Math 1A, Spring 2008, Wilkening

Sample Midterm 2

You are allowed one 8.5×11 sheet of notes with writing on both sides. This sheet must be turned in with your exam. *Calculators are not allowed.*

1. (1 point) write your name, section number, and GSI's name on your exam and write your name on your sheet of notes.

2. (4 points) Find the equation of the tangent line to the curve $y^2 = x^3 + 3x^2$ at the point $(1, -2)$.

Answer:

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(x^3 + 3x^2)$$

$$2y\frac{dy}{dx} = 3x^2 + 6x$$

$$2(-2)\frac{dy}{dx} = 3(1)^2 + 6(1) = 9$$

$$\frac{dy}{dx} = -\frac{9}{4}$$

so the equation of the tangent line is $y + 2 = -\frac{9}{4}(x - 1)$.

3. (5 points) Find the relative maxima, minima and inflection points of the function

$$f(x) = xe^{-x^2/2}$$

Answer:

$$f'(x) = x(-x)e^{-x^2/2} + e^{-x^2/2} = (1 - x^2)e^{-x^2/2}$$

$f'(x) = 0$ if $x = \pm 1$. (because $(1 - x^2) = 0$ if $x = \pm 1$)
$f'(x) > 0$ if $-1 < x < 1$. (because $(1 - x^2) > 0$ if $-1 < x < 1$.)
$f'(x) < 0$ if $x < -1$ or $x > 1$. (because $(1 - x^2) < 0$ if $x < -1$ or $x > 1$.)
(note: $e^{-x^2/2}$ is always > 0)
so f has a local minimum at $x = -1$ and a local maximum at $x = 1$

$$f''(x) = (1 - x^2)(-x)e^{-x^2/2} + (-2x)e^{-x^2/2} = (x^3 - 3x)e^{-x^2/2}$$

$f''(x) = 0$ if $x = 0$ or $x = \pm\sqrt{3}$
$f''(x) > 0$ if $-\sqrt{3} < x < 0$ or $x > \sqrt{3}$
$f''(x) < 0$ if $x < -\sqrt{3}$ or $0 < x < \sqrt{3}$
so $x = -\sqrt{3}$, $x = 0$, and $x = \sqrt{3}$ are all inflection points of f

4. (5 points) Find the function $u(t)$ that satisfies

$$\frac{du}{dt} = -3(u-5), \qquad u(0) = 1$$

and evaluate $u(\ln 2)$.

Answer:
Let $v = u - 5$. Then $\frac{dv}{dt} = \frac{d}{dt}(u-5) = \frac{du}{dt} - 0 = -3(u-5) = -3v$ and $v(0) = -4$.

Then, $v(t) = -4e^{-3t}$ so $u(t) = -4e^{-3t} + 5$ so

$$u(\ln 2) = -4e^{-3\ln 2} + 5 = -4(e^{\ln 2})^{-3} + 5 = -4(2)^{-3} + 5 = \frac{9}{2}$$

5. (5 points) Let $f(x) = \sqrt{4+x}$. Find the linearization L of f at 0 and use the mean value theorem to show that $f(x) < L(x)$ for $x > 0$.

Answer:
$f(x) = \sqrt{4+x} = (4+x)^{1/2}$
$f(0) = \sqrt{4+0} = 2$
$f'(x) = \frac{1}{2}(4+x)^{-1/2}$
$f'(0) = \frac{1}{2}(4+0)^{-1/2} = \frac{1}{2}(\frac{1}{2}) = \frac{1}{4}$

$L(x) = f(0) + f'(0)(x-0) = 2 + \dfrac{1}{4}x$
If $x > 0$ then by the mean value theorem, there is some c in $(0, x)$ so that

$$f'(c) = \frac{f(x) - f(0)}{x - 0}$$

But if $c > 0$ then $f'(c) = \frac{1}{2\sqrt{4+c}} < \frac{1}{4}$ so $\frac{f(x)-f(0)}{x-0} < \frac{1}{4}$ so $f(x) - f(0) < \frac{1}{4}x$ so $f(x) < \frac{1}{4}x + f(0) = \frac{1}{4}x + 2 = L(x)$

6. (5 points) Evaluate the limit

$$\lim_{x\to\infty} \frac{\tanh x - 1}{\tan^{-1} x - \pi/2}$$

Answer:

$$\lim_{x\to\infty} \tanh x - 1 = \lim_{x\to\infty} \frac{\sinh x}{\cosh x} - 1 = \lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} - 1 = \lim_{x\to\infty} \frac{1 - e^{-2x}}{1 + e^{-2x}} - 1 = 1 - 1 = 0$$

$$\lim_{x\to\infty} \tan^{-1} x - \pi/2 = \pi/2 - \pi/2 = 0$$

so we can use l'Hospital's rule to get:

$$\lim_{x\to\infty} \frac{\tanh x - 1}{\tan^{-1} x - \pi/2} = \lim_{x\to\infty} \frac{\text{sech}^2 x}{\frac{1}{1+x^2}} = \lim_{x\to\infty} \frac{\frac{1}{\cosh^2 x}}{\frac{1}{1+x^2}} = \lim_{x\to\infty} \frac{1+x^2}{\cosh^2 x}$$

Again,

$$\lim_{x\to\infty} 1 + x^2 = \infty$$

and

$$\lim_{x\to\infty} \cosh^2 x = \lim_{x\to\infty} \left(\frac{e^x + e^{-x}}{2}\right)^2 = \left(\frac{\infty + 0}{2}\right)^2 = \infty$$

So we can use l'Hospital's rule again to get

$$\lim_{x\to\infty} \frac{1+x^2}{\cosh^2 x} = \lim_{x\to\infty} \frac{2x}{2(\cosh x)(\sinh x)} = \lim_{x\to\infty} \frac{2}{2(\cosh^2 x + \sinh^2 x)} = \frac{2}{\infty} = 0$$

(note: We used l'Hospital's rule twice and the product rule here. Also, $\cosh^2 x$ and $\sinh^2 x$ both approach ∞ as $x \to \infty$.)