Midterm 1, Math 1A, section 1, Fall 2011 solutions

1. Find $f+g$, $f-g$, fg and f/g and their domains for the functions

$$f(x) = x+2, \quad g(x) = x^2-4.$$

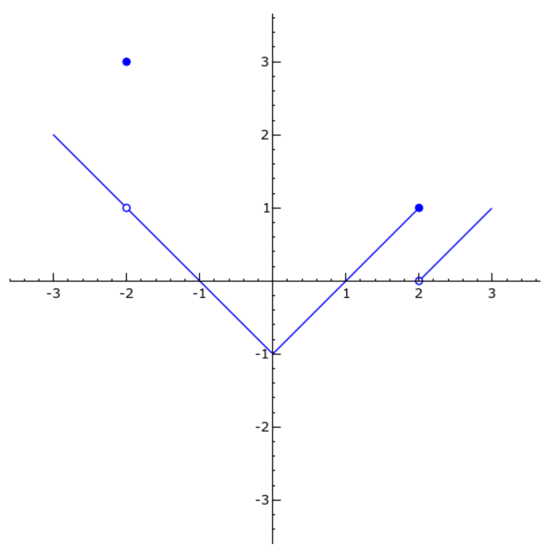
Solution: $f+g = x+2+x^2-4 = x^2+x-2$. Domain $=\mathbb{R}$, since this is a polynomial. $f-g = x+2-(x^2-4) = -x^2+x+6$. Domain $=\mathbb{R}$, since it is a polynomial $fg = (x+2)(x^2-4) = (x^3-4x)+(2x^2-8) = x^3+2x^2-4x-8$. Domain $=\mathbb{R}$, since it is a polynomial.
$f/g = \frac{x+2}{x^2-4}$. The domain is the set of real numbers x such that $x^2-4\neq 0$, since a rational function has domain all numbers at which the denominator does not vanish. Since $x^2-4 = (x-2)(x+2)$, we see that $x^2-4\neq 0$ when $x\neq 2$, $x\neq -2$. Thus the domain is the set of real numbers different from $2,-2$. One may simplify to $\frac{1}{x-2}$ by canceling the $x+2$ factor, as long as you restrict the domain to $\mathbb{R}-\{2,-2\}$.

2. Sketch the graph of an example of a function f that satisfies:

$$\lim_{x\to 2^+} f(x) = 0,\ \lim_{x\to 2^-} f(x) = 1,\ \lim_{x\to -2} f(x) = 1,\ f(2) = 1,\ f(-2) = 3, f(1) = 0, f'(1) = 1.$$

Explain whether f is left continuous and/or right continuous at $x = 1, 2$.

Solution:



The function is continuous at $x = 1$ since by hypothesis $f'(1) = 1$ exists, and therefore by Theorem 2.8.4, f is (both left and right) continuous at $x = 1$. We have $\lim_{x\to 2^-} f(x) = 1 = f(2)$, so f is continuous from the left at $x = 2$, but $\lim_{x\to 2^+} f(x) = 0 \neq 1 = f(2)$, so f is not continuous from the right at $x = 2$.

3. Evaluate

$$\lim_{x\to\infty}\left(\frac{1-2x^2}{3x+6x^2} - 2\right).$$

Solution: Since $x\to\infty$, we may assume $x > 0$, and thus we may divide by x^2 in the numerator and denominator to get an algebraically equivalent expression when $x > 0$, and use the sum, constant, product and quotient limit laws:

$$\lim_{x\to\infty}\left(\frac{1-2x^2}{3x+6x^2}-2\right) = -2+\lim_{x\to\infty}\frac{(1-2x^2)/x^2}{(3x+6x^2)/x^2} = -2+\lim_{x\to\infty}\frac{1/x^2-2}{3/x+6} = -2+\frac{(\lim_{x\to\infty}1/x)^2-2}{3(\lim_{x\to\infty}1/x)+6}.$$

We will verify that the denominator does not converge to 0, which will justify the use of the quotient law, and we plug in $\lim_{x\to\infty}1/x=0$ to obtain

$$\lim_{x\to\infty}\left(\frac{1-2x^2}{3x+6x^2}-2\right) = -2+\frac{0^2-2}{3\cdot 0+6} = -2+\frac{-2}{6} = -2-1/3 = -7/3.$$

4. Show that there exists a positive number c such that $\sin(c)=c^3-1$.

   Solution: Consider the function $g(x)=\sin(x)-(x^3-1)=\sin(x)+1-x^3$. Since $\sin(x)$ and x^3-1 are both continuous by Theorem 2.5.7 (polynomial and trig functions are continuous) and sums of continuous functions are continuous, $g(x)$ is continuous. Now $g(0)=\sin(0)-0^3+1=1>0$ and $g(2)=\sin(2)-(2)^3+1=-7+\sin(-2)\le -6<0$, by the intermediate value theorem, there is a number $c\in(0,2)$ such that $g(c)=0$. But then this c satisfies $\sin(c)=c^3-1$.

5. Use the ϵ-δ definition of the limit to prove that

$$\lim_{x\to -2}(-3x+1)=7.$$

   Solution: For any $\epsilon>0$, we want to find $\delta>0$ such that if $0<|x-(-2)|<\delta$, then $|(-3x+1)-7|<\epsilon$. We work backwards:

$$|(-3x+1)-7| = |-3x-6| = 3|x-(-2)|<\epsilon.$$

   Divide both sides of the inequality by 3 (which preserves the inequality) to get the equivalent inequality $|x-(-2)|<\epsilon/3 \Leftrightarrow |(-3x+1)-7|<\epsilon$. Thus, if we take $\delta=\epsilon/3$, we have if $0<|x-(-2)|<\delta=\epsilon/3 \implies |(-3x+1)-7|<\epsilon$.

6. Find the tangent line to the graph of $y=e^x/x$ at the point $(1,e)$. State the differentiation rules you are using.

   Solution:

   The domain of e^x/x is $x\ne 0$, since e^x is defined for all x, but we cannot divide by 0. Also, $(e^x)'=e^x$ by the exponential rule, and $x'=1$ by the power rule. We compute

$$\frac{dy}{dx}=(e^x/x)'=\frac{(e^x)'x-e^x(x)'}{x^2}=\frac{e^x x-e^x}{x^2}=e^x\frac{x-1}{x^2}$$

   by the quotient rule for $x\ne 0$.

   Plug in $\frac{dy}{dx}(1)=e^1\cdot\frac{1-1}{1^2}=0$, so the tangent line is horizontal. Plug into the point-slope formula for the tangent line, we obtain $y-e=0(x-1)=0$, so the equation of the tangent line is $y=e$.

7. Find the first and second derivatives of the function $f(x)=2e^x-x^3+2\sqrt{x}$. State which differentiation rules you are using and what the domains of f and its derivatives are.

Solution: The domain of f is function is $x \geq 0$, since this is the domain of $\sqrt{x} = x^{\frac{1}{2}}$, and the other components have domain $\mathbb{R}$. When we take a sum or product of functions, the domain of the result is the intersection of the domains of each component function.

We use the exponential, power, constant multiple, and sum rules for differentiating:

$$f'(x) = 2(e^x)' - (x^3)' + 2(x^{\frac{1}{2}})' = 2e^x - 3x^2 + 2(\frac{1}{2}x^{\frac{1}{2}-1}) = 2e^x - 3x^2 + x^{-\frac{1}{2}}.$$

This has domain $x > 0$, since $x^{-\frac{1}{2}}$ has this domain. We differentiate again to obtain:

$$f''(x) = (f'(x))' = (2e^x - 3x^2 + x^{-\frac{1}{2}})' = 2(e^x)' - 3(x^2)' + (x^{-\frac{1}{2}})',$$

by the sum and contant multiple rules. We then apply the exponential and power rules again for $x > 0$:

$$f''(x) = 2e^x - 3(2x) + (-\frac{1}{2}x^{-\frac{3}{2}}) = 2e^x - 6x - \frac{1}{2}x^{-\frac{3}{2}}.$$

This has domain $x > 0$ for similar reasons.