}}_{=\mathrm{pt}}), \quad \simeq \mathbb{Q}$$

that are invariants of the symplectic deformation type of (M, ω). They coincide with the ones defined in [RuTi2] in case (M, ω) is semi-positive.

main construction is

$[\overline{\mathcal{M}}] \in H_*(\overline{\mathcal{M}})$

then $ev : \overline{\mathcal{M}} \to M$ yields $GW : PD(S_1) \longmapsto \langle [\underbrace{\mathcal{C}^{hol}(M, J)}_{=\overline{\mathcal{M}}}], ev^* PD(S_1) \rangle$

$= \text{“}\# \{ [u] \in \overline{\mathcal{M}} \mid u(\infty) \in S_1 \}\text{”}$

Guiding Questions for studying regularization approaches via abstract perturbations / "virtual" fundamental class

$\overline{\mathcal{M}} = s^{-1}(0)$ $[(s+\nu)^{-1}(0)]$ / $\boxed{\text{Euler}(s) = [\overline{\mathcal{M}}]}$

- what is the abstract form of section s ? → "Banach orbifold"
  - e.g. when $\bar{\partial}_J \pitchfork 0$? → "Kuranishi section"
- why does regularization theorem hold ? → "Euler class"
- how is s constructed for pseudoholomorphic curve moduli spaces ?
  - from local Fredholm descriptions
  - in basic example $\overline{\mathcal{M}}_{0,1}(A,J) = \dfrac{\{u : \mathbb{P}^1 \to M \mid u_*[\mathbb{P}^1] = A,\ \bar{\partial}_J u = 0\}}{\{\varphi : \mathbb{P}^1 \xrightarrow{\sim} \mathbb{P}^1,\ \varphi(\infty) = \infty\}}$

L11

Definition 3.1 A *(marked, parametrized) complex curve* in ~~X~~ M is a triple $(C, \mathbf{x}, \varphi)$ with $(C, \mathbf{x})$ a marked prestable curve and $\varphi : C \to X$ a continuous map.

A morphism $(C, \mathbf{x}, \varphi) \to (C', \mathbf{x}', \varphi')$ is a holomorphic map $\Psi : C \to C'$ with $\varphi' \circ \Psi = \varphi$, $\Psi(\mathbf{x}) = \mathbf{x}'$ (as tuples). This defines the sets $\mathrm{Hom}((C, \mathbf{x}, \varphi), (C', \mathbf{x}', \varphi'))$ and $\mathrm{Aut}(C, \mathbf{x}, \varphi) \subset \mathrm{Aut}(C, \mathbf{x})$. $\to \mathrm{Hom}(\varphi, \varphi) = \{\varphi \circ \Psi = \varphi\}$ finite

$(C, \mathbf{x}, \varphi)$ is called *stable* if the restriction of φ to any unstable component $D \subset (C, \mathbf{x})$ is non-constant. We write $\mathcal{C}(X)$ for the space of isomorphism classes of complex curves in X and —by abuse of notation— $(C, \mathbf{x}, \varphi) \in \mathcal{C}(X)$. $= \mathcal{C}^0(\mathbb{P}^1, X)/\mathrm{Aut}$ ◇

Definition 3.2 Let $2 < p < \infty$. The subset of ~~$\mathcal{C}(X)$~~ $\mathcal{C}(M;p)$ of stable *(marked, parametrized) complex curves of Sobolev class L^p_1* in M is defined as set of (isomorphism classes of) curves $(C, \mathbf{x}, \varphi) \in \mathcal{C}(X)$ with: $\to W^{1,p}(\mathbb{P}^1, M)/\mathrm{Aut}$

1. $\varphi \in$ ~~$L^p_1(C, \varphi^* T_M)$~~ $W^{1,p}(C, M)$
2. $\mathrm{area}_\rho(\varphi|_D) > 0$ for any unstable component $D \subset (C, \mathbf{x})$. ◇

The condition of positive area is of course independent of the choice of ρ.

basic example

$C = \mathbb{P}^1$

$x = \infty$

$\varphi : \mathbb{P}^1 \to M$ $\mathcal{C}^0$

mor: $\mathrm{Aut}(\mathbb{P}^1, \infty)$

not finite

$\Updownarrow$

domain $(\mathbb{P}^1, \infty)$

unstable

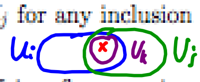
Definition 3.5 Let (M, J) be an almost complex manifold, J the almost complex structure. $(C, \mathbf{x}, \varphi) \in \mathcal{C}(M)$ is called *pseudo holomorphic (with respect to J)*, or *J-holomorphic* if for any irreducible component D of C, $\varphi|_D : D \to M$ is a morphism of almost complex manifolds. The subset

$$\overline{\mathcal{M}} = \mathcal{C}^{\mathrm{hol}}(M, J) := \{(C, \mathbf{x}, \varphi) \in \mathcal{C}(M) \ J\text{-holomorphic}\}/\text{isomorphism} \subset \mathcal{C}(M;p) = W^{1,p}(\mathbb{P}^1, M)/\mathrm{Aut}$$

of $\mathcal{C}(M)$ is the space of (marked, parametrized) stable pseudo holomorphic curves on M with respect to J. ◇

Now let X be a Hausdorff space. If $\mathcal{U} = \{U_i\}_{i\in I}$ is a covering of X by open sets we can form a category $\mathcal{T}(\mathcal{U}) \subset \mathcal{S}ets$ with objects U_i and a morphism $U_i \to U_j$ for any inclusion $U_i \subset U_j$. If for any i, j there exists k with $U_k \subset U_i \cap U_j$ we call $\mathcal{U}$ *fine*.

and $x \in U_i \cap U_j$ $\quad x \in$



$X = \mathcal{C}(M;p)$

$= W^{1,p}(\mathbb{P}^1, M)/Aut$

$= \bigcup_i \mathcal{U}_i$ fine

Definition 1.2 A *(Banach)* manifold *structure* on a Hausdorff space X is a fine covering $\mathcal{U}$ of X and a functor $\mathcal{O} : \mathcal{T}(\mathcal{U}) \to \mathcal{L}\mathcal{US}$ with $\mathcal{Q} \circ \mathcal{O} = \mathrm{Id}_{\mathcal{T}(\mathcal{U})}$ and with $\mathcal{O}(\iota)$ an open embedding for any $\iota \in \mathrm{Hom}(\mathcal{T}(\mathcal{U}))$.

Definition 1.1 Let $\mathcal{L}\mathcal{US}$ be the category whose objects consist of [redacted] : $\hat{U}$ [redacted]) *(local uniformizing systems)* with

- $\hat{U}$ is an open set in some Banach space T
- [redacted]

Obj: $\mathcal{U}_i \xrightarrow{\mathcal{O}} \hat{\mathcal{U}}_i \subset T$

Mor: $\mathcal{U}_k \subset \mathcal{U}_i \xrightarrow{\mathcal{O}} \hat{\mathcal{U}}_k \hookrightarrow \hat{\mathcal{U}}_i$

$Mor((\hat{U},\cdot,\cdot),(\hat{V},\cdot,\cdot)) = \{\hat{f} : \hat{U} \to \hat{V}$ continuous, $\ldots\}$ $\Big(\xleftrightarrow{e.g.}$ $v \mapsto v \circ \varphi_r\, e^0\Big)$

Definition 1.7 Let $p : E \to X$ be a continuous surjection of topological spaces. A *(Banach)* [redacted] *bundle structure* on p is a morphism of Banach orbifold structures

$$P : (E, \{p^{-1}(U)\}_{U\in\mathcal{U}}, \mathcal{O}^E) \longrightarrow (X, \mathcal{U}, \mathcal{O})$$

on E and X with

- $\kappa = \mathrm{Id}_I$
- if $U \in \mathcal{U}$ and $\mathcal{O}(U) =$ [redacted] $\hat{U}$, [redacted] then $\mathcal{O}^E(p^{-1}(U)) = (\hat{U} \times E_0)$ [redacted] with E_0 a Banach space [redacted]

$E = \mathcal{E}$

$p \downarrow \quad \uparrow s = \bar{\partial}_J$

$X = W^{1,p}(\mathbb{P}^1, M)/Aut$

A *section* of E is a morphism $s : X \to E$ of orbifolds with $p \circ s = \mathrm{Id}_X$.

L11

what is the abstract form of section s? (& how is it constructed in basic example)

$$\begin{array}{c}\mathcal{E}\\ \downarrow \uparrow s=\bar\partial_J\\ W^{1,p}/Aut \xrightarrow{p} \mathcal{M}_{0,1}=pt\end{array}$$

$\mathcal{S}$/locally $L\times V$

$pt = L\times V$

$\hat V=\{\xi\in W^{1,p}(\mathbb{P}^1, v^*TM) \mid \xi\binom{0}{1}\in T_{v\binom{0}{1}}H_0\}$

$\mathcal{M}_{g,k}=pt$ in basic example

Definition 1.11 Let $p: X\to S$ be a submersion of Banach orbifolds, s a section of a Banach orbibundle E over X. A *Fredholm structure for s relative S* is a choice of orbifold structures for X, and E such that any local trivialization $\hat E_U=\hat U\times E_0$, centered in some $z\in Z(s)$ has the form

$Z(s)$ " $s^{-1}(0)$

$$\hat U = L\times V$$

L: $\mathbb{R}^b$; V: Banach space

with

- L is an open subset of a finite dimensional vector space
- $\Pi\circ\hat s: L\times V\to E_0$ is differentiable relative L with relative differential

$$D_V(\Pi\circ\hat s): L\times V \longrightarrow L(T,E_0),\quad T=T_0V$$

continuous at 0 and with $\sigma=D_V(\Pi\circ\hat s)(0)$ Fredholm (think index Ds = index $D_V s + b$)

$D\bar\partial_J(\ell, v+\cdot): TV\to E_0$

Ex (gluing): $L=\{\text{gluing parameters}\}$

Definition 1.15 Let X be a Banach orbifolds and s an Fredholm section of a Banach orbibundle E over X. A *Kuranishi structure for s* is a morphism

$$\tau: F\longrightarrow \mathcal{E}$$

"global stabilization"

from a finite rank bundle F defined over an open suborbifold $X'\subset X$ containing $Z(s)$ such that for any distinguished local trivialization $\hat E_U=\hat U\times E_0$, centered in some $z\in Z(s)$, $\hat U= L\times V$:

- ~~im $\hat\tau_z$ spans coker~~ σ, $\sigma=$ ~~$D_V(\Pi\circ\hat s)(0)$~~
- $\hat\tau$ is continuously differentiable relative L.

Two Kuranishi structures $\tau: F\to E$, $\tau': F'\to E$ are *compatible* iff $\tau+\tau': F\oplus F'\to E$ is a Kuranishi structure too. An S-Fredholm section together with an equivalence class of compatible Kuranishi structures is called *S-Kuranishi* section. ⋄

$$\begin{array}{ccc} F & \xrightarrow{\tau} & \mathcal{E}\\ \downarrow & & \downarrow \uparrow s\\ X' & \hookrightarrow & X = W^{1,p}/Aut\\ \cup & & \cup \\ & s^{-1}(0)\simeq\bar{\mathcal{M}} & \end{array}$$

$W^{1,p}/Aut \simeq \bigcup_i \hat V_i$ (as above)

$\bar\partial_J|_{\hat V_i}$ Fredholm $\Rightarrow \exists$ local stabilization

$$\begin{array}{ccc} F_i & \xrightarrow{\beta_i\tau_i} & \mathcal{E}\\ \downarrow & & \downarrow\\ \hat V_i & \hookrightarrow & W^{1,p}/Aut\end{array}$$

$\Rightarrow$ multiply by cutoff function β_i to extend to $W^{1,p}/Aut$?

$\Rightarrow$ covers cokernel on $\hat V_i'\subset\hat V_i$ open

$s^{-1}(0)$ compact

$\Rightarrow$ cover by $\bigcup_{i\text{ finite}}\hat V_i'$

$$\Rightarrow\quad \begin{array}{ccc} \bigoplus_i F_i & \xrightarrow{\Sigma\beta_i\tau_i} & \mathcal{E}\\ \downarrow & & \downarrow\\ \text{nbhd of } s^{-1}(0) = \bigcup_i \hat V_i' & \longrightarrow & W^{1,p}/Aut\end{array}$$

covers cokernel everywhere

will apply to stabilized section

$$s_F : F \longrightarrow \mathcal{E}$$
$$([u], f) \longmapsto \bar\partial_J[u] + \tau(f)$$

Definition 1.13 A [redacted]-Fredholm section s of E [redacted] is ~~transverse~~ along a closed subset $A \subset Z(s)$ iff for any $z \in A$ there exists a distinguished local trivialization centered in z with σ surjective. ◇

"$D_V s$ on $L \times V \hookrightarrow W^{4,p}/\mathrm{Aut}$

By applying the implicit function theorem locally relative [redacted] L, we see that the zero locus of transverse sections is a finite dimensional topological orbifold, locally uniformized by [redacted] $L \times K$, $K = \ker D_V s$ [redacted]

$$s_F^{-1}(0) = \{([u], f) \mid \bar\partial_J[u] = -\tau(f)\}$$
$\cong$ if τ injective
$$\{[u] \mid \bar\partial_J[u] \in \tau(F)\}$$

Proposition 1.14 *Let s be a [redacted]-Fredholm section of a Banach orbibundle E over a Banach orbifold X [redacted] and assume s is transverse along A. Then in a neighbourhood of A, $Z(s)$ has naturally a structure of topological orbifold,* [redacted] ◇

reminder

Definition 1.11 Let [redacted] X [redacted] be a [redacted] Banach orbifolds, s a section of a Banach orbibundle E over X. A *Fredholm structure* for s [redacted] is a choice of orbifold structures for X, [redacted] and E such that any local trivialization $\hat E_U = \hat U \times E_0$ [redacted] centered in some $z \in Z(s)$ has the form

$$\hat U = [redacted] L \times V$$

with

[redacted]

- L is an open subset of a finite dimensional vector space
- $\Pi \circ \hat s :$ [redacted] $L \times V \to E_0$ is differentiable relative [redacted] L with relative differential

$$D_V(\Pi \circ \hat s) : [redacted] L \times V \longrightarrow \mathbb{B}(T, E_0), \quad T = T_0 V$$

continuous at 0 and with $\sigma = D_V(\Pi \circ \hat s)(0)$ Fredholm. "bounded linear operators"

in operator topology on $\mathbb{B}(TV, E_0)$

$$F \underset{\text{locally}}{\simeq} L \times V \times F_0$$
finite dim.

⚠ in gluing analysis would need "$R \longrightarrow D_{u \#_R v} \bar\partial_J$" continuous wrt R
e.g. unif. bounded as $R \to \infty$
<u>not</u> true in polyfold setup

L11

why does regularization theorem hold?

Theorem 1.21 *Let X be a topological Banach orbifold,* [redacted]*, and s an oriented* [redacted] *Kuranishi section of a Banach orbibundle E over X of constant index d* [redacted] *We assume $Z = Z(s)$ to be compact.*

Then there exists a localized $\boxed{\text{Euler class}}$ *$[E, s] \in H_{d+b}(Z)$, depending only on E and the Kuranishi section s, with the following properties:*

1. *If s is transverse (so Z is an oriented topological orbifold of dimension d* [redacted]*) then $[E, s] = [Z]$.* " $s^{-1}(0) \simeq \overline{\mathcal{M}}$ "

[redacted]

$$\begin{tikzcd} & \mathcal{E} \arrow[d] & F \arrow[l] \arrow[d, "q"] \\ s=\bar\partial_J & W^{k,p}/\mathrm{Aut} = X & \supset X' \end{tikzcd}$$

$L \times V$ $\widetilde{\text{locally}}$ global stabilization

d = Fredholm index of $D_u s = D_u \bar\partial_J$

$b = \dim L$

Proposition 1.17 *Let s be a* [redacted] *Kuranishi section of a Banach orbibundle E over a Banach orbifold X* [redacted]*. Let $\tau : F \to E$ represent the Kuranishi structure, $q : F \to X$ the bundle projection. Then the section* stabilized section"

$$\tilde{s} := q^* s + \tau \quad = " \bar\partial_J + \tau "$$

of q^*E over the total space F is transverse*. In particular, $\tilde{Z} := Z(\tilde{s})$ is a topological submanifold of F* [redacted].

$$\begin{tikzcd} F \arrow[r, "\tau"] \arrow[d, "q"] & \mathcal{E} \\ X' \arrow[ur, "s=\bar\partial_J"'] & \end{tikzcd}$$

$\tilde{Z} = \{(x, f) \mid s(x) + \tau(f) = 0\}$

$$\begin{tikzcd} \tilde{Z} \arrow[d, "s_{can}"] & (x,f) \arrow[d] \\ F & f \end{tikzcd}$$

Remark 1.18 Let $s_{\rm can}$ be the tautological section of $q^*F \to F$. So on local uniformizers $s_{\rm can}$ is given by putting

$$\Pi \circ \hat{s}_{\rm can} := \mathrm{pr}_{F_0} : \hat{U} \times F_0 \longrightarrow F_0\,.$$

The zero locus of $s_{\rm can}$ is just the zero section of F, and can be identified with X. Restricting to $\tilde{Z} = Z(\tilde{s})$ we obtain

$$Z(s) = Z(s_{\rm can}|_{\tilde{Z}})\,.$$

In this way we have exhibited the zero locus of s as zero locus of a *finite rank* orbibundle over a finite dimensional orbifold. $\diamond$

$\Rightarrow s_{can}^{-1}(0) \simeq s^{-1}(0) \simeq \overline{\mathcal{M}}$

global finite dimensional reduction

$\leadsto$ $F|_{\tilde{Z}} \downarrow \tilde{Z}$, s_{can} has an Euler class $\underbrace{E(s_{can})}_{=: [\overline{\mathcal{M}}]} \in H_*(\underbrace{s_{can}^{-1}(0)}_{\simeq \overline{\mathcal{M}}})$

how is S constructed for pseudoholomorphic curve moduli spaces?

Theorem 5.1 *Let (M, J) be an almost complex manifold and $p > 2$. Then the space $\mathcal{C}(M;p)$ of stable parametrized complex curves of Sobolev class L^p_1 has naturally a structure of ~~Banach~~ manifold [illegible]. The orbifold topology is finer than the C^0-topology introduced in Section 3.2.*

$\mathcal{C}(M;p) = W^{1,p}(\mathbb{P}^1, M) / v \sim v \circ \psi \quad \forall \psi \in \mathrm{Aut}(\mathbb{P}^1, \infty)$

(fix $v_*[\mathbb{P}^1] = A$ s.t. $v = v \circ \psi \Rightarrow \psi = id$ to get "trivial isotropy")

We are going to show now that our charts $S \times \bar{V}$ for $\mathcal{C}(M;p)$ together with its naturally associated trivialization of E (i.e. using the same retraction κ) endow $s_{\bar{\partial},J}$ with a Fredholm structure, with $(S = W, L, V)$ in Definition 1.11 equal to $(\mathrm{pt}, S, \bar{V})$.

We want to model local uniformizing systems at $\underset{(\mathbb{P}^1, \infty, v)}{(C, \mathbf{x}, \varphi)}$ on $S \times L^p_1(C; \varphi^* T_M)$, where S is naturally viewed as open neighbourhood of the origin in the tangent space $\hat{T}_{\mathcal{M}_{g,k},(C,\mathbf{x})}$ of the differentiable orbifold $\mathcal{M}_{g,k}$ at $(C, \mathbf{x})$.

$S \times W^{1,p}(\mathbb{P}^1, v^* TM)$

$S = \mathrm{pt}$

$\hat{V} = \{ \xi \mid \xi(0) \in T_{v(0)} H_0, \ \xi(1) \in T_{v(1)} H_1 \} \subset W^{1,p}(\mathbb{P}^1, v^* TM)$

$\hat{V} \to W^{1,p}/\mathrm{Aut} \ni [\underbrace{\exp_v(\xi)}_{w}]$

$w(0) \in H_0$, $w(1) \in H_1$ } hypersurface slicing conditions

Change of coordinates: The general case.
Now assume $(C, \mathbf{x})$ unstable and let $S \times \bar{V} \subset S \times V$ be a rigidifying slice. Let $q : S \times V \to \mathcal{C}(M;p)$ induce the structure map. As shown in Section 5.3, $S \times \bar{V}$ is the topological quotient of $S \times V$ by the equivalence relation R generated by the germ of action of $\mathrm{Aut}^0(C, \mathbf{x})$. Given $(C', \mathbf{x}', \varphi') \in \operatorname{im} q$ (v') and sufficiently close to $(C, \mathbf{x}, \varphi)$ (v) as above (depending on the injectivity radius of ρ etc.), we choose the local uniformizing system with center $(C', \mathbf{x}', \varphi')$ to be a slice $S' \times \bar{V}'$ in $S' \times V'$ with S', V' sufficiently small as before. Again, $S' \times \bar{V}'$ is the quotient of $S' \times V'$ by the equivalence relation R', generated by the germ of action of $\mathrm{Aut}^0(C', \mathbf{x}', \varphi')$.

Now $(C', \mathbf{x}')$ belongs to some $s_0 \in S$, and with a choice of s_0 the unstable components of $(C', \mathbf{x}')$ can be identified with a subset $D_1, \ldots, D_a$ of the set of unstable components of $(C, \mathbf{x})$ via κ. Let $G := \{\Psi \in \mathrm{Aut}^0(C, \mathbf{x}) \mid \Psi|_{D_i} = \mathrm{Id},\ i = 1, \ldots, a\}$. By our explicit description of the semiuniversal deformation it is not hard to see that the local action of G fibers S smoothly near s_0, and that the restriction of $q : \mathcal{C} \to S$ to a smooth analytic slice of the action of G at s_0 is a semiuniversal deformation of $(C', \mathbf{x}')$, hence locally isomorphic to S'. Again we can thus identify C' with a locally analytic subset of $\mathcal{C}$, this time of codimension equal to the dimension of G. The map $\boxed{\hat{\sigma} : S' \times \bar{V}' \hookrightarrow S \times \bar{V}}$ is then defined as composition of $(s, v') \mapsto (s, \Pi_s^{-1} \Theta \Pi'_s v')$ with the quotient map $S \times V \to S \times \bar{V}$. Equivariance and continuity of $\hat{\sigma}$ are thus inherited by the corresponding properties of the unrigidified map $S' \times V' \to S \times V$. Again we can change the roles of $S' \times \bar{V}'$ and $S \times \bar{V}$ to conclude that $\hat{\sigma}$ is an open embedding.

$\varphi(s, v) \in \check{L}^p_1(C_s; M)$ defined by

$$\varphi(s, v)(z) := \exp^{\rho}_{\varphi(\kappa_s(z))} \Pi_s(v)(z),$$

$\hat{V} \ni \xi \mapsto w$, $\quad \xi' \in \hat{V}' \mapsto w'$

$[\exp_v \xi] = [\exp_{v'} \xi'] \in W^{1,p}/\mathrm{Aut}$

$$\Theta_{PP'} : B_r(0) \subset T_{M,P'} \longrightarrow T_{M,P}, \quad v \longmapsto (\exp^{\rho}_P)^{-1}(\exp^{\rho'}_{P'} v)$$

reparametrization?
$w' \mapsto w' \circ \psi = w$

but only need this $\mathcal{C}^0$ for construction of Banach manifold & Fredholm structure