Sample Midterm 2 Solutions from 2009, Math 1A

1. Differentiate $e^{\cos(\ln(x))}$

   $(e^{\cos(\ln(x))})' = e^{\cos(\ln(x))}(\cos(\ln(x)))' = e^{\cos(\ln(x))} \cdot -\sin(\ln(x)) \cdot 1/x$ by two applications of the chain rule and formulae for derivatives of trig, exponential, and logarithmic functions.

2. Find the equation of the tangent line to the curve $y = \ln(x)/x$ at the point $(1, 0)$.

   $y' = 1/x^2 - \ln(x)/x^2 = (1 - \ln(x))/x^2$. $y'(1) = (1 - \ln(1))/1^2 = 1$, so the equation of the line is $y = 0 + (x - 1) = x - 1$.

3. Find $\frac{dy}{dx}$ by implicit differentiation if $e^{x/y} = x - y$.

   We implicitly differentiate:

   $$\frac{d}{dx}(e^{x/y}) = e^{x/y}(1/y - x/y^2\frac{dy}{dx}) = \frac{d}{dx}(x - y) = 1 - \frac{dy}{dx}.$$

   Solving for $\frac{dy}{dx}$ and multiplying both sides by $y \neq 0$, we get

   $$e^{x/y} - y = (e^{x/y}x/y - y)\frac{dy}{dx},$$

   so

   $$\frac{dy}{dx} = (e^{x/y} - y)/(e^{x/y}x/y - y).$$

4. The concentration y of a chemical at time t satisfies the equation $dy/dt = -.0005y$. Find a formula for y in terms of t given that $y = 1$ at time $t = 0$.

   The concentration satisfies the law of natural growth, so $y = y(0)e^{-.0005t} = e^{-.0005t}$.

5. Use differentials or a linear approximation to estimate $\sqrt{4.1}$.

   We compute the differential for $y = x^{\frac{1}{2}}$ about 4:

   $dy = \frac{1}{2}4^{-\frac{1}{2}}dx = \frac{1}{4}dx$. Use $dx = .1$, we get $dy = .25 \cdot .1 = .025$. So the approximation is $\sqrt{4.1} \approx y + dy = \sqrt{4} + dy = 2.025$.

6. Find the absolute maximum and absolute minimum values of $f(x) = x^3 - 12x$ on the interval $[-3, 5]$.

   We compute $f'(x) = 3x^2 - 12 = 3(x - 2)(x + 2)$. So f has critical points ± 2, both of which are in the interval $[-3, 5]$. We compute $f(-3) = -27 - 12(-3) = 9, f(-2) = -8 - 12(-2) = 16, f(2) = -16, f(5) = 5(25 - 12) = 65$. Thus, the absolute minimum of f on the interval $[-3, 5]$ is -16, and the absolute maximum is 65.

7. Show that the equation $e^x = -x$ has exactly one real root.

   Solution: We need to show that $e^x + x = 0$ has exactly one solution in x. Let $f(x) = e^x + x$. Then we compute $f'(x) = e^x + 1 > 1$ for all x. We see that $f(-1) = e^{-1} - 1 < 0$, and $f(0) = e^0 - 0 = 1$. By the IVT since f is differentiable and therefore continuous on the interval $[-1, 0]$, there is $-1 < c < 0$ such that $f(c) = 0$. There can be only one real root by Rolle's theorem, since if there were $b \neq c$ such that $f(b) = 0$, then there would be an a between b and c with $f'(a) = 0$, but $f'(a) > 1$, a contradiction.

8. Find the limit $\lim_{x\to 0} \frac{\tan(x)-x}{x^3}$.

Solution: We see that $\lim_{x\to 0} \tan(x) - x = \tan(0) - 0 = 0$ by DSP for continuous functions, and $\lim_{x\to 0} x^3 = 0$. So the limit is indeterminate of type $\frac{0}{0}$.

$\lim_{x\to 0} \frac{(\tan(x)-x)'}{(x^3)'} = \lim_{x\to 0} \frac{\sec^2(x)-1}{3x^2}$. The numerator and denominator still have limit 0 by the DSP, so this is indeterminate of type $\frac{0}{0}$, with denominator non-zero for $x \neq 0$.

$\lim_{x\to 0} \frac{(\sec^2(x)-1)'}{(3x^2)'} = \lim_{x\to 0} \frac{2\sec^2(x)\tan(x)}{6x}$. The numerator and denominator still have limit 0 by the DSP, so this is indeterminate of type $\frac{0}{0}$, with denominator non-zero for $x \neq 0$.

$\lim_{x\to 0} \frac{2\sec^2(x)\tan(x)}{6x} = \lim_{x\to 0} \frac{\sin(x)}{x} \frac{1}{3\cos^3(x)} = 1 \cdot \frac{1}{3\cos^3(0)} = \frac{1}{3}$ using the fact that $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$ and limit laws.

Thus, we conclude by two applications of l'Hospital's rule that $\lim_{x\to 0} \frac{\tan(x)-x}{x^3} = \lim_{x\to 0} \frac{2\sec^2(x)\tan(x)}{6x} = \frac{1}{3}$.

9. Sketch the curve $y = (x^3 - x)^{\frac{1}{3}}$.

Solution: The domain of this function is all x, since it is composed of functions whose domain is all x. The function is an odd function, so we need only analyze the behavior for $x \geq 0$. The x-intercepts are obtained by setting $x^3 - x = 0$, so $x = 0, 1, -1$, and the y-intercept is 0.

We compute $y' = \frac{1}{3}(x^3 - x)^{-\frac{2}{3}}(3x^2 - 1) = \frac{x^2 - \frac{1}{3}}{(x^3-x)^{\frac{2}{3}}}$ using the chain rule.

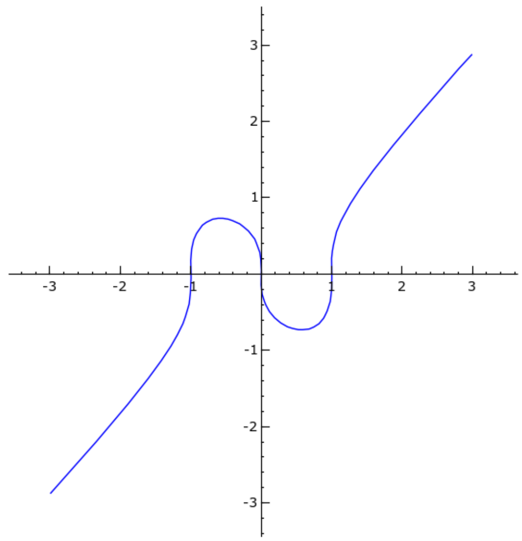
This is undefined when the denominator $= 0$ (since the numerator is non-zero), which is when $x^3 - x = x(x^2 - 1) = 0$, so at $x = 0, 1, -1$. However, the function y is defined and continuous at these points.

We also have $y' = 0$ when the numerator is zero, so $x^2 - \frac{1}{3} = 0$, or $x^2 = \frac{1}{3}$, so $x = \pm 1/\sqrt{3}$.

So the critical points are $x = 0, \pm 1, \pm 1/\sqrt{3}$.

The denominator of y' is > 0, since it is a square, and therefore the sign of y' is determined by the numerator. So we see that $y'(x) > 0$ when $x^2 - \frac{1}{3} > 0$, so for $|x| > 1/\sqrt{3}$, and $y'(x) < 0$ when $x^2 - \frac{1}{3} < 0$, so for $|x| < 1/\sqrt{3}$. So by the I/D test, y is increasing for $|x| > 1/\sqrt{3}$, and is decreasing for $|x| < 1/\sqrt{3}$. By the first derivative test, we conclude that $1/\sqrt{3}$ is a local minimum, and $-1/\sqrt{3}$ is a local maximum (since y is odd).

We also have $\lim_{x\to\infty} (x^3 - x)^{\frac{1}{3}} = \infty$. There is a slant asymptote $y = (x^3 - x)^{\frac{1}{3}} \sim (x^3)^{\frac{1}{3}} = x$, so the graph behaves like $y = x$ for x large.



10. Sketch the curve $y = x^{\frac{1}{x}}$, for $x > 0$.

    Solution: The domain of the function is given to us, $x > 0$.

    We use logarithmic differentiation to compute $y' = y(\ln y)' = y(\ln(x^{1/x}))' = y(1/x \ln(x))' = y(-1/x^2 \ln(x) + 1/x \cdot 1/x) = x^{1/x}(1 - \ln(x))/x^2$.

    This is defined for all $x > 0$. We compute $y' = 0$ when $1 - \ln(x) = 0$, so for $\ln(x) = 1$, or $x = e$. Thus, the only critical point of y is $x = e$.

    We also see that $y' > 0$ when $1 - \ln(x) > 0$, so for $x < e$, and $y' < 0$ when $x > e$. By the I/D test, y is increasing for $0 < x < e$, and is decreasing for $e < x$. By the first derivative test, $x = e$ is a local maximum of y.

    We also have $\lim\limits_{x\to\infty} x^{1/x}$ is indeterminate of type ∞^0. We convert this to $\lim\limits_{x\to\infty} e^{\ln(x)/x}$, whose exponent is indeterminate of type ∞/∞. Thus $\lim\limits_{x\to\infty} (\ln(x))'/x' = \lim\limits_{x\to\infty} 1/x = 0$, so by l'Hospital's rule, $\lim\limits_{x\to\infty} \ln(x)/x = 0$, and $\lim\limits_{x\to\infty} x^{1/x} = e^0 = 1$. So y has a horizontal asymptote as $x \to \infty$.

    We have $\lim\limits_{x\to 0^+} x^{1/x} = 0^\infty = 0$.