Math 215a Homework #3, Due Wednesday 10/17 at 1:10 PM

1. (a) Show that chain homotopy of chain maps is an equivalence relation.

   (b) Show that composition of chain maps induces a well-defined map on equivalence classes.

2. Hatcher, section 2.1, exercise 8.

3. Let
$$0 \longrightarrow V_1 \longrightarrow V_2 \longrightarrow \cdots \longrightarrow V_n \longrightarrow 0$$
be an exact sequence of finite dimensional vector spaces over a field F. Show that $\sum_{i=1}^n (-1)^i \dim(V_i) = 0$.

4. Show that if
$$0 \longrightarrow A \longrightarrow B \longrightarrow \mathbb{Z}^k \longrightarrow 0$$
is exact, then $B \simeq A \oplus \mathbb{Z}^k$.

For the next three problems, use the Mayer-Vietoris sequence.

5. Compute the homology of the space obtained by taking three copies of D^n and identifying their boundaries with each other.

6. Compute the homology of the nonorientable surface obtained by taking the connect sum[1] of a genus g surface with a projective plane.

7. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2 \mathbb{Z}$; this induces a self-homeomorphism ϕ_A of $\mathbb{R}^2/\mathbb{Z}^2 = S^1 \times S^1$. Let Y be the 3-manifold[2] obtained by taking two copies of $S^1 \times D^2$ and identifying the boundary tori via ϕ_A. Compute the homology of Y, in terms of A.

[1] The *connect sum* of two surfaces is defined by cutting an open disc out of each surface, and then gluing the boundary circles together.

[2] For example, if A is the identity matrix, then $Y \simeq S^1 \times S^2$; if $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ then $Y \simeq S^3$. Can you see why?