**Another Sample Midterm 2**

1. (1 point) write your name, section number, and GSI's name on your exam and write your name on your sheet of notes.

2. (3 points) Suppose $f$ is twice differentiable on the interval $[0, 4]$ and satisfies

$$f'(0) = 1 \qquad f'(1) = 0 \qquad f'(2) = 0 \qquad f'(3) = -1 \qquad f'(4) = 0$$
$$f''(0) = -1 \qquad f''(1) = -2 \qquad f''(2) = 0 \qquad f''(3) = 1 \qquad f''(4) = 1$$

At the endpoints $x = 0$ and $x = 4$, these are one-sided derivatives. Fill in the following table with YES, NO, or CBT (cannot be determined).

| $c =$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $f$ has a local max at $c$ | No | Yes, $f''(1) < 0$ | CBT | No, $f'(3) \neq 0$ | No |
| $f$ has a local min at $c$ | Yes, $f'(0) > 0$ | No | CBT | No, $f'(3) \neq 0$ | Yes, $f''(4) > 0$ |

3. (5 points) Let $f(x) = x^x$. Compute $f'(2)$, $f'(4)$ and $(f \circ f)'(2)$. Note that $4^4 = 256$.

$$y = f(x) = x^x \implies \qquad \ln(y) = \ln(x^x) \implies \qquad \ln(y) = x\ln(x)$$
$$\frac{1}{y}\frac{dy}{dx} = \ln(x) + \frac{x}{x} \implies \qquad \frac{1}{y}\frac{dy}{dx} = \ln(x) + 1 \implies \qquad \frac{dy}{dx} = y(\ln(x) + 1)$$
$$\implies f'(x) = x^x(\ln(x) + 1)$$

$$(f \circ f)' = (f(f(x)))' = f'(f(x)) \cdot f'(x)$$

$$f'(2) = 2^2(\ln(2) + 1) = \boxed{4(\ln(2) + 1)} \qquad f'(4) = 4^4(\ln(4) + 1) = \boxed{256(\ln(4) + 1)}$$
$$(f \circ f)'(2) = f'(f(2)) \cdot f'(2) = f'(4) \cdot f'(2) = \boxed{1024(\ln(2) + 1)(\ln(4) + 1)}$$

4. (5 points) Use a linear approximation to estimate: $\frac{1}{\pi}\tan^{-1}\left(1+\frac{\pi}{100}\right)$.

$$f(a+h) \approx f(a) + h \cdot f'(a)$$

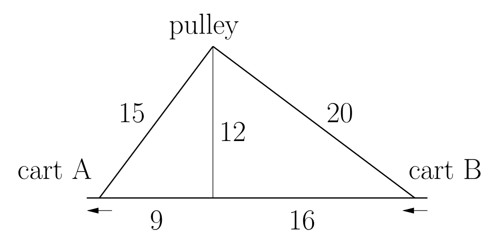
$$\boxed{f(x) = \frac{1}{\pi}\tan^{-1}(x), \quad a = 1, \quad h = \frac{\pi}{100}}$$

$$\tan(\frac{\pi}{4}) = 1 \quad \Longrightarrow \quad \tan^{-1}(1) = \frac{\pi}{4} \quad \Longrightarrow \quad f(1) = \frac{1}{4}$$

$$f'(x) = \frac{1}{\pi}\frac{1}{1+x^2} \quad \Longrightarrow \quad f'(1) = \frac{1}{2\pi}$$

$$\boxed{f\left(1+\frac{\pi}{100}\right) \approx \frac{1}{4} + \frac{\pi}{100}\cdot\frac{1}{2\pi} = \frac{51}{200}}$$

5. (6 points) Two carts are connected by a 35 foot rope that passes over a pulley 12 feet above the floor. Cart A is being pulled to the left at a speed of 2 ft/sec. How fast is cart B moving at the instant cart A is 9 feet from the point on the floor beneath the pulley?




Let $x_A, x_B$ be the distance that carts A and B are from the point under the pulley.

Let $L_A, L_B$ be the length of rope on from the pulley to cart A and B respectively.

$$x_A^2 + 12^2 = L_A^2 \qquad x_B^2 + 12^2 = L_B^2 \qquad L_A + L_B = 35$$

$$2x_A\frac{dx_A}{dt} = 2L_A\frac{dL_A}{dt} \qquad 2x_B\frac{dx_B}{dt} = 2L_B\frac{dL_B}{dt} \qquad \frac{dL_A}{dt} + \frac{dL_B}{dt} = 0$$

$$\frac{x_A\frac{dx_A}{dt}}{L_A} = \frac{dL_A}{dt} \qquad \frac{x_B\frac{dx_B}{dt}}{L_B} = \frac{dL_B}{dt} \qquad \frac{dL_A}{dt} = -\frac{dL_B}{dt}$$

$$\Longrightarrow \quad \frac{x_A\frac{dx_A}{dt}}{L_A} = -\frac{x_B\frac{dx_B}{dt}}{L_B}$$

Plugging in the numbers from the problem

$$\frac{9\cdot 2}{15} = -\frac{16\frac{dx_B}{dt}}{20} \quad \Longrightarrow \quad \boxed{\frac{dx_B}{dt} = -1.5 \text{ ft/sec}}$$

6. (5 points) Show that there is exactly one $x \in \mathbb{R}$ satisfying

$$x^5 + e^x - 2 = 0.$$

Let $f(x) = x^5 + e^x - 2$. $f(0) = 0^5 + e^0 - 2 = -1 < 0$, $f(2) = 2^5 + e^2 - 2 > 2^5 - 2 > 0$

Since $f(x)$ is continuous on $[0, 2]$ and $f(0) < 0 < f(2)$ by the intermediate value theorem there is some $c \in (0, 2)$ so that $f(c) = 0$. Hence there is at least one $x$ satisfying the equation.

Now suppose that $f(a) = f(b), a \neq b$. Since $f(x)$ is continuous and differentiable on $\mathbb{R}$ it is continuous on $[a, b]$ and differentiable on $(a, b)$ so by Rolle's theorem there is some $c \in (a, b)$ with $f'(c) = 0$.

But this is impossible since $f'(x) = 5x^4 + e^x > 0$ for every $x$. Hence $f$ is one-to-one and there is exactly one solution.

7. (5 points) Do *one* of the following:

(a) Show that

$$\tanh(\sinh^{-1} x) = \frac{x}{\sqrt{1 + x^2}} \qquad (x \in \mathbb{R}).$$

$\tanh(\sinh^{-1} x) = \frac{\sinh(\sinh^{-1} x)}{\cosh(\sinh^{-1} x)} = \frac{x}{\cosh(\sinh^{-1} x)}$

Let $y = \sinh^{-1}(x) \implies \sinh(y) = x$ we want to simplify $\cosh(y)$.

Use identity $\cosh^2(y) - \sinh^2(y) = 1$ to get $\cosh^2(y) = 1 + x^2$

Hence $\tanh(\sinh^{-1} x) = \frac{x}{\sqrt{1+x^2}}$

(b) If $g(x) = 1 + x + e^x$, find $g^{-1}(2)$ and $(g^{-1})'(2)$.

$g(0) = 1 + 0 + e^0 = 2$ so $g^{-1}(2) = 0$.

$$(g^{-1})'(2) = \frac{1}{g'(g^{-1}(2))}$$

$$g'(x) = 1 + e^x$$

$$(g^{-1})'(2) = \frac{1}{1 + e^0} = \boxed{\frac{1}{2}}$$