Math 282b Homework # 1, due 2/2/01

1. Let $\Sigma_{g,k}$ denote a genus g surface with k points removed. Compute $H_*(\Sigma_{g,k})$ for $k > 0$.

2. Let X be a topological space, let $\phi : X \to X$ be a homeomorphism, and let ϕ_k denote the induced map on $H_k(X)$. Define the *mapping torus*

$$Y = X \times [0,1]/(x,1) \sim (\phi(x),0).$$

Show that there is a short exact sequence

$$0 \to \mathrm{Coker}(1-\phi_k) \to H_k(Y) \to \mathrm{Ker}(1-\phi_{k-1}) \to 0.$$

3. Show that singular homology defined using simplices is naturally isomorphic to singular homology defined using cubes[1].

4. (a) Show that the Grassmannian $G(k,n)$, consisting of k-dimensional subspaces of $\mathbb{R}^n$, is a manifold.

   (b) Show that for $S \in G(k,n)$ there is a canonical isomorphism

$$T_S G(k,n) = \mathrm{Hom}(S, \mathbb{R}^n/S).$$

[1] The cubical singular chain complex $(C_*^{\square}(X), \partial)$ of a topological space is defined as follows. For $i = 1, \ldots, n$ define face maps $F_{i,n}^{\pm} : I^{n-1} \to I^n$ by

$$F_{i,n}^{-}(x_1, \ldots, x_{n-1}) = (x_1, \ldots, x_{i-1}, 0, x_i, \ldots, x_{n-1}),$$
$$F_{i,n}^{+}(x_1, \ldots, x_{n-1}) = (x_1, \ldots, x_{i-1}, 1, x_i, \ldots, x_{n-1}).$$

Let $C_n^{\square}(X)$ denote the free $\mathbb{Z}$-module generated by continuous maps $I^n \to X$, modulo constant maps when $n > 0$. For $\sigma : I^n \to X$, define

$$\partial\sigma = \sum_{i=1}^{n} \left(\sigma \circ F_{i,n}^{+} - \sigma \circ F_{i,n}^{-}\right) \in C_{n-1}^{\square}(X).$$