Abstract approaches to regularizing moduli spaces of pseudoholomorphic curves

→ How to deal with isotropy $\rightsquigarrow$ in the following will simplify to manifolds

→ Approach #1 : "Euler class on Banach orbifolds"
[Siebert]

$\boxed{\text{How to deal with isotropy}}$

Recall $u:(\Sigma,j)\to(M,J)$ stable (J,j)-holomorphic map

$:\Leftrightarrow$ isotropy/stabilizer $\operatorname{Stab}(u)=\{\varphi \circlearrowleft (\Sigma,j)\mid u\circ\varphi=u\}$ finite

Ex: $u=v\circ(z\mapsto z^k):\mathbb{P}^1\to M$, v injective

$\Rightarrow \operatorname{Stab}_u=\operatorname{Stab}_{(z\mapsto z^k)}=\{\varphi(z)=az\mid a^k=1\}\simeq\mathbb{Z}_k$

Geometric Regularization: If $\bar\partial_J \pitchfork 0$, then $\mathcal{M}=\bar\partial_J^{-1}(0)/\operatorname{Aut}(\Sigma,j)$ locally has $\underset{U}{}$ orbifold structure: "$\operatorname{nbhd}([u])\simeq T(\operatorname{Aut}\cdot u)^\perp\subset\bar\partial_J^{-1}(0)/\operatorname{Stab}_u$"

Abstract Regularization: Generally don't have equivariant transversality even under finite group acting. E.g.

$$S^1\times\mathbb{R} \circlearrowleft \mathbb{Z}_2 \qquad (-1)\cdot(z,x)=(z,-x)$$
$$\downarrow\ \uparrow s(z)=(z,0)$$
$$S^1$$

$\tilde{s}$ $\mathbb{Z}_2$-equivariant $\Leftrightarrow$ $\tilde{s}=0$

But "multivalued perturbations give well defined $\#\, f^{-1}(0)/\operatorname{Aut}\in\mathbb{Q}$"

$\nu(z)=\{\varepsilon,-\varepsilon\} \rightsquigarrow (s+\nu)^{-1}(0)=\emptyset \rightsquigarrow \#=0$

$\nu(z)=\{\operatorname{Im} z,-\operatorname{Im} z\} \rightsquigarrow (s+\nu)^{-1}(0)/\mathbb{Z}_2=\{1,1,-1,-1\} \rightsquigarrow \#=\frac{1}{2}-\frac{1}{2}+\frac{1}{2}-\frac{1}{2}=0$

$z\in S^1\subset\mathbb{C}$

Ex 1

$$\begin{array}{c} S^2\times\mathbb{C} \\ \downarrow \\ S^2 \end{array} \circlearrowleft \mathbb{Z}_4 = \sqrt[4]{1} \qquad \gamma\cdot(\underset{\mathbb{C}\cup\infty}{\overset{\cap}{x}}, z) = (\gamma x, \gamma^2 z)$$

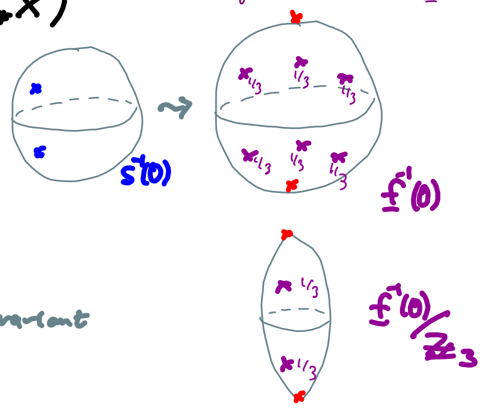
$f: S^2 \to \mathbb{C}$ equivariant $\Rightarrow f(0) = -f(0) \Rightarrow 0 \in f^{-1}(0)$
$\Rightarrow f(z) = f(-z) \Rightarrow df(0) = 0$ $\Big\}$ nontransverse

multisection: $\underline{f}(x) = \{\varepsilon, -\varepsilon\} \Rightarrow \underline{f}^{-1}(0)/\mathbb{Z}_4 = \emptyset$

Ex 2

$$\begin{array}{c} TS^2 \\ \downarrow \\ S^2 \end{array} \circlearrowleft \mathbb{Z}_3 \qquad \gamma\cdot(x, X) \longmapsto (\gamma x, \gamma_* X)$$

weight $= \dfrac{\#\text{values} = 0}{\#\text{values of } \underline{f}}$



From $s: S^2 \to TS^2 \pitchfork 0$ construct transverse

multisection $\underline{f}(x) = \{s(x), \gamma_* s(\gamma^{-1}x), \gamma_*^2 s(\gamma^{-2}x)\}$

$\underline{f}(\gamma x) = \{s(\gamma x), \gamma_* s(x), \gamma_*^2 s(\gamma^{-1}x)\}$

$\gamma_* \underline{f}(x) = \{\gamma_* s(x), \gamma_*^2 s(\gamma^{-1}x), \gamma_*^3 s(\gamma^{-2}x) = s(\gamma x)\}$ $\rightsquigarrow$ equivariant

$$\#\, \underline{f}^{-1}(0)/\mathbb{Z}_3 = \frac{1}{3} + \frac{1}{3} = \frac{2}{3} \quad \left(= \frac{\text{Eulerchar}(S^2)}{\text{order}(\mathbb{Z}_3)} \right)$$

more formally, a smooth transverse multisection $\underline{f}: S^2 \to$ finite subsets of TS^2 is locally of the form $\underline{f}(x) = \{f_1(x), \dots f_n(x)\}$ with n smooth transverse sections $f_1 \dots f_n : S^2 \to TS^2$

$\boxed{\text{How to deal with isotropy}}$ → analysis : "permute marked points"
→ topology : groupoid language

An orbifold is
→ Hausdorff space X with local charts $\mathcal{U}/\Gamma \hookrightarrow X$ $\quad \mathcal{U} \subset \mathbb{R}^n$ open, Γ finite
& "smooth transition maps"

↓

the realization of a proper étale groupoid → category $\mathrm{Obj}\,\mathfrak{X}$, $\mathrm{Mor}\,\mathfrak{X}$, composition, identities
s.t. all morphisms invertible

$|\mathfrak{X}|$

proper: $(\text{source} \times \text{target})^{-1}(\text{compact})$ is compact

étale: Obj, Mor smooth manifolds
structure maps local diffeo
- source $\mathrm{Mor} \to \mathrm{Obj}$
- target $\mathrm{Mor} \to \mathrm{Obj}$
- composition
- identity

$|\mathfrak{X}| = \mathrm{Obj}\,\mathfrak{X}/\sim \qquad x \sim y \iff \mathrm{Mor}(x,y) \neq \emptyset$

is Hausdorff iff "$\sim$" proper

Rmk: orbifold atlas $(\mathcal{U}_i, \Gamma_i, \psi_i : \mathcal{U}_i/\Gamma_i \hookrightarrow X)$ induces a groupoid with

$$\mathrm{Obj}\,\mathfrak{X} = \coprod_i \mathcal{U}_i \quad , \quad \mathrm{Mor}\,\mathfrak{X} = \coprod_i \mathcal{U}_i \times \Gamma_i \;\cup\; \coprod_{i \neq j} \{(u_i, u_j) \mid \psi_i([u_i]) = \psi_j([u_j])\}$$

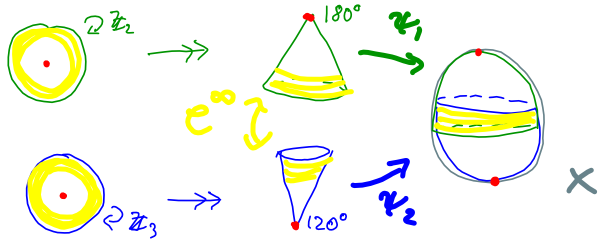
$$u \xleftarrow{s} (u,\gamma) \xrightarrow{t} \gamma \cdot u \qquad\qquad u_i \xleftarrow{s} (u_i,u_j) \xrightarrow{t} u_j$$

isotropy groups : $\mathrm{Mor}(x,x) \simeq \mathrm{Stab}(x) \subset \Gamma_i$ when $x \in \mathcal{U}_i$

Ex: $(\mathbb{Z}_2, \mathbb{Z}_3)$-football: $X = S^2$

ψ_1: $\text{disk}/\mathbb{Z}_2 \to$ upper hemisphere $+$ annulus

ψ_2: $\text{disk}/\mathbb{Z}_3 \to$ lower hemisphere $+$ annulus



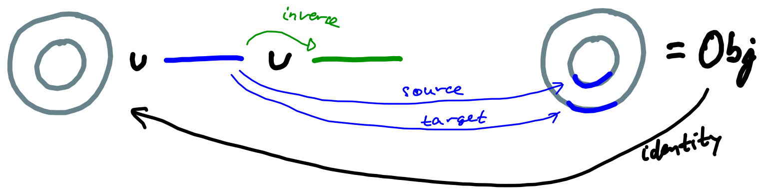
$\mathrm{Obj}\, \mathfrak{X} = D_1 \sqcup D_2$

$\mathrm{Mor}\, \mathfrak{X} = D_1 \times \mathbb{Z}_2 \sqcup D_2 \times \mathbb{Z}_3 \sqcup \{(z_1, z_2) \in \underbrace{A_1 \times A_2}_{\text{annuli}} \mid z_1^2 = z_2^3\}$

$|\mathfrak{X}| = {D_1 \sqcup D_2}/{\mathrm{Mor}} \simeq S^2$

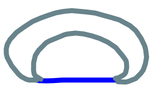
Ex: branched manifold : realization of $\overbrace{\text{étale}}^{\text{a certain}}$ groupoid $\mathfrak{X}$

Mor =

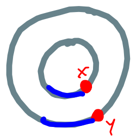


= Obj

$|\mathfrak{X}| =$



not Hausdorff
no fundamental cycle



$\nexists$ nbhds $\mathcal{U}_x, \mathcal{U}_y$: $\mathcal{U}_x \cap \mathcal{U}_y = \emptyset$

<u>Philosophy</u>: $\overline{\mathcal{M}} = |s^{-1}(0)|$ $\begin{matrix}\mathcal{E}\\ \downarrow \uparrow s \\ \mathcal{X}\end{matrix}$ étale categories nontrivial isotropy $\hat{=}$ $Mor(x,x) \supsetneq \{1_x\}$

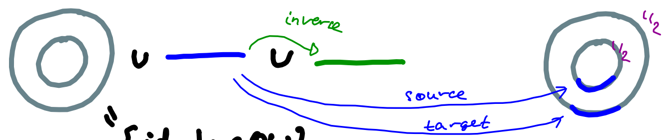
regularization thm for trivial isotropy $\Longrightarrow$ reg. thm. for finite isotropy

↳ manifolds ⇝ $\mathbb{Z}$-fundamental cycle ↳ branched weighted mflds ⇝ $\mathbb{Q}$-fundamental cycle

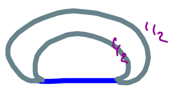
<u>Ex:</u> weighted branched manifold : realization of Étale groupoid $\mathcal{X}$ with weights "$\bar{f}^{-1}(0)$"

$\frac{1}{n}\#\{i \mid f_i(x)=0\}$ when $\bar{f}=\{f_1,\dots,f_n\}$ near x

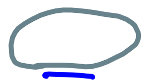
[McDuff]

Mor =  = Obj

"$\{id_x \mid x \in Obj\}$

$|\mathcal{X}|$ = 

$\frac{1}{2}+\frac{1}{2}=1$ = sum of weights of preimages in Obj $\mathcal{X}$

fundamental cycle $[\mathcal{X}] = \frac{1}{2}$  $+ \frac{1}{2}$  $\in H_*(|\mathcal{X}|;\mathbb{Q})$