Sample Midterm 2 Solutions, Math 1A

1. Let $p \neq 0$. Show by implicit differentiation that the tangent line to the curve

$$x^p + y^p = 1, \ x > 0, \ y > 0$$

at the point (x_0, y_0) is given by the equation $x_0^{p-1}x + y_0^{p-1}y = 1$. Show that the x-intercept a and y-intercept b of the tangent line satisfy $a^{p/(1-p)} + b^{p/(1-p)} = 1$ if $p \neq 1$.

Solution: Implicitly differentiate, using the power and chain rules:

$$\frac{d}{dx}(x^p + y^p) = px^{p-1} + py^{p-1}y' = \frac{d}{dx}(1) = 0.$$

Solving for y', we get $y' = -(x/y)^{p-1}$. We plug in the point (x_0, y_0) such that $x_0^p + y_0^p = 1$ into the point-slope formula for the tangent line:

$$y - y_0 = -(x_0/y_0)^{p-1}(x - x_0).$$

Multiplying by y_0^{p-1} and putting the constants to one side, we obtain

$$y_0^{p-1}y + x_0^{p-1}x = y_0^p + x_0^p = 1.$$

Setting x and $y = 0$, we see that the intercepts are $(a, b) = (x_0^{1-p}, y_0^{1-p})$. Then we see that

$$a^{p/(1-p)} + b^{p/(1-p)} = (x_0^{1-p})^{p/(1-p)} + (y_0^{1-p})^{p/(1-p)} = x_0^p + y_0^p = 1.$$

2. A ladder $10ft.$ long leans against a vertical wall. If the bottom of the ladder slides away from the base of the wall at a speed of $2ft./s.$, how fast is the angle between the ladder and the wall changing when the bottom of the ladder is $6ft.$ from the base of the wall?

Solution: Let α be the angle between the ladder and the wall, let $x\ ft.$ be the distance from the base of the ladder and the wall, and let $t\ s.$ be the time. The given information implies that $\frac{dx}{dt} = 2$. Then $\sin(\alpha) = x/10$, so $\alpha = \arcsin(x/10)$. Then we differentiate using the chain rule:

$$\frac{d\alpha}{dt} = \frac{d}{dt}\arcsin(x/10) = \frac{1}{\sqrt{1-(x/10)^2}}\frac{dx}{dt}\frac{1}{10} = \frac{2}{\sqrt{10^2 - x^2}}.$$

When $x = 6$, we have

$$\frac{d\alpha}{dt} rad/s. = \frac{2}{\sqrt{10^2 - 6^2}} rad/s. = \frac{1}{4} rad/s.$$

3. Prove that $\ln(x) \leq x - 1$ for $x > 0$.

Solution: Note that we have equality at $\ln(1) = 1 - 1 = 0$.

We compute $(\ln(x))' = 1/x, (\ln(x))'' = -1/x^2$, for $x > 0$. Then the tangent line to $\ln(x)$ at $x = 1$ is $y = x - 1$. Since $(\ln(x))'' = -1/x^2 < 0$ for all $x > 0$, the function is concave down on this interval. Therefore, the graph $y = \ln(x)$ lies below the tangent line by the concavity test, so $\ln(x) \leq x - 1$.

4. Let

$$g(x) = \begin{cases} e^{-1/x}, & x > 0 \\ 0, & x \leq 0 \end{cases} \tag{1}$$

Show that g is differentiable and $g'(0) = 0$.

Solution: For $x > 0$, g is obtained as a composition of differentiable functions, and therefore is differentiable by the chain rule. For $x < 0$, g is constant, so is differentiable with derivative 0. So we need only show that $h'(0) = 0$. We have

$$\lim_{h \to 0^-} \frac{g(h) - g(0)}{h} = \lim_{h \to 0^-} \frac{0 - 0}{h} = 0.$$

$$\lim_{h \to 0^+} g(h) = \lim_{h \to 0^+} \frac{e^{-1/h} - 0}{h}.$$

This second limit is indeterminate of the form $0/0$. Let $u = 1/h$, then

$$\lim_{h \to 0^+} \frac{e^{-1/h}}{h} = \lim_{u \to \infty} \frac{e^{-u}}{1/u} = \lim_{u \to \infty} \frac{u}{e^u}.$$

This limit is indeterminate of the form ∞/∞, so we compute

$$\lim_{u \to \infty} \frac{du/du}{d(e^u)/du} = \lim_{u \to \infty} \frac{1}{e^u} = 1/\infty = 0 = \lim_{h \to 0^+} \frac{g(h) - g(0)}{h}$$

by l'Hospital's rule.

Thus, both limits agree, so we have $h'(0) = 0$.

5. Bismuth-210 has a half-life of 5.0 days. A sample of Bismuth has a mass of $128mg$.

   (a) Find a formula for the mass remaining after t days.
   (b) Find the mass remaining after 30 days.
   (c) When is the mass reduced to $1mg$?

Solution:

   (a) The mass is given by $M(t) = 128 * 2^{-t/5}$.
   (b) $M(30) = 128 * 2^{-30/5} = 128/2^6 = 2$.
   (c) $M(35) = 128 * 2^{-35/5} = 1$.

6. Find the maxima and minima of $y = x^3 - 3x + 1$ on the interval $[0, 3]$.

Solution: We compute the critical points of $x^3 - 3x + 1$ by computing the derivative and setting it equal to zero.

$$\frac{d}{dx}(x^3 - 3x + 1) = 3x^2 - 3 = 3(x - 1)(x + 1) = 0.$$

The solutions of this equation are ± 1. Then $y(1) = -1$, $y(-1) = 3$. Plugging in the endpoints, $y(0) = 1$, $y(3) = 19$. So the maximum is 19, and the minimum is -1, by the Closed Interval Method.

7. Find the intervals on which f is increasing and decreasing, find the intervals of concavity and the inflection points, for the function $f(x) = (x^2 + 4x + 5)e^{-x}$.

   Solution: Using the product rule, power rule, and chain rule, we have

$$f'(x) = (x^2 + 4x + 5)(-e^{-x}) + (2x + 4)e^{-x} = -(x^2 + 2x + 1)e^{-x} = -(x + 1)^2 e^{-x}.$$

   Then we have $f'(x) \leq 0$, with equality only if $x = -1$. Thus, $f(x)$ is decreasing for all x.

   To determine the concavity, we compute $f''(x) = -2(x+1)e^{-x} - (x+1)^2(-e^{-x}) = (x^2-1)e^{-x}$. Then since $e^{-x} > 0$ for all x, $f''(x) > 0$ for $x^2 - 1 > 0$, and $f''(x) < 0$ for $x^2 - 1 < 0$. So f is concave up for $|x| > 1$, and f is concave down for $|x| < 1$ by the Concavity Test. The inflection points are $x = \pm 1$ since the concavity changes at these points.

8. Find

$$\lim_{x \to 0} \frac{x^2 \sin(1/x)}{\sin(x)},$$

   or prove that the limit doesn't exist.

   Solution: We have $\lim_{x\to 0} \frac{x}{\sin(x)} = \lim_{x \to 0} 1/\frac{\sin(x)}{x} = 1$ by one of the limits from Chapter 1 (or one may use l'Hospital's rule). Also, $|x \sin(1/x)| \leq |x|$ for $x \neq 0$, and thus by the squeeze theorem we have

$$0 = \lim_{x\to 0} -|x| \leq \lim_{x \to 0} x \sin(1/x) \leq \lim_{x\to 0} |x| = 0.$$

   Thus, $\lim_{x\to 0} \frac{x}{\sin(x)} \lim_{x\to 0} x\sin(1/x) = 1 \times 0 = 0 = \lim_{x \to 0} \frac{x^2 \sin(1/x)}{\sin(x)}$ by the product formula for computing limits.