Sample Final Exam 1

You are allowed one 8.5×11 sheet of notes with writing on both sides. This sheet must be turned in with your exam. *Calculators are not allowed.*

0. (1 point) write your name, section number, and GSI's name on your exam.

1. (3 points) give precise definitions of the following statements:
(a) $\displaystyle\lim_{x \to 3^-} f(x) = 17.$ (δ-ϵ definition)
(b) $f(x)$ is continuous at x_0
(c) $f(x)$ has an absolute maximum at x_0 over the interval $[a, b]$

Answer: (a) For every $\epsilon > 0$ there exists a $\delta > 0$ such that:
if $0 < 3 - x < \delta$ then $|f(x) - 17| < \epsilon$
(b) $f(x_0)$ is defined and $\lim_{x \to x_0} f(x) = f(x_0)$
(c) For every x, if $a \le x \le b$ then $f(x) \le f(x_0)$
(Assuming that $f(x)$ is defined on $[a, b]$ and x_0 is in $[a, b]$)

2. (5 points) Evaluate the integral: $\displaystyle\int_0^{\sinh^{-1}(4/3)} e^{\cosh x} \sinh x\, dx$

Answer:
$u = \cosh x$,
$du = \sinh x\, dx$,
$\cosh(0) = 1$,
$\cosh(\sinh^{-1}(4/3)) = \sqrt{1 + \sinh^2(\sinh^{-1}(4/3))} = \sqrt{1 + (4/3)^2} = \sqrt{25/9} = 5/3$

$$\int_0^{\sinh^{-1}(4/3)} e^{\cosh x} \sinh x\, dx = \int_1^{5/3} e^u\, du = e^u \Big]_1^{5/3} = e^{5/3} - e$$

3. (6 points) Let $f(x) = \begin{cases} \dfrac{\sin x}{x} & x \neq 0, \\ 1 & x = 0. \end{cases}$

(a) Use the definition of the derivative to evaluate $f'(0)$.
(b) compute $f'(x)$ for $x \neq 0$.
(c) show that $f'(x)$ is continuous at $x = 0$.

Answer: (a)

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{\frac{\sin x}{x} - 1}{x} = \lim_{x\to 0} \frac{\sin x - x}{x^2} =^1 \lim_{x\to 0} \frac{\cos x - 1}{2x} =^1 \lim_{x\to 0} \frac{-\sin x}{2} = 0$$

(^{1}: these two equalities are both by l'Hopital's rule, since bottom and top both $\to 0$ as $x \to 0$)
(b) For $x \neq 0$, $f'(x) = \frac{x\cos x - \sin x}{x^2}$ (Using the quotient rule)
(c) We need to show that $\lim_{x\to 0} f'(x) = f'(0)$, so:

$$\lim_{x\to 0} f'(x) = \lim_{x\to 0} \frac{x\cos x - \sin x}{x^2} =^1 \lim_{x\to 0} \frac{(-x\sin x + \cos x) - \cos x}{2x} = \lim_{x\to 0} \frac{-\sin x}{2} = 0 = f'(0)$$

(^{1}: Again, by l'Hopital's rule)

4. (5 points) Let $f(x) = \frac{x}{\sqrt{1+x^2}}$. Explain what happens if you try to use Newton's method to solve $f(x) = 0$ with a starting guess $x_0 = 1$.

Answer:

$$f'(x) = \frac{\sqrt{1+x^2} - x(2x)(\frac{1}{2\sqrt{1+x^2}})}{1+x^2} = \frac{1 + x^2 - x(2x)(\frac{1}{2})}{(1+x^2)^{3/2}} = \frac{1}{(1+x^2)^{3/2}}$$

$f(1) = \frac{1}{\sqrt{2}}$, $f'(1) = \frac{1}{2\sqrt{2}}$, so:

$$x_1 = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{\frac{1}{\sqrt{2}}}{\frac{1}{2\sqrt{2}}} = 1 - 2 = -1$$

But then, $f(-1) = \frac{-1}{\sqrt{2}}$, $f'(-1) = \frac{1}{2\sqrt{2}}$, so:

$$x_2 = -1 - \frac{f(-1)}{f'(-1)} = -1 - \frac{\frac{-1}{\sqrt{2}}}{\frac{1}{2\sqrt{2}}} = -1 + 2 = 1$$

so $x_1 = -1,\ x_2 = 1,\ x_3 = -1,\ x_4 = 1$, etc
Newton's Method does not converge on an answer, if you start with $x_0 = 1$

5. (5 points) Compute the volume of the solid obtained by revolving the region between the parabola $y = (x-2)^2$ and the line $y = x$ about the y-axis.

Answer: First, we find the points of intersection:
$x = (x-2)^2, x = x^2 - 4x + 4, 0 = x^2 - 5x + 4 = (x-4)(x-1)$
So $x = 1$ and $x = 4$ are the points of intersection.
We now do the integral by shells:

$$\int_1^4 2\pi x(x - (x-2)^2)\, dx = \int_1^4 2\pi(-x^3 + 5x^2 - 4x)\, dx = 2\pi(-\frac{1}{4}x^4 + \frac{5}{3}x^3 - 2x^2)\Big]_1^4 =$$

$$2\pi\left(\left(-\frac{1}{4}4^4 + \frac{5}{3}4^3 - 2(4^2)\right) - \left(-\frac{1}{4} + \frac{5}{3} - 2\right)\right)$$

you can leave the answer like this since the algebra is messy... (it evaluates to $45\pi/2$)

6. (3 points) Suppose $g(y)$ is defined to be the value of x such that $x^5 + x + 1 = y$. For example, $g(-1) = -1$, $g(1) = 0$, $g(3) = 1$ and $g(35) = 2$. Evaluate $(g^{-1})'(1)$.

Answer:
let $f(x) = x^5 + x + 1$ then $g = f^{-1}$ and so $f = g^{-1}$ so $(g^{-1})'(1) = f'(1)$
$f'(x) = 5x^4 + 1$ so
$f'(1) = 5 + 1 = 6$

7. (5 points) An airplane flies from New York to Los Angeles, a 3000 mile trip. It costs \$2500 per hour to pay the crew and use the plane. The cost of fuel (in cents per mile) is equal to the velocity of the plane (in miles per hour). For example, it costs \$6 per mile to travel 600 miles/hour. How fast should the plane fly to minimize the cost of the trip?

Answer:
Cost $= 3000(.01)(\text{velocity}) + 2500(\text{time})$
time $= 3000/\text{velocity}$
$C(v) = 30v + 2500(3000/v)$
$C'(v) = 30 - 2500(3000/v^2)$
Setting $C'(v) = 0$ gives:
$30 = 2500(3000/v^2), \quad v^2 = 2500(100), \quad v = 500$

Note that $C'' > 0$ for $v > 0$, so C' is an increasing function. Thus, $v = 500$ is the only critical point (with $v > 0$) and C has a local and global minimum there.

8. (5 points) Compute the derivative: $\quad \dfrac{d}{dx}\displaystyle\int_{-x}^{\sqrt{\ln x}} e^{-t^2}\,dt, \qquad (x > 1)$

Answer:
let $F(x)$ be an antiderivative of e^{-x^2}. (e.g. let $F(x) = \int_0^x e^{-t^2}\,dt$.)
Then

$$\begin{aligned}
\frac{d}{dx}\left[F(\sqrt{\ln x}) - F(-x)\right] &= F'(\sqrt{\ln x})\frac{1}{2}(\ln x)^{-1/2}\left(\frac{1}{x}\right) - F'(-x)(-1) \\
&= \frac{F'(\sqrt{\ln x})}{2x\sqrt{\ln x}} + F'(-x) = \frac{e^{-(\sqrt{\ln x})^2}}{2x\sqrt{\ln x}} + e^{-(-x)^2} \\
&= \frac{1}{2x^2\sqrt{\ln x}} + e^{-x^2}
\end{aligned}$$

9. (3 points) Evaluate the limit by interpreting it as an integral:

$$\lim_{n\to\infty}\left[\left(1+\frac{2}{n}\right)^9\frac{2}{n} + \left(1+\frac{4}{n}\right)^9\frac{2}{n} + \left(1+\frac{6}{n}\right)^9\frac{2}{n} + \cdots + \left(1+\frac{2n}{n}\right)^9\frac{2}{n}\right].$$

Solution: first write the sum using sigma notation:

$$\text{answer} = \lim_{n\to\infty}\sum_{i=1}^{n}\left(1+i\frac{2}{n}\right)^9\frac{2}{n} \tag{1}$$

The generic definite integral (using sample points $x_i^* = x_i$) looks like

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\Delta x, \qquad \left(\Delta x = \frac{b-a}{n},\ x_i = a + i\Delta x\right). \tag{2}$$

so we interpret (1) to be of the form (2) with $a = 1$, $b = 3$, $f(x) = x^9$. Hence,

$$\text{answer} = \int_1^3 x^9\,dx = \left.\frac{x^{10}}{10}\right]_1^3 = \frac{1}{10}(3^{10} - 1).$$

Note: you could also use $a = 0$, $b = 2$, $f(x) = (1+x)^9$ to get the same result.