The virtual fibering conjecture and related questions

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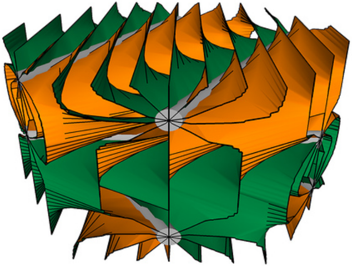
From braids to Teichmuller spaces



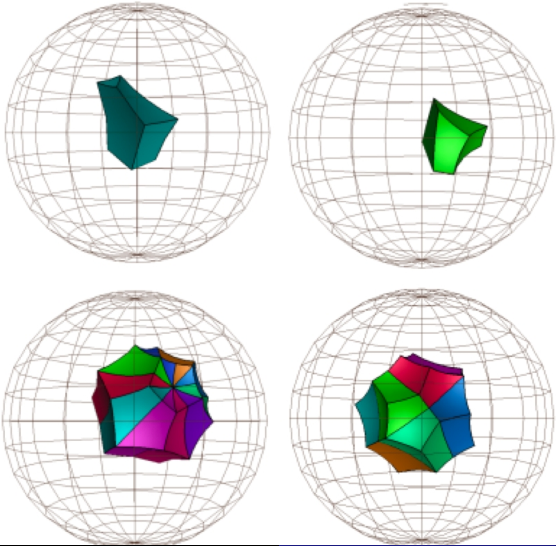
Outline of the talk

- Definitions
  - Hyperbolic manifolds
  - RF and LERF
  - The virtual Haken conjecture and related conjectures
  - Cubulations
  - Heegaard gradient
- Implications
- Sketch of techniques

Hyperbolic 3-manifolds

- A 3-manifold M is **hyperbolic** if it admits a complete Riemannian metric with constant curvature -1. Then $M = \mathbb{H}^3/\Gamma$, where $\mathbb{H}^3$ is hyperbolic 3-space, and Γ is a discrete torsion-free subgroup of $\mathrm{PSL}(2, \mathbb{C})$ (if M is orientable). If Γ is finitely generated, then it is called a **Kleinian group**
- More generally, a hyperbolic **3-orbifold** is a quotient $\mathbb{H}^3/\Gamma$, where $\Gamma < \mathrm{Isom}(\mathbb{H}^3)$ is discrete but might have torsion
- The orbifold theorem and geometrization conjecture imply that hyperbolic 3-orbifolds are the most common, yet least understood 3-orbifolds.
- Classic examples (classified by Andreev) include polyhedra with "mirrored" sides. Then Γ is the group generated by reflections in the sides of the polyhedron all of whose dihedral angles are of the form π/m.

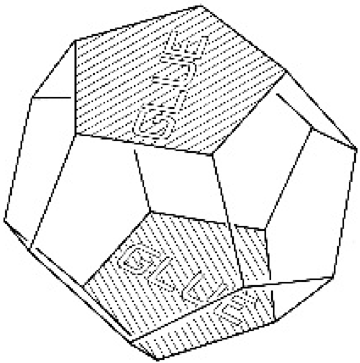
Reflection polyhedra in $\mathbb{H}^3$ (pictures by Roland Roeder)



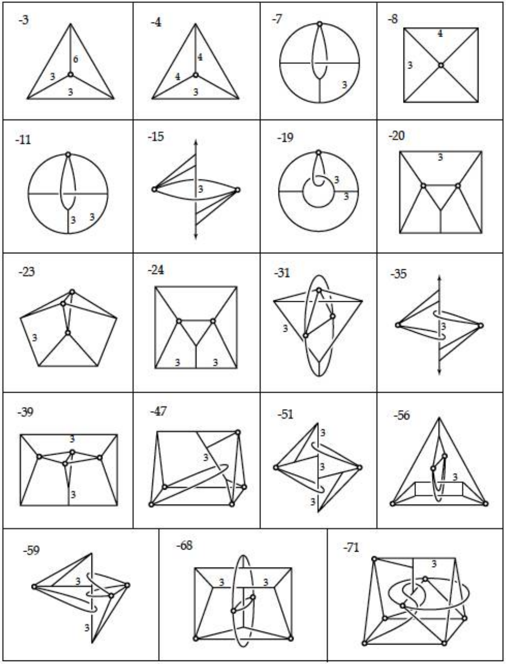
Hyperbolic 3-orbifolds

- Other examples include **arithmetic orbifolds** which are the quotient of $\mathbb{H}^3$ by **arithmetic groups**, such as the **Bianchi groups** $\Gamma = \mathrm{PSL}(2, \mathcal{O}_d)$, where $\mathcal{O}_d$ is the ring of integers in the field $\mathbb{Q}(\sqrt{-d})$, where $d \in \mathbb{N}$.
- The construction of arithmetic groups uses number-theoretic techniques to construct discrete lattices $\Gamma < \mathrm{PSL}(2, \mathbb{C})$.
- Many classic examples of hyperbolic 3-manifolds are arithmetic, including the **Seifert-Weber dodecahedral space**, the **figure eight knot complement**, and the **Whitehead link complement**

Some hyperbolic spaces



Some Bianchi orbifolds (picture by Hatcher)



Residual Finiteness

- A group G is **residually finite (RF)** if for every $1 \neq g \in G$, there exists a finite group K and a homomorphism $\phi : G \to K$ such that $\phi(g) \neq 1 \in K$.
- Alternatively,

$$\{1\} = \bigcap_{[G:H]<\infty} H. \tag{1}$$

- Examples include
- - finitely generated linear groups (Malcev)
- - 3-manifold groups (Hempel)
- - mapping class groups of surfaces

Locally Extended Residual Finiteness

A group G is **locally extended residually finite (LERF)** if for every finitely generated subgroup $L < G$, for all $g \in G - L$, there exists $\phi : G \to K$ finite $\phi(g) \notin \phi(L)$. Alternatively,

$$L = \bigcap_{L \leq H \leq G, [G:H] < \infty} H \tag{2}$$

If this holds for some L, then L is said to be **separable**.

Loally Extended Residual Finiteness

LERF means that *finitely generated subgroups of G are separable.* Examples include

- \- free groups (Hall) and surface groups (Scott)
- \- Bianchi groups (Agol-Long-Reid) and certain other arithmetic subgroups of $\mathrm{PSL}(2, \mathbb{C})$.
- \- 3-dimensional hyperbolic reflection groups (Haglund-Wise)
- There are examples of 3-manifold groups which are not LERF which are *graph manifold* groups
- LERF allows one to lift π_1-injective immersions to embeddings in finite-sheeted covers

Virtual Haken

- A 3-manifold M is **aspherical** if every $\pi_i(M) = 0$ for $i \geq 2$.
- A 3-manifold M is **Haken** if it is aspherical and M contains an embedded π_1-injective surface (e.g. a knot complement).
- A 3-orbifold M is **virtually Haken** if there is a finite-sheeted manifold cover $\tilde{M} \to M$ such that $\tilde{M}$ is Haken.
- A 3-orbifold M **contains an essential surface** if there is a map $f : \Sigma_g \to M$, such that f is π_1-injective.
- Waldhausen conjectured that every aspherical 3-manifold M is virtually Haken (the *virtual Haken conjecture).*
- A virtually Haken 3-orbifold contains an essential surface.
- Examples of virtually Haken hyperbolic orbifolds include reflection orbifolds and Bianchi orbifolds

Virtual fibering

- A manifold M **fibers over the circle** if there is a submersion $\eta : M \to S^1$. Each preimage $\eta^{-1}(x)$ is a codimension-one submanifold of M called the **fiber**.
- If M is 3-dimensional and fibers over S^1, then the fiber is a surface Σ, and M is obtained as the mapping torus of a homeomorphism $\phi : \Sigma \to \Sigma$,

$$M \cong T_\phi = \frac{\Sigma \times [0,1]}{\{(x,0) \sim (\phi(x),1)\}}.$$

- M is **virtually fibered** if there exists a finite-sheeted cover $\tilde{M} \to M$ such that $\tilde{M}$ fibers
- There are known examples of **Seifert-fibered spaces** and **graph manifolds** which are not virtually fibered
- Thurston asked whether every hyperbolic 3-manifold is virtually fibered?

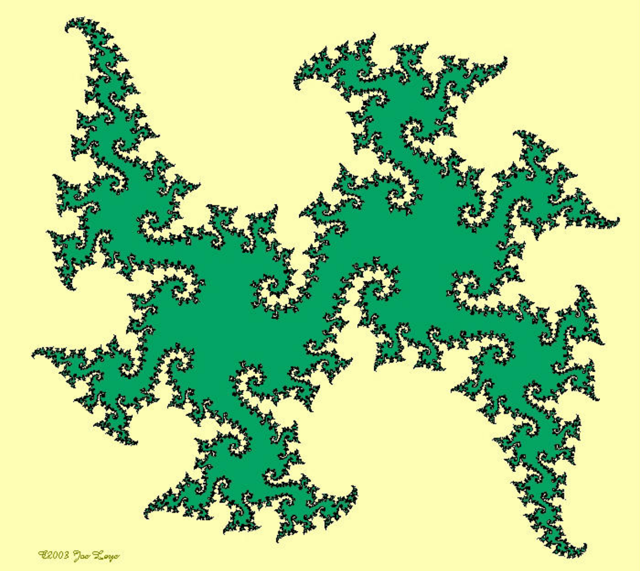
Surfaces in hyperbolic 3-manifolds

If M is a finite volume hyperbolic 3-manifold, and $f : \Sigma_g \to M$ is an essential immersion of a surface of genus $g > 0$, then there is a dichotomy for the geometric structure of the surface discovered by Thurston, and proven by Bonahon in general.
Either f is

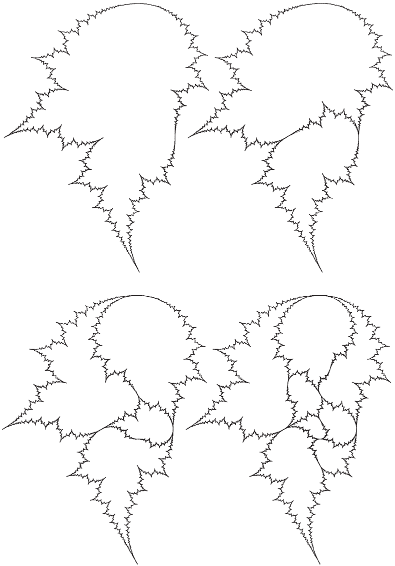
- **geometrically finite** or
- **geometrically infinite**.

The first case includes **quasifuchsian** surfaces.
In the geometrically infinite case, the surface is **virtually the fiber** of a fibering of a finite-sheeted cover of M, and the subgroup $f_\#(\pi_1(\Sigma_g)) < \pi_1(M)$ is separable. The **Tameness theorem** (A., Calegari-Gabai) plus the **covering theorem** of Canary implies a similar dichotomy for finitely generated subgroups of $\pi_1(M)$: either a subgroup is geometrically finite, or it corresponds to a virtual fiber. This result is used in proving that certain Kleinian groups are LERF.

Quasi-fuchsian surface group limit set



Part of the Peano curve "limit set" of the figure eight fiber

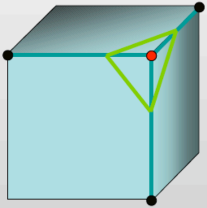


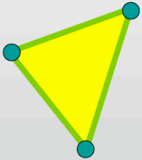
Cubulations

A topological space X is **CAT(0) cubed** if X is a cubical complex such that putting the standard Euclidean metric on each cube gives a locally CAT(0) metric (we won't explain what this means since it is not so relevant to this talk). Gromov showed that this condition is equivalent to a purely combinatorial condition on the links of vertices of X, they are **flag**.

gromov's link condition:

link: simplicial complex of incident cells…





theorem [gromov]:

cube complex is npc ⇔ link of each vertex is a flag complex

if the edges look like a k-simplex, there
really is a k-simplex spanning them…

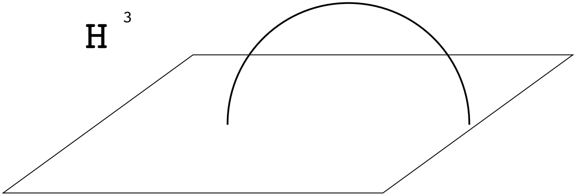
Cubulations

A topological space Y is **cubulated** if it is homotopy equivalent to a compact CAT(0) cube complex $X \simeq Y$. We will be interested in 3-manifolds which are cubulated.

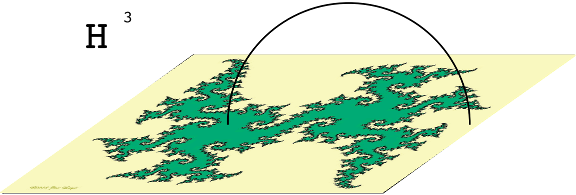
Remark: If $Y = M^3$, and $X \simeq Y$ is a CAT(0) cubing, then $dim X$ may be > 3. Tao Li has shown that there are hyperbolic 3-manifolds Y such that there is no **homeomorphic** CAT(0) cubing $X \cong Y$.

Theorems of Sageev allow one to conclude that a hyperbolic 3-manifold M is cubulated if and only if there are "enough" essential surfaces in M.

Essential surface meeting a geodesic



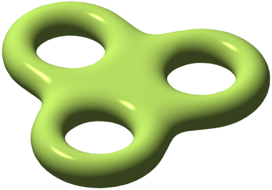
Essential surface meeting a geodesic



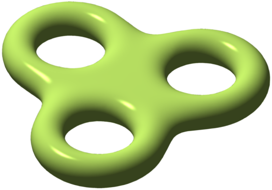
Heegaard gradient

For M a closed 3-manifold, the **Heegaard genus** $g(M)$ is the minimal genus of a surface $\Sigma_g \subset M$ such that Σ_g bounds handlebodies to each side (Σ_g is a **Heegaard surface**).

homemorphism



$\leftrightarrow$



The **Heegaard gradient** of M is

$$\nabla g(M) = \inf_{\tilde{M} \to M \text{ finite}} \frac{g(\tilde{M})}{[M : \tilde{M}]}$$

Heegaard gradient

This notion was introduced by Lackenby to probe the virtual Haken conjecture

If M is fibered, then it is easy to see that $\nabla g(M) = 0$.

Conjecture:(Lackenby) M is virtually fibered if and only if $\nabla g(M) = 0$.

Lackenby has made some progress on this conjecture under strong assumptions on the asymptotic behavior of Heegaard genus in finite regular covers.

Virtual betti number and largeness of groups

Groups are assumed to be finitely generated here.

- A group G has **positive** β_1 if $\beta_1(G) = rank H_1(G;\mathbb{Q}) > 0$. Equivalently there exists a homomorphism $\phi : G \twoheadrightarrow \mathbb{Z}$.
- A group G has **virtual positive** β_1 if there exists finite index $\tilde{G} < G$ with $\beta_1(\tilde{G}) > 0$.
- A group G has **virtual infinite** β_1 if for any $k > 0$, there exists finite index $\tilde{G} < G$ with $\beta_1(\tilde{G}) > k$.
- A group G is **large** if there is a finite index subgroup $\tilde{G} < G$ and a homomorphism $\phi : G \twoheadrightarrow \mathbb{Z} * \mathbb{Z}$.
- We say that a manifold M has these corresponding properties if $\pi_1(M)$ does

Previous theorem on virtual fibering

Theorem (A.)

If M^3 is hyperbolic, M is cubulated, and $\pi_1(M)$ is LERF, then M is virtually fibered.

This theorem is based on the results of Haglund-Wise. Since M is cubulated, $M \simeq X$, where X is a CAT(0) compact cube complex. Since $\pi_1 M = \pi_1 X$ is LERF, Haglund and Wise prove that there is a finite-sheeted cover $\tilde{X}$ which is **special**, and implies that $\pi_1(\tilde{X}) < RAAG$, where $RAAG$ is a **right-angled Artin group**. A strong form of residual solvability for RAAG's called **RFRS** passes to $\pi_1(\tilde{X})$ and implies that M is virtually fibered (A.).

Previous Theorem on virtual fibering

Corollary (A.)

Reflection 3-orbifolds and arithmetic 3-manifolds containing totally geodesic surfaces virtually fiber.

These examples include the Bianchi orbifolds and the Seifert-Weber dodecahedral space.
This follows because Haglund-Wise showed that these have virtually special cubulations.

Now we describe the technical condition **RFRS** which is used to prove virtual fibering, and which implies more.

RFRS and virtual fibering

The **rational derived series of a group** G is defined inductively as follows.
If $G^{(1)} = [G, G]$, then $G_r^{(1)} = \{x \in G | \exists k \neq 0, x^k \in G^{(1)}\}$.
If $G_r^{(n)}$ has been defined, define $G_r^{(n+1)} = (G_r^{(n)})_r^{(1)}$.
The rational derived series gets its name because
$G_r^{(1)} = \ker\{G \to \mathbb{Q} \otimes_{\mathbb{Z}} G/G^{(1)}\} = \ker\{G \to H_1(G; \mathbb{Q})\}$.
The quotients $G/G_r^{(n)}$ are solvable.

Definition

A group G is *residually finite* $\mathbb{Q}$*-solvable* or *RFRS* if there is a sequence of subgroups $G = G_0 > G_1 > G_2 > \cdots$ such that $\cap_i G_i = \{1\}$, $[G : G_i] < \infty$ and $G_{i+1} \geq (G_i)_r^{(1)}$.

RFRS and virtual fibering

By induction, $G_i \geq G_r^{(i)}$, and thus G/G_i is solvable with derived series of length at most i. We remark that if G is RFRS, then any subgroup $H < G$ is as well.

Examples of RFRS groups are free groups, surface groups, RAAGs.

For a 3-manifold M with RFRS fundamental group, the condition is equivalent to there existing a **cofinal tower** of finite-index covers

$$M \leftarrow M_1 \leftarrow M_2 \leftarrow \cdots$$

such that M_{i+1} is obtained from M_i by taking a finite-sheeted cyclic cover dual to an embedded non-separating surface in M_i.

Equivalently, $\pi_1(M_{i+1}) = ker\{\pi_1(M_i) \to \mathbb{Z} \to \mathbb{Z}/k\mathbb{Z}\}$.

This condition implies that M has virtual infinite β_1, unless $\pi_1(M)$ is virtually abelian.

RFRS and virtual fibering

Theorem (A. 2007)

If M is aspherical and $\pi_1(M)$ is RFRS, then M virtually fibers.

The proof makes use of **sutured manifold theory**, an inductive technique for studying foliations of 3-manifolds introduced by Gabai.
This criterion was one ingredient of the proof of

Theorem (Friedl-Vidussi 2008)

If M is a closed 3-manifold such that $M \times S^1$ is a **symplectic** *4-manifold, then M is fibered.*

The converse is an old result of Thurston, and this result was conjectured by Kronheimer and Taubes.

RFRS and virtual fibering

The RFRS condition for aspherical 3-manifolds has stronger implications for the structure of $H^1(M)$. For any $\alpha \in H^1(M)$, there is a finite-sheeted cover $\pi : \tilde{M} \to M$ (coming from the RFRS condition) such that $\tilde{\alpha} = \pi^*(\alpha) \in H^1(\tilde{M})$ is a limit of cohomology classes which correspond to fiberings of $\tilde{M}$. (this is actually the criterion that Friedl-Vidussi use)
This implies that $\tilde{\alpha}$ is dual to a **depth one taut foliation** $\mathcal{F}$ of $\tilde{M}$, with a closed leaf of $\mathcal{F}$ corresponding to the Poincare dual of $\tilde{\alpha}$.

Main Theorem

Theorem (A.)

If M is hyperbolic, Haken, and $\pi_1(M)$ is LERF, then M is cubulated. Therefore, M virtually fibers.

Remark: To prove that M is cubulated in the case that M already fibers, one first applies a theorem of Cooper-Long-Reid to reduce to the case that M has a quasi-fuchsian surface.

Implications

Corollary

If M is hyperbolic and $\pi_1 M$ is LERF, then the Heegaard gradient $\nabla g(M) = 0$.

Proof.

A result of Lackenby, Long, and Reid (based on work of L. Bowen) implies that M has a sequence of covers $\{M_i\}$ with Cheeger constant $h(M_i) \to 0$. Then a result of Lackenby implies that either these covers have Heegaard gradient $\nabla g(M) \leq \inf_i g(M_i)/[M : M_i] = 0$, or else M_i is Haken for large enough i. In the second case, our theorem implies that M is virtually fibered, and therefore there is a sequence of covers with Heegaard gradient $\to 0$, so again $\nabla g(M) = 0$. $\square$

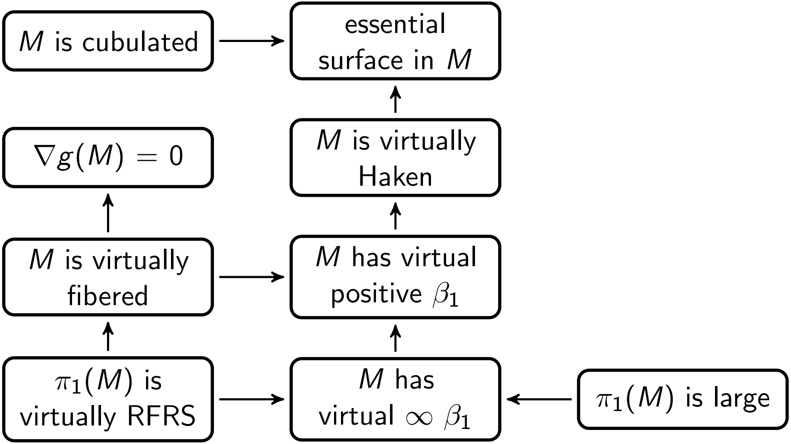
Implications

Theorem (A., Groves, Manning, Martinez-Pedrosa)

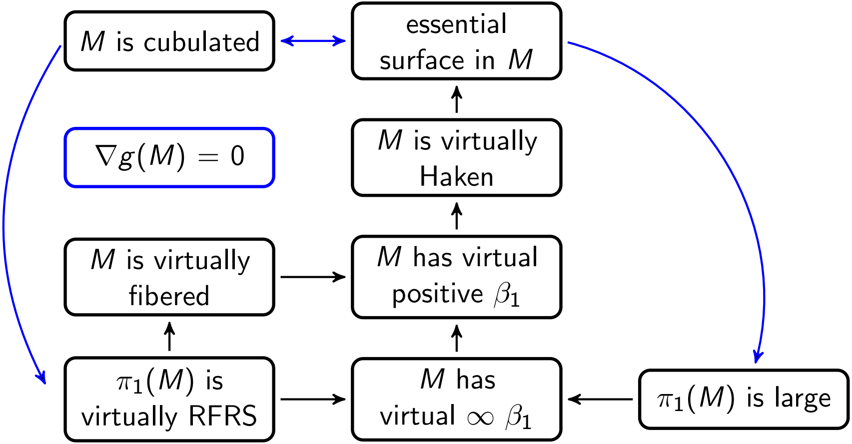
If **Gromov-hyperbolic groups** *are RF, then Kleinian groups are LERF*

So it may be possible to show that hyperbolic 3-manifold groups are LERF by showing that Gromov-hyperbolic groups are RF
Caveat: This approach seems quite unlikely to work, since many experts believe that there are non-RF Gromov-hyperbolic groups.

Relations between various conjectures for M hyperbolic



Relations between various conjectures for $\pi_1(M)$ LERF



Previous work on these questions

There has been much work devoted to understanding the various conjectures summarized in the previous diagram.

- Much older work has been done analyzing incompressible surfaces in Haken 3-manifolds, as well as demonstrating infinite classes of non-Haken (hyperbolic) 3-manifolds.
- Freedman-Freedman and Cooper-Long proved that many Dehn fillings on cusped hyperbolic non-fibered 3-manifolds are virtually Haken
- Cooper-Long and independently Li proved that all but finitely many Dehn fillings on a cusped hyperbolic 3-manifold have essential surfaces.
- Cooper-Long-Reid proved that non-cocompact Kleinian groups are large.

Previous work on these questions

- Masters showed that fibered manifolds with a fiber of genus 2 have virtual infinite β_1.
- Masters-Zhang showed that cusped hyperbolic 3-manifolds contain essential quasifuchsian surfaces.
- There have been several classes of 3-manifolds shown to virtually fiber, including 2-bridge links, some Montesinos links, and certain alternating links (Agol-Boyer-Zhang, Aitchison-Rubinstein, Bergeron, Chesebro-DeBlois-Wilton, Gabai, Leininger, Reid, Walsh, Wise).

Previous work on these questions

- For arithmetic 3-manifolds, the virtual betti number conjecture is equivalent to the large conjecture by work of Lackenby-Long-Reid and Cooper-Long-Reid. The virtual positive β_1 conjecture for arithmetic 3-manifolds would follow from conjectures in number theory, such as the Taniyama-Shimura conjecture for number fields, or the Langlands functoriality conjecture, via the Jacquet-Langlands correspondence. Previous work in various cases has proved this for certain classes of arithmetic 3-manifolds (Millson, Schwermer, et al).
- These are some highlights, but there are many other important works on this suite of questions!!

Cubulations without LERF

For certain hyperbolic 3-manifolds, we can prove that there exists a cubulation without assuming LERF.
Examples which were already known include hyperbolic alternating link complements and reflection orbifolds.

Theorem

Arithmetic 3-manifolds are cubulated.

This theorem was already known for arithmetic 3-manifolds containing immersed geodesic surfaces, including the Bianchi orbifolds and Seifert-Weber dodecahedral space.

Cubulations without LERF

Theorem

Arithmetic 3-manifolds are cubulated.

Proof.

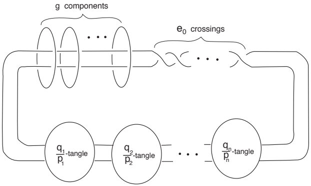
Lackenby proved that arithmetic 3-manifolds contain essential surfaces. One can use the commensurator of the group to move an essential surface around up to finite index, until one obtains a finite collection of surfaces which "fill" the manifold. Then apply a construction of Sageev to obtain a cubulation. □

Cubulations without LERF

Theorem

Hyperbolic Montesinos link complements are cubulated.

Remark: Certain classes of Montesinos links were known to be virtually fibered. Walsh proved that 2-bridge links virtually fibered. Some other classes of examples and certain branched covers were shown to be virtually fibered by Deblois and A.-Boyer-Zhang.



Conjecture: Fibered hyperbolic 3-manifolds are cubulated.

Cubulations without LERF

Theorem (Hsu-Wise)

Graphs of free groups along cyclic edge groups which are Gromov-hyperbolic are cubulated.

By results of Wise, these groups are known to be LERF, and therefore virtually a subgroup of RAAG and virtually RFRS. This example includes fundamental groups of many 3-manifolds with boundary. The interior of these manifolds have a finite-sheeted cover fibering over S^1 with fiber a non-compact surface.

Techniques used in the proof of the main theorem

We need to prove that a Haken 3-manifold has enough immersed π_1-injective surfaces so that Sageev's theorem provides a faithful proper cocompact action on a CAT(0) cube complex, and therefore a cubulation.

The Haken hypothesis provides one surface. We need to use this surface to create surfaces which intersect every curve in the 3-manifold essentially. We make use of

- **Hierarchies** for Haken 3-manifolds
- A combination theorem for Kleinian groups due to Baker and Cooper
- $\mathrm{SL}(2, \mathbb{C})$ **character varieties**

$\mathrm{SL}(2,\mathbb{C})$ character varieties

Lemma

Let M be a compact 3-manifold with hyperbolic interior and let $X(M)$ be the character variety of conjugacy classes of homomorphisms of $\pi_1(M) \to \mathrm{SL}(2,\mathbb{C})$. There exists finitely many (boundary) incompressible surfaces $\{\Sigma_i\}, \Sigma_i \subset M$ such that every element $g \in \pi_1(M)$ with $tr(g)$ not locally constant on $X(M)$ intersects some Σ_i non-trivially.

The idea of the proof is to find a curve $C \subset X(M)$ on which every such $tr(g)$ restricts to a non-constant character. Then construct surfaces dual to the finitely many ideal points of C using Bass-Serre theory and Stallings' method. These surfaces will have the desired properties.

$\mathrm{SL}(2,\mathbb{C})$ **character varieties**

This lemma is used to prove that we may obtain surfaces intersecting every curve in the manifold in certain cases.

Question:

If M is a manifold with boundary and hyperbolic interior, is there a finite-sheeted cover $\tilde{M} \to M$ for which every element $g \in \pi_1(\tilde{M})$ has non-constant $tr(g) \in X(\tilde{M})$?

The answer to this question would help simplify our argument, and would eliminate some of the reliance on hierarchies.

The End

Can one prove Haken hyperbolic 3-manifold groups are LERF?