# SHORT GEODESICS IN HYPERBOLIC COMPRESSION BODIES ARE NOT KNOTTED

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## 1

Let $\bar{N}$ be a compact, orientable, irreducible and atoroidal 3-manifold with boundary $\partial\bar{N}$. A simple closed curve $\gamma \subset \bar{N}$ is said to be *unknotted* with respect to $\partial\bar{N}$ if it can be isotoped into $\partial\bar{N}$. Equivalently, $\gamma$ is contained in an embedded surface which is isotopic to the boundary. More generally, a finite collection $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ of simple curves is *unlinked* in $\bar{N}$ if there is a collection of disjoint embedded surfaces $S_1, \dots, S_n$ which are parallel to $\partial\bar{N}$ and with $\gamma_i \subset S_i$ for all $i$.

In this note we are interested in those curves which are short geodesics with respect to a complete hyperbolic metric on the interior $N$ of a compression body $\bar{N}$. We prove:

**Theorem 1.1.** *If $\bar{N}$ is a compression body then there is a constant $\epsilon$ which depends only on $\chi(\bar{N})$ such that for every complete hyperbolic metric on the interior $N$ of $\bar{N}$ we have: Every finite collection of geodesics which are shorter than $\epsilon$ is unlinked with respect to $\partial\bar{N}$.*

In [Ota95], Otal proved Theorem 1.1 in the case that the manifold $\bar{N}$ is homeomorphic to the trivial interval bundle over a closed surface. Otal also considered in [Ota03] the case that $\bar{N}$ is a general compression body, proving that geodesics which are homotopic to short curves on the boundary of the convex core are not knotted. In general, short geodesic $\gamma_*$ may be miles far away from the boundary of the convex-hull, forcing that every curve on the boundary of the convex-core and homotopic to $\gamma_*$ is long.

We describe now the strategy of the proof of Theorem 1.1: A short geodesic with respect to a complete hyperbolic metric on $N$ is simple by the Margulis lemma; moreover, if it is sufficiently short it is contained in an enormous Margulis tube $T(\gamma)$. We obtain a complete metric on the manifold $N \setminus \gamma$ which coincides with the hyperbolic metric on $N \setminus T(\gamma)$ and which has sectional curvature in $[-2, -\frac{1}{2}]$. We show that there is a simplicial ruled surface $\Sigma$ homotopic to $\partial\bar{N}$ in $N \setminus \gamma$ and which intersects very deeply the part where we changed the metric.

An area bound on $\Sigma$, the Margulis Lemma and the annulus theorem imply that there is a simple curve $\gamma'$ in $\partial\bar{N}$ which can be isotoped into a regular neighborhood of $\gamma$ within $\bar{N}\setminus\gamma$. This proves that some power of $\gamma$ is unknotted. A little bit of nice cup-and-paste topology shows that $\gamma'$ can in fact be chosen to be isotopic to $\gamma$. In particular $\gamma$ itself is not knotted.

The paper is organized as follows: In section 2 we discuss some well-known facts about compression bodies, manifolds of negative curvature and simplicial ruled surfaces. In section 3 we use the geometric argument sketched above to prove that, up to taking powers, every short geodesic in unknoted. Afterwards, in section 4 we prove that it is actually not necessary to take powers. Finally, in section 5 we sketch the proof of a result, Theorem 5.1, which asserts that sufficiently short geodesics in closed hyperbolic 3-manifolds are unknotted with respect to strongly irreducible Heegaard surfaces.

Jean-Pierre Otal deserves all my gratitude for many very stimulating discussions. I also would like to thank Ian Agol for founding and filling a gap in the original proof of Lemma 3.1. Finally, I would like to thank Dave Gabai for the remark leading to the proof of Theorem 5.1.

The first draft of this paper was written during the winter of 2004, while the author was visiting the University of Chicago. At the time the author was a member of the mathematics institute of the Rheinische-Friedrich-Wilhelms Universität Bonn and was being supported by the Sonderforschungsbereich 611 and by a post-doctoral research grant of the Deutsche Forschungs Gemeinschaft. I apologize to all these institutions for the delay with which I am expressing my gratitude.

## 2. Preliminaries

In this section we recall some facts about 3-dimensional topology, negatively curved manifolds and simplicial ruled surfaces. Throughout the paper we will only consider orientable 3-manifolds.

2.1. **Some topology.** A compact 3-manifold $\bar{N}$ is irreducible if every embedded sphere bounds a ball and it is atoroidal if every properly embedded incompressible torus is boundary parallel. A meridian is an essential simple closed curve in $\partial\bar{N}$ which is homotopically trivial in $\bar{N}$. A component $S$ of $\partial\bar{N}$ containing a meridian is said to be compressible. The Dehn lemma shows that every meridian is the boundary of a properly embedded disk. See [Hem76, Jac80] for more on the topology of 3-manifolds.

The manifold $\bar{N}$ is a compression body if it is irreducible and atoroidal and has a privileged boundary component $\partial_{\text{ext}}\bar{N}$, called the exterior boundary, such that $\pi_1(\partial_{\text{ext}}\bar{N})$ surjects onto $\pi_1(\bar{N})$. The union of the remaining boundary components is said to be the interior boundary of $\bar{N}$; remark that the interior boundary is incompressible. A compression body with incompressible boundary is homeomorphic to a trivial interval bundle over a closed orientable surface; such compression bodies are said to be trivial. Abusing terminology, we will sometimes say that the disjoint union of compression bodies is a compression body.

Compression bodies arise naturally studying compact irreducible 3-manifolds $\bar{N}$. Let namely $S$ be a component of $\partial\bar{N}$ and let $C'_S$ be the 2-complex obtained by attaching to $S$ a maximal collection of disjoint non-parallel properly embedded essential disks $(D,\partial D)\subset(\bar{N},S)$. Taking a regular neighborhood of $C'_S$ and capping off every spherical boundary component we obtain a submanifold $C_S$ of $\bar{N}$ homeomorphic to a compression body with exterior boundary $S$. In [Bon83], Bonahon proved that the isotopy class of $C_S$ in $\bar{N}$ depends only on the component $S\subset\partial\bar{N}$. The compression body $C_S$ is said to be the *relative compression body* of $S$ in $\bar{N}$.

Let from now on $\bar{N}$ be a compression body with exterior boundary $\partial_{\text{ext}}\bar{N}$ and interior $N$. If $\Gamma$ is a finite collection of disjoint simple curves in $N$ then let $\mathcal{N}(\Gamma)$ be one of its regular neighborhoods. If $\bar{N}\setminus\mathcal{N}(\Gamma)$ is irreducible then we will denote by $C_\Gamma$ the relative compression body in $\bar{N}\setminus\mathcal{N}(\Gamma)$ corresponding to $\partial_{\text{ext}}\bar{N}$. The following lemma follows easily from the construction of the relative compression body.

**Lemma 2.1.** *Let $\bar{N}$ be a compression body with exterior boundary $\partial_{\text{ext}}\bar{N}$ and interior $N$. Denote by $\tilde{N}$ the universal cover of $N$.*

*If $\Gamma$ is a collection of disjoint simple curves in $N$ with $\bar{N}\setminus\mathcal{N}(\Gamma)$ irreducible then the following holds:*

(1) *$\bar{N}\setminus C_\Gamma$ is a, possibly disconnected, compression body.*
(2) *The collection $\Gamma$ is unlinked in $\bar{N}$ if and only if it is unlinked in $\bar{N}\setminus C_\Gamma$.*
(3) *If $\Gamma'\subset\Gamma$ is such that $\bar{N}\setminus\mathcal{N}(\Gamma')$ is irreducible then $C_\Gamma$ can be isotoped such that $C_\Gamma\subset C_{\Gamma'}$.*
(4) *Every component $U$ of $\bar{N}\setminus C_\Gamma$ is homeomorphic to a compression body. Moreover, $\pi_1(U)$ injects into $\pi_1(\bar{N})$ and the cover $\tilde{N}/\pi_1(U)$ of $N$ determined by $U$ is homeomorphic to the interior of $U$. More precisely, the lift of the surface $\partial U$ to $\tilde{N}/\pi_1(U)$ is boundary parallel.* □

2.2. **Some geometry.** We refer to [BP92] and [BGS85] for basic facts about hyperbolic and negatively curved manifolds respectively.

Let $N$ be a (geodesically) complete, oriented, 3-dimensional Riemannian manifold with pinched negative curvature $-2 \leq \kappa_N \leq -\frac{1}{2}$. Before going further we recall that it is due to Margulis that there is a constant $\mu$, depending only on the dimension and the pinching constants, with the property that every component of the $\mu$-thin part of $N$, i.e. the set where the injectivity radius is less than $\mu$, is either homeomorphic to a solid torus or to a trivial interval bundle over a torus. This implies that primitive geodesics shorter than $\mu$ are simple and disjoint.

If $\Gamma$ is a collection of primitive geodesics shorter than some $\epsilon < \mu$ then we will denote by $N^{\Gamma<\epsilon}$ the union of those components of the $\epsilon$-thin part of $N$ which contain a component of $\Gamma$; $N^{\Gamma<\epsilon}$ is a collection of disjoint solid tori.

The reason why we are going to work with manifolds of variable negative curvature is that a tube around a simple geodesic in a hyperbolic manifold can be replaced by a cusp with variable negative curvature; we say that $\gamma$ can be drilled out. If the geodesic $\gamma$ has a tubular neighborhood with huge tube radius then $\gamma$ can be drilled out with sectional curvature close to $-1$. Short primitive geodesics in hyperbolic 3-manifolds have very large tube neighborhoods by the Margulis lemma and thus we obtain:

**Lemma 2.2.** *For every $\epsilon_0$ positive there is $\epsilon > 0$ such that the following holds: If $\Gamma$ is a collection of geodesics in a hyperbolic manifold $N$ which are shorter than $\epsilon$, then there is a complete Riemannian metric $\rho$ on $N \setminus \Gamma$ with curvature pinched in $[-2, -\frac{1}{2}]$ and which coincides with the original hyperbolic metric outside of $N^{\Gamma<\epsilon_0}$.* □

See for example [Ago02] for a proof of Lemma 2.2.

It is well-known that if the interior of a compact manifold $\bar{N}$ admits a complete metric of negative curvature then $\bar{N}$ is irreducible and atoroidal. In particular we deduce from Lemma 2.2:

**Lemma 2.3.** *There is $\epsilon$ positive such that whenever $\bar{N}$ is a compact 3-manifold, $N$ is a complete hyperbolic manifold homeomorphic to the interior of $\bar{N}$ and $\Gamma \subset N$ is a finite collection of geodesics shorter than $\epsilon$, then the compact manifold $\bar{N} \setminus \mathcal{N}(\Gamma)$ is irreducible and atoroidal.* □

2.3. **Simplicial ruled surfaces.** Let $N$ be a complete manifold with curvature pinched by $-2$ and $\frac{-1}{2}$ and $\Delta \subset \mathbb{R}^2$ a triangle which is foliated by segments with an endpoint at a vertex $v$ of $\Delta$ and the other endpoint at the the edge of $\Delta$ opposite to $v$. An immersion $f : \Delta \to N$ is said to be a *ruled triangle* if every edge of $\Delta$ and every leaf of the foliation is mapped to a geodesic segment. Sometimes we will also allow that the map $f$ is only defined on $\Delta - v$. In this case we have the additional condition on $f$ to be proper and that $f$ maps the edges adjacent to $v$ to asymptotic geodesic rays in $N$. Remark that the pull back of the Riemannian metric of $N$ via $f$ is a smooth metric on $\Delta$ with upper curvature bound $-\frac{1}{2}$.

Let $\bar{S}$ be a closed surface, $\mathcal{V} \subset \bar{S}$ a finite collection of points and $S = \bar{S} \setminus \mathcal{V}$. A proper continuous map $\phi : S \to N$ is a *pre-simplicial ruled surface* if the following conditions hold:

- The boundary of every small disk in $S$ centered at a point in $\mathcal{V}$ is mapped by $\phi$ to an essential curve in $N$.
- There is a triangulation $\mathcal{T}$ of $\bar{S}$ which contains $\mathcal{V}$ in the set of vertices such that $\phi|_\Delta$ is a simplicial ruled triangle for every face $\Delta$ of $\mathcal{T}$.

Remark that it follows from the definition that every pre-simplicial ruled surface maps small loops around the set $\mathcal{V}$ of punctures to parabolic elements in $\pi_1(N)$. Moreover, the Riemannian metric of $N$ induces a metric on $S$ which is smooth with curvature bounded from above by $-\frac{1}{2}$ on every face of $\mathcal{T}$. In particular, this metric has well-defined cone-angles at every point. A pre-simplicial hyperbolic surface is a *simplicial ruled surface* if the cone angles are at least $2\pi$ at every point.

If $\phi : S \to N$ is a simplicial hyperbolic surface, then the distance induced on the universal cover of $S$ is complete and $CAT(-\frac{1}{2})$. In particular, it follows from the Gauß-Bonnet theorem [Bon86, Can93] that $\text{vol}(S) \leq 4\pi|\chi(S)|$.

Later we will only consider a particular type of simplicial ruled surfaces $\phi : S \to N$. We will namely assume that the triangulation $\mathcal{T}$ has only one vertex $v$ in $S \setminus \mathcal{V}$ and that there is a privileged edge $I$ of $\mathcal{T}$ which is mapped by $\phi$ to a closed geodesic $\eta_*$ [Can93]. Such simplicial simplicial hyperbolic surfaces are said to be *good* and to *realize* the geodesic $\eta_*$.

**Lemma 2.4.** [Bon86, Can93] *Let $\bar{S}$ be a closed surface and $\mathcal{V} \subset \bar{S}$ a finite collection of points. Set $S = \bar{S} \setminus \mathcal{V}$ and let $\eta \subset S$ be an essential simple closed curve.*

*If $f : S \to N$ is a $\pi_1$-injective map which maps small loops around the punctures of $S$ to parabolic elements in $\pi_1(N)$ and such that $f(\eta)$ is homotopic to a geodesic $\eta_*$ in $N$, then there is a good simplicial ruled surface $\phi : S \to N$ homotopic to $f$ which realizes $\eta_*$.*

An important draw back of simplicial ruled surfaces is that they do not have curvature bounded from below. In particular, the Margulis lemma does not apply. However, such a Margulis Lemma exists if we consider only $\pi_1$-injective simplicial ruled surfaces. More precisely, for every complete Riemannian 3-manifold $M$ with pinched negative curvature $-2 \leq \kappa_M \leq -\frac{1}{2}$ and for every $\pi_1$-injective simplicial ruled surface $\phi : S \to M$ the following holds: two essential loops $\gamma$ and $\gamma'$ in $S$ based at a point $x$ and with length less than $\mu$ generate an abelian subgroup of $\pi_1(S)$. Recall that $\mu$ is the Margulis constant for 3-manifolds with curvature pinched in $-2 \leq \kappa \leq -\frac{1}{2}$.

In particular we obtain that the $\mu$-thin part of every simplicial ruled surface is a union of disjoint embedded annuli; the unbounded components are said to be cusps and the union of the cusps is said to be the cuspidal part. Moreover, from the point of view of coarse geometry, a lower bound of the curvature can often be replaced by a Margulis Lemma. In particular the Gauß-Bonnet theorem, the Margulis Lemma and basic convexity properties of distance functions on metric spaces with negative curvature imply:

**Lemma 2.5.** *For all $A$ there are positive constants $\mu' \leq \mu$, $L$ and $\epsilon_L$ such that for every complete Riemannian 3-manifold $M$ with pinched negative curvature $-2 \leq \kappa_N \leq -\frac{1}{2}$ and for every $\pi_1$-injective simplicial ruled surface $\phi : S \to N$ with $|\chi(S)| \leq A$ we have:*

(1) *A geodesic in $S$ which is homotopic to a simple closed curve in $S$, does not enter the $\mu'$-cuspidal part of $S$.*
(2) *For every point $x \in S$ there is a non-homotopically trivial loop $\gamma_x$ based at $x$ which is shorter than $L$. If moreover $x$ is not contained in the $\mu'$-cuspidal part of $S$ and $S$ is not a thrice punctured sphere then the loop $\gamma_x$ can be chosen to be simple and not homotopic into a cusp of $S$.*
(3) *If $\eta$ and $\eta'$ are two loops in $N$ based at a point $x$ with $l_N(\eta) \leq L$ and $l_N(\eta') \leq \epsilon_L$ then $\eta$ and $\eta'$ generate an abelian subgroup of $\pi_1(N)$.* □

Before moving on observe that simplicial ruled surfaces can be constructed in many other situations: the key fact needed is that the universal cover of the manifold in question is for example CAT(-1). Compare with [Som06].

## 3. The geometric part of the proof

Let $\bar{N}$ be a compression body with exterior boundary $\partial_{\text{ext}}\bar{N}$ and interior $N$. We start fixing some constants:

**Constants:** Let $\mu'$, $L$ and $\epsilon_L$ be the constants provided by Lemma 2.5 for $A = |\chi(\partial\bar{N})|$. Let then $\epsilon < \epsilon_L$ is the constant provided by Lemma 2.2 for $\epsilon_0 = \epsilon_L$.

The main result of this section is the following Lemma:

**Lemma 3.1.** *Let $\rho_0$ be a complete hyperbolic metric in $N$ and $\Gamma$ a collection of geodesics in $(N, \rho_0)$ shorter than $\epsilon$. Then there is a properly embedded annulus $(A, \partial_1 A, \partial_2 A)$ in $(\bar{N} \setminus \mathcal{N}(\Gamma), \partial\bar{N}, \partial\mathcal{N}(\Gamma))$. Moreover, $\partial_2 A$ is not the meridian of the corresponding component of $\mathcal{N}(\Gamma)$.*

To begin with, observe that the claim is trivial if $\bar{N}$ is a solid torus. We assume from now on that this is not the case.

Recall that $C_\Gamma$ is the relative compression body corresponding to the surface $\partial_{\text{ext}}\bar{N}$ in $\bar{N} \setminus \mathcal{N}(\Gamma)$. By Lemma 2.1, the claim of Lemma 3.1 holds if and only if it holds in the cover of $N$ determined by $C_\Gamma$; this cover is again homeomorphic to the interior of a compression body. In other words, we may assume without loss of generality that $\partial\bar{N} \setminus \mathcal{N}(\Gamma)$ has incompressible boundary.

Let $\rho$ be the complete metric with curvature pinched in $[-2, -\frac{1}{2}]$ on $N \setminus \Gamma$ provided by Lemma 2.2. We will denote by $N_\Gamma$ the Riemannian manifold $(N \setminus \Gamma, \rho)$ and by $N_\Gamma^{\Gamma<\epsilon_L}$ the union of the components of the $\epsilon_L$-thin part of $N_\Gamma$ which corresponds to the rank-two cusps of $N_\Gamma$ corresponding to $\Gamma$. We will refer to these cusps and the associated subgroups of $\pi_1(N_\Gamma)$ as the *new* cusps and *new* rank-two parabolic groups. Recall that by construction the metrics $\rho$ and $\rho_0$ coincide on $N_\Gamma \setminus N_\Gamma^{\Gamma<\epsilon_L}$.

After these preliminary remarks, we can start with the proof of Lemma 3.1.

*Proof of Lemma 3.1.* Assume to begin with that $\bar{N}$ is not a trivial compression body; equivalently $\partial_{\text{ext}}\bar{N}$ is compressible.

By the remark above, we may assume without loss of generality that $\partial_{\text{ext}}\bar{N}$ is incompressible in $\bar{N} \setminus \mathcal{N}(\Gamma)$. There may be however properly embedded essential disks which intersect $\Gamma$ exactly once. Choose a maximal collection $\mathcal{D}$ of disjoint, non-parallel properly embedded disks $D_1, \ldots, D_k$ with $|D_i \cap \Gamma| = 1$ for all $i$ and let $S$ be a component of $\partial_{\text{ext}}\bar{N} \setminus \partial\mathcal{D}$. A simple parity argument shows that the closure $\bar{S}$ of $S$, with respect to the interior distance of $S$, has an even number of boundary components. In particular, $S$ cannot be a trice punctured sphere. Observe also that if $I$ is an arc in $\bar{S}$ joining two different components of $\partial\bar{S}$ then one of the components of a regular neighborhood of $I \cup \partial\bar{S}$ is not boundary parallel in $\bar{S}$ and bounds a properly embedded essential disk $(D, \partial D) \subset (\bar{N}, S)$. Summing up, the following holds for the subsurface $S$ of $\partial_{\text{ext}} \bar{N}$:

(1) $S$ contains an essential, non-boundary parallel, simple closed curve which is homotopically trivial in $\bar{N}$.
(2) $S$ is incompressible in $\bar{N} \setminus \Gamma$ and hence in $N_\Gamma$.
(3) Every boundary parallel curve in $S$ can be freely homotoped within $N \setminus \Gamma$ into $\mathcal{N}(\Gamma)$. In particular, every boundary parallel curve in $S$ represents a parabolic element in $\pi_1(N_\Gamma)$.

Moreover, by the maximallity of the collection $\mathcal{D}$ we have:

(4) If an essential simple closed curve $\gamma \subset S$ is homotopic in $\bar{N} \setminus \mathcal{N}(\Gamma)$ to a curve $\gamma' \subset \partial\mathcal{N}(\Gamma)$, then $\gamma'$ is not the meridian of the corresponding component of $\mathcal{N}(\Gamma)$.

We claim:

**Claim.** The surface $S$ contains an essential simple closed curve $\eta$ which is homotopic in $N_\Gamma$ to a geodesic $\eta_*$ with $\eta_* \cap N_\Gamma^{\Gamma<\epsilon_L} \neq \emptyset$.

*Proof of the claim.* By (1), the surface $S$ contains an essential simple closed curve $\eta$ which is compressible in $\bar{N}$. If $\eta$ is not homotopic in $N_\Gamma$ to a geodesic $\eta_*$, then $\eta$ is either homotopically trivial in $\bar{N} \setminus \mathcal{N}(\Gamma)$, which is impossible by (2), or $\eta$ represents a parabolic element in $\pi_1(N_\Gamma)$. Moreover, this parabolic element must belong to one of the new rank-two parabolic groups again because $\eta$ is homotopically trivial in $\bar{N}$. In other words, $\eta$ is homotopic within $\bar{N} \setminus \mathcal{N}(\Gamma)$ to a curve $\eta'$ in $\partial\mathcal{N}(\Gamma)$. By (4), the curve $\eta'$ represents a non-trivial multiple of one of the components of $\Gamma$. However, this is not possible since all the components of $\Gamma$ are essential in $\pi_1(\bar{N})$. This proves that $\eta$ is homotopic in $N_\Gamma$ to a geodesic $\eta_*$.

Since $\eta$ is homotopically trivial in $\bar{N}$, the same is true for $\eta^*$ and hence $\eta^*$ cannot be a geodesic in $(N, \rho_0)$. In particular, $\eta^*$ must enter the region where the metrics $\rho$ and $\rho_0$ do not coincide. This means that $\eta_* \cap N_\Gamma^{\Gamma<\epsilon_L} \neq \emptyset$. This concludes the proof of the claim. □

By (2) and (3), Lemma 2.4 applies. Hence, there is a simplicial hyperbolic surface $\phi : S \to N_\Gamma$ realizing $\eta_*$. In particular, there is $x \in \eta \subset S$ with $\phi(x) \in N_\Gamma^{\Gamma<\epsilon_L}$. By Lemma 2.5, we find a simple loop $\gamma_x$ based at $x \in \eta$ with length, with respect to $\phi$, less than $L$ and which is not boundary parallel $S$. Observe that by (2), the loop $\gamma_x$ is homotopically essential in $N_\Gamma$.

By construction, there is a loop $\gamma' \subset N_\Gamma^{\Gamma<\epsilon_L}$ based at $\phi(x)$, shorter than $\epsilon_L$ and representing an essential element in one of the new rank-two parabolic groups in $\pi_1(N_\Gamma)$. It follows from the last claim of Lemma 2.5 that the loops $\gamma'$ and $\phi(\gamma_x)$ generate an abelian subgroup of $\pi_1(N_\Gamma, \phi(x))$. Hence, $\gamma_x$ belongs also to one of the new rank-two parabolic groups. It follows that the loop $\gamma_x \subset S$ is freely homotopic in $\bar{N} \setminus \mathcal{N}(\Gamma)$ into $\partial\mathcal{N}(\Gamma)$.

We obtains now from the annulus theorem a properly embedded annulus $(A, \partial_1 A, \partial_2 A)$ in $(\bar{N} \setminus \mathcal{N}(\Gamma), S, \partial\mathcal{N}(\Gamma))$ with $\partial_1 A = \gamma_x$. It follows from (4) that $\partial_2 A$ is not the meridian of the corresponding component of $\mathcal{N}(\Gamma)$. This concludes the proof of Lemma 3.1 in the case that $\bar{N}$ is not a trivial interval bundle.

If $\bar{N}$ is a trivial interval bundle a similar, in fact simpler, discussion applies. The same argument applies once we find a curve $\eta$ in $\partial\bar{N}$ such that the corresponding geodesic $\eta_*$ in $N_\Gamma$ enters $N_\Gamma^{\Gamma<\epsilon_L}$. Let $(Z, \partial Z)$ be a properly embedded essential annulus in $(\bar{N}, \partial\bar{N})$ which intersects the collection $\Gamma$ essentially and whose soul is homotopic to a geodesic $z_*$ in $N$. Then at least one of the two boundary components of $Z$ cannot be homotoped to $z_*$ in the complement of $\Gamma$; let $\eta$ be this component of $\partial Z$. □

## 4. The topological part of the proof

After the preparatory work in the last section we prove Theorem 1.1

**Theorem 1.1.** *If $\bar{N}$ is a compression body then there is a constant $\epsilon$ which depends only on $\chi(\bar{N})$ such that for every complete hyperbolic metric on the interior $N$ of $\bar{N}$ we have: Every finite collection of geodesics which are shorter than $\epsilon$ is unlinked with respect to $\partial\bar{N}$.*

If $\bar{N}$ has abelian fundamental group then Theorem 1.1 is trivial. We assume from now on that this is not the case.

Continuing with the same notation as in the previous section, let $\rho$ be a complete hyperbolic metric on $N$ and let $\Gamma$ be a finite collection of closed primitive geodesics in $(N, \rho)$ which are shorter than the constant $\epsilon$ in the statement of Lemma 3.1.

If $\Gamma'$ is a subcollection of $\Gamma$ and $\gamma$ is a component of $\Gamma'$ then we will say that $\gamma$ in *unlinked with* $\Gamma'$ if it can be isotoped into $\partial\bar{N}$ within $\bar{N} \setminus (\Gamma' \setminus \gamma)$. Theorem 1.1 follows by induction if we show that every finite subcollection $\Gamma' \subset \Gamma$ contains an unlinked curve. This is what we intend to do.

We start fixing a subcollection $\Gamma'$ of $\Gamma$. In order to save notation, we assume that $\Gamma' = \Gamma$. Recall that $C_\Gamma$ is the relative compression body of $\partial\bar{N}$ in $\bar{N} \setminus \mathcal{N}(\Gamma)$. Lemma 2.1 implies that a component $\gamma$ of $\Gamma$ is unlinked with $\Gamma$ within $\bar{N}$ if and only if it is within $\bar{N} \setminus C_\Gamma$. In particular, we may assume as in the proof of Lemma 3.1, that $\partial\bar{N}$ is incompressible in $\bar{N} \setminus \mathcal{N}(\Gamma)$.

Lemma 3.1 implies that there is a properly embedded annulus $(A, \partial_1 A, \partial_2 A)$ in $(\bar{N} \setminus \mathcal{N}(\Gamma), \partial\bar{N}, \partial\mathcal{N}(\Gamma))$. Let $\gamma$ be the component in $\Gamma$ with $\partial_2 A \subset \partial\mathcal{N}(\gamma)$. Also by Lemma 3.1 we may assume that $\partial_2 A$ does not represent a trivial element $[\partial_2 A]$ in $\pi_1(\mathcal{N}(\gamma))$. If the element $[\partial_2 A]$ is indivisible, then $\partial_2 A$ is a longitude of $\mathcal{N}(\gamma)$. This implies that $\gamma$ and $\partial_2 A$ are isotopic within $\mathcal{N}(\gamma)$. Since $\partial_2 A$ and $\partial_1 A$ are isotopic in $\bar{N} \setminus \mathcal{N}(\Gamma)$ this implies that $\gamma$ is unlinked with $\Gamma$. Before going further, remark that a simple curve on a surface is indivisible; in particular Theorem 1.1 is proved in the case that $\bar{N}$ is a trivial interval bundle.

In the case that $\bar{N}$ has compressible boundary some more work has to be done. Assume that $[\partial_2 A]$ is divisible and set $\Gamma' = \Gamma \setminus \gamma$. We identify $\mathcal{N}(\gamma)$ with the normal bundle of $\gamma$ and project $\partial_2 A$ along the fibers. We obtain a properly embedded 2-complex $(X, \partial X)$ in $(\bar{N} \setminus \mathcal{N}(\Gamma'), \partial\bar{N})$ and such that $X \setminus (X \cap \mathcal{N}(\gamma)) = A$. Let $\mathcal{N}(X)$ be a regular neighborhood of $X$ in $\bar{N}$ and observe that the intersection of the boundary $\partial\mathcal{N}(X)$ of $\mathcal{N}(X)$ with the interior of $\bar{N}$ is an open annulus whose closure we denote by $A_X$. The annulus $A_X$ is incompressible and it is $\partial$-incompressible because $\bar{N}$ is not a solid torus and $\partial\bar{N}$ is incompressible in $\bar{N} \setminus \mathcal{N}(\Gamma)$.

We apply now Lemma 3.1 to the collection $\Gamma'$ and find a new annulus $(A', \partial_1 A', \partial_2 A')$ in $(\bar{N} \setminus \mathcal{N}(\Gamma'), \partial\bar{N}, \partial\mathcal{N}(\Gamma'))$. Since the annulus $A_X$ is incompressible and $\partial$-incompressible we observe that the annulus $A'$ can be chosen to be disjoint of $A_X$, and, *a fortiori*, disjoint of $\mathcal{N}(X)$.

If the curve $\partial_2 A'$ determines a primitive element in the fundamental group of the corresponding component $\mathcal{N}(\gamma')$ of $\mathcal{N}(\Gamma')$ then we deduce that $\gamma'$ is unlinked with $\Gamma$ and we are done. If this is not the case then we can repeat the process above again and again and again... At the end of the day we reduce to the following case: $\Gamma = \{\gamma_1, \dots, \gamma_k\}$ and there are properly embedded disjoint annuli $(A_i, \partial_1 A_i, \partial_2 A_i)$ in $(\bar{N} \setminus \mathcal{N}(\Gamma), \partial\bar{N}, \partial\mathcal{N}(\Gamma))$ for $i = 1, \dots, k$ such that the curve $\partial_2 A_i \subset \partial\mathcal{N}(\gamma_i)$ represents a divisible element in $\pi_1(\mathcal{N}(\gamma_i))$. For each $\gamma_i$ we construct a 2-complex $X_i$ as above and choose a meridian $m \subset \partial\bar{N}$ which intersects $\partial X_1 \cup \dots \cup \partial X_k$ in a minimal number of points. Applying Dehn's lemma we get a properly embedded disk $D$ with boundary $m$. Up to isotopy we may assume that the intersection of $X_i$ with $D$ is a collection of properly embedded graphs with one vertex each. This implies that there is an embedded segment $I \subset m$ whose interior is disjoint of $X_1, \dots, X_k$ and whose endpoints lie both in $\partial X_i \cap m$ for some $i$. Then, one of the curves $m'$ which is obtained from $\partial X_i$ by surgery along $I$ is simple, essential and, by construction, disjoint of $\partial X_1 \cup \dots \cup \partial X_k$ and compressible in $\bar{N}$. This contradicts the assumption that $\bar{N} \setminus \mathcal{N}(\Gamma)$ is incompressible and concludes the proof of Theorem 1.1. $\square$

## 5. An extension of Theorem 1.1

So far, we have been interested in compact 3-manifolds with boundary. However, it makes also perfect sense to ask if curves are knotted with respect to embedded surfaces in closed manifolds. In this setting we have:

**Theorem 5.1.** *For every $g$ there is a constant $\epsilon$ such that the following holds: If $N$ is a closed hyperbolic 3-manifold, $S \subset N$ is a genus $g$ strongly irreducible Heegaard surface and $\Gamma \subset N$ is the collection of those primitive geodesics in $N$ which are shorter than $\epsilon$, then $\Gamma$ is unlinked with respect to $S$.*

See [Sch02] for basic facts and definitions about Heegaard splittings of 3-manifolds.

We will limit ourselves to sketch the proof of Theorem 5.1. This may be surprising but there is a, from the point of view of the author, very good reason for this decision. Our original proof of Theorem 5.1, the one we are going to sketch, follows the lines of the proof of Theorem 1.1. However, after perhaps one or two moderately interesting observations, one is forced to choose constant upon constant and the proof becomes an impenetrable morass of unpleasant details. This morass is what has kept this paper for four years in the limbo. And it would continue in the limbo if William Breslin and Joel Hass wouldn't have developed a different, and in some sense much more natural, approach to Theorem 5.1. Once this is said, we start the discussion of Theorem 5.1.

*Sketch of the proof of Theorem 5.1.* The starting point is a theorem of Pitts and Rubinstein (see [Rub05]) asserting that $N$ contains a minimal surface $F$ such that one of the following holds:

1. $S$ is isotopic to $F$ and hence $N \setminus F$ consists of two handlebodies of genus $g$, or
2. $F$ is one sided and $S$ is isotopic to the surface obtained by taking the boundary of a regular neighborhood of $F$ and attaching a vertical handle. In this case $S \setminus F$ is homeomorphic to a handlebody of genus $g - 1$.

In both cases, the proof of Theorem 5.1 remains almost the same. For the sake of simplicity we will consider only case (1). In other words, we assume that the Heegaard surface $S$ is minimal.

The first source of technical difficulties in the proof of Theorem 5.1 is that the surface $S$ can intersect the collection $\Gamma$. Our first goal is to by-pass this problem.

**Claim 1.** *There is a constant $\epsilon_1$ depending only on $g$ such that the following holds. The surface $S$ is isotopic to a surface $S'$ with $S' \cap N^{<\epsilon_1} = \emptyset$ by an isotopy which is supported in $N^{<\mu}$. Moreover, if $\gamma \subset N$ is a geodesic with $S \cap N^{\gamma<\epsilon_1} \neq \emptyset$, then $S' \cap N^{\gamma<\frac{\mu}{2}}$ is an annulus contained in $\partial N^{\gamma<\frac{\mu}{2}}$ and whose soul is homotopically essential in $N^{\gamma<\frac{\mu}{2}}$.*

Recall that $\mu$ is the 3-dimensional Margulis constant.

*Proof of claim 1.* Up to replacing $\mu$ by an infinitesimally smaller constant we may assume that $S$ intersects $\partial N^{\gamma<\frac{\mu}{2}}$ transversally. In particular, $S \cap \partial N^{\gamma<\frac{\mu}{2}}$ is a multicurve.

Let $H_1, H_2$ be the components of $N \setminus S$. If we are lucky, then $H_i \cap \partial N^{\gamma<\frac{\mu}{2}}$ is incompressible in $H_i$. If this is the case, a theorem of Scharlemann [Sch98] asserts that the subsurface $S \cap N^{\gamma<\frac{\mu}{2}}$ consists of a collection of boundary parallel annuli in $N^{\gamma<\frac{\mu}{2}}$ together with at most a component obtained by attaching two of these annuli by an unknotted handle. The claim follows directly from this description.

However, in general we are not lucky and there is a component $Y$ of $H_i \cap \partial N^{\gamma<\frac{\mu}{2}}$ such that $\pi_1(Y) \to \pi_1(H_i)$ is not injective. Our goal is to prove that we can avoid this problem by isotopying $S$ within $N^{\gamma<\mu}$.

To begin with we claim that every component $Y$ of $H_i \cap \partial N^{\gamma<\frac{\mu}{2}}$ is incompressible in $H_i \setminus N^{\gamma<\frac{\mu}{2}}$. If this fails to be true there is then a properly embedded essential disk in $D$ in $H_i \setminus N^{\gamma<\frac{\mu}{2}}$ with boundary in $Y$. By the solution of the Plateau problem, $\partial D$ bounds also a minimal disk $D'$ in $H_i$. The convexity of $N^{\gamma<\frac{\mu}{2}}$ implies that $D'$ is embedded and contained in $N^{\gamma<\frac{\mu}{2}}$. Since $H_i$ is irreducible, this implies that the two disks $D$ and $D'$ bound a ball. Since the surface $S$ cannot be contained in a ball, this contradicts the assumption that $D$ was essential.

It remains to understand how a component $Y$ of $H_i \cap \partial N^{\gamma<\frac{\mu}{2}}$ can be compressible in $H_i \cap N^{\gamma<\frac{\mu}{2}}$. To begin with observe that since $S$ is a genus $g$ minimal surface in the hyperbolic 3-manifold $N$, $S$ has at most area $4\pi(g-1)$ by the Gauß-Bonnet theorem. In particular, the monotonicity formula proves that there is a constant $d$ such that every component $X$ of $S \cap N^{\gamma<\frac{\mu}{2}}$ whose fundamental group has trivial image in $\pi_1(N)$ has at most diameter $d$. There is therefore a constant $\epsilon_1$ depending only on $g$ such that none of this components intersects $N^{\gamma<\epsilon_1}$.

The claim follows directly if $S \cap N^{\gamma<\epsilon_1} = \emptyset$; assume that this is not the case. By the preceding discussion this implies that there is a component of $S \cap N^{\gamma<\frac{\mu}{2}}$ which intersects every properly embedded essential disk in $N^{\gamma<\frac{\mu}{2}}$. In particular, if $(D, \partial D)$ is a properly embedded disk in $(H_i \cap N^{\gamma<\frac{\mu}{2}}, H_i \cap \partial N^{\gamma<\frac{\mu}{2}})$ then $D$ is boundary parallel in $N^{\gamma<\frac{\mu}{2}}$ and hence $\partial D$ bounds a disk $D' \subset \partial N^{\gamma<\frac{\mu}{2}}$. The disks $D$ and $D'$ bound a ball $B$ and we can isotope the surface $S$ to a surface $S_1$ within $N^{\gamma<\mu}$ in such a way that $S_1 \cap N^{\gamma<\frac{\mu}{2}} = S \cap (N^{\gamma<\frac{\mu}{2}} \setminus B)$. Observe that the surface $S_1$ has still the property that $S_1 \cap \partial N^{\gamma<\frac{\mu}{2}}$ is incompressible in $(N \setminus N^{\gamma<\frac{\mu}{2}}) \setminus S_1$ and that a component of $S_1 \cap N^{\gamma<\frac{\mu}{2}}$ intersects every essential disk in $N^{\frac{\mu}{2}}$. In particular, if $\partial N^{\gamma<\frac{\mu}{2}} \setminus S_1$ is compressible in $N^{\gamma<\frac{\mu}{2}} \setminus S$ we can repeat this process and get a new surface $S_2$, and then $S_3$ and so on. This process has to end because at every step we are reducing the number of components of $S_i \cap \partial N^{\gamma<\frac{\mu}{2}}$.

At the end of the day we obtain a surface $\bar{S}$ isotopic to $S$ by an isotopy supported by $N^{\gamma<\mu}$, such that $S \cap N^{\gamma<\epsilon_1} \subset \bar{S} \cap N^{\gamma<\frac{\mu}{2}}$ and such that $\partial N^{\gamma<\frac{\mu}{2}} \setminus \bar{S}$ is incompressible in $N \setminus \bar{S}$. In particular, Scharlemann's theorem [Sch98] applies to the surface $\bar{S} \cap N^{\gamma<\frac{\mu}{2}}$. As above, it follows directly that $\bar{S}$ is now isotopic within $N^{\gamma<\mu}$ to a surface $S'$ with the desired properties. □

Assuming that the constant $\epsilon$ in Theorem 5.1 is chosen to be smaller than the constant $\epsilon_1$ provided by claim 1 we have now that $S' \cap \Gamma = \emptyset$; here $S'$ is the surface also provided by claim 1. The surface $S'$ divides the manifold $N$ into two handlebodies $H_1$ and $H_2$. In order to prove Theorem 5.1 it suffices to prove that $\Gamma \cap H_i$ is unknotted in $H_i$ with respect to $\partial H_i$ for $i = 1, 2$. In the light of the proof of Theorem 1.1, it suffices in fact to prove the following claim:

**Claim 2.** *For every subcollection $\Gamma' \subset \Gamma \cap H_1$ there is a properly embedded annulus $(A, \partial_1 A, \partial_2 A)$ in $(H_1 \setminus \mathcal{N}(\Gamma'), \partial H_1, \partial\mathcal{N}(\Gamma'))$. Moreover, $\partial_2 A$ is not the meridian of the corresponding component of $\mathcal{N}(\Gamma')$.*

*Proof.* If the subcollection $\Gamma'$ contains a component $\gamma$ with $S \cap N^{\gamma<\epsilon_1} \neq \emptyset$, then the existence of the desired annulus follows directly from the last statement in claim 1. Assume from now on that this is not the case. Then we can isotope $S'$ back to the minimal surface $S$ within $N \setminus N^{\Gamma'<\epsilon_1}$. In other words, we can assume that $S' = S$.

In order to prove the claim we intend to use the same strategy as in the proof of Lemma 3.1. For the sake of concreteness we assume:

> **Assumption.** Every properly embedded essential disk in $H_1$ intersects the collection $\Gamma'$ at least twice.

Fix once and for ever an essential curve $\eta \subset \partial H_1$ which is homotopically trivial within $H_1$ and choose a constant $\epsilon_2$ much much smaller than the constant $\epsilon_1$. Assuming that $\epsilon < \epsilon_2$, we obtain from Lemma 2.2 a metrically complete metric $\rho$ on $H_1 \setminus \Gamma'$ which coincides with the original metric outside of $H_1^{\Gamma'<\epsilon_2}$. The manifold $V = (H_1 \setminus \Gamma', \rho)$ has minimal boundary and, in the interior, its sectional curvature is pinched by $-2$ and $\frac{-1}{2}$. A theorem of Alexander, Berg and Bishop [ABB93] implies then that the universal cover of $V$ is a metrically complete CAT(-1)-space. In particular, the same arguments as in the proof of Lemma 3.1 apply and show that $\eta$ is homotopic in $V$ to a geodesics $\eta_*$ which has to enter the part of the manifold where we changed the metric.

Since the universal cover of $V$ is a CAT(-1)-space, the same arguments used to prove Lemma 2.4 yield that there is a simplicial ruled surface $\phi : \partial V \to V$ which realizes $\eta$. We want now to conclude the proof of the claim using the same words as in the proof of Lemma 3.1. There is however a last technical difficulty. Namely that in the proof of Lemma 3.1 we used Lemma 2.5, which follows from the Margulis lemma. Unfortunately, in our current situation the Margulis lemma does not apply because of the presence of the boundary. However, we are only interested in applying Lemma 2.5 to points and loops contained in a fixed size neighborhood of the part of $V$ where we changed the metric. If we choose the constant $\epsilon_2$ to be really much much more smaller than $\epsilon_1$, this part of the manifold is miles far away from the boundary $\partial V$ of $V$. It is now not difficult to prove the appropriate version of Lemma 2.5 and conclude the proof of the claim as in the final part of the proof of Lemma 3.1.

Before moving on, observe that we can use the same arguments as in the first part of the proof of Lemma 3.1 to avoid the assumption above. This concludes the proof of claim 2. □

Using claim 2 and repeating word by word the arguments given in section 4 we obtain that $\Gamma \cap H_1$ is unlinked in $H_1$. Similarly, we obtain that $\Gamma \cap H_2$ is unlinked in $H_2$. Theorem 5.1 follows. □

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