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Reidemeister torsion in generalized Morse theory

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Abstract. In two previous papers with Yi-Jen Lee, we defined and computed a notion of Reidemeister torsion for the Morse theory of closed 1-forms on a finite dimensional manifold. The present paper gives an a priori proof that this Morse theory invariant is a topological invariant. It is hoped that this will provide a model for possible generalizations to Floer theory.

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In two papers with Yi-Jen Lee [HL1, HL2], we defined a notion of Reidemeister torsion for the Morse theory of closed 1-forms on a finite dimensional manifold. We consider the flow dual to the 1-form via an auxiliary metric. Our invariant, which we call I, multiplies the algebraic Reidemeister torsion of the Novikov complex, which counts flow lines between critical points, by a zeta function which counts closed orbits of the flow. For a closed 1-form in a real multiple of an integral cohomology class, i.e. d of a circle-valued function, we proved in the above papers that I equals a form of topological Reidemeister torsion due to Turaev. This implies *a posteriori* that I is invariant under homotopy of the circle-valued function and the auxiliary metric.

In this paper we reprove these results using an opposite approach: we first prove *a priori* that I is a topological invariant, depending only on the cohomology class of the closed 1-form. We then deduce that I agrees with Turaev torsion, by using invariance to reduce to the easier case of an exact 1-form. This approach has two advantages. First, it works for closed 1-forms in an arbitrary cohomology class, thus extending the results of our previous papers. Second, and perhaps more importantly, the proof of invariance here should provide a model for the possible construction of torsion in Floer theory.

The contents of this paper are as follows. In §1 we review the definition of the invariant I, state the main results, and outline the proofs. In §2 and §3 we prove that I is invariant. The strategy is to study how I varies in a generic one parameter family of 1-forms and metrics. In §2, we prepare for this analysis by classifying the bifurcations that generically occur, and we also deal with the complication that infinitely many bifurcations may occur in a finite time. The heart of the paper is in §3, where we analyze what happens in each individual bifurcation. While the torsion of the Novikov complex and the zeta function can change, we will see that their product I does not. In §4 we use topological invariance to give a quick proof that I agrees with Turaev torsion. In §5 we discuss open questions and possible generalizations. Appendix A reviews algebraic aspects of Reidemeister torsion that are used throughout the paper. Appendix B reviews how to remove a certain ambiguity in Reidemeister torsion using Turaev's Euler structures.

1 The invariant I

We begin by reviewing the definition of the invariant I from [HL2]. We will emphasize geometric aspects which are important for the present paper, and we generalize [HL2] slightly by allowing different abelian covers in Choice 1.2. After defining I, we will state the main results and outline the proofs.

1.1 Setup and geometric definitions

Let X be a smooth, finite dimensional, closed (connected) manifold with $\chi(X) = 0$. We consider a closed 1-form α and a Riemannian metric g on X. Let $V := g^{-1}\alpha$ denote the vector field dual to α via g. We wish to count closed orbits and flow lines of V, which are defined as follows.

A *closed orbit* is a nonconstant map $\gamma : S^1 \to X$ such that $\gamma'(t) = -\lambda V(\gamma(t))$ for some $\lambda > 0$. There is a minus sign because we work with the "downward" flow as in classical Morse theory. We consider two closed orbits to be equivalent if they differ by a rotation of S^1. The *period* $p(\gamma)$ is the largest integer k such that γ factors through a k-fold covering $S^1 \to S^1$.

For counting purposes, we can attach a sign to a generic closed orbit as follows. For $q \in \gamma(S^1)$, let $U \subset X$ be a hypersurface intersecting γ transversely near q, and let $\phi_q : U \to U$ be the return map (defined near q) which follows the flow $-V$ a total of $p(\gamma)$ times around the image $\gamma(S^1)$. The linearized return map induces a map

$$d\phi_q : T_qX/T_q\gamma(S^1) \to T_qX/T_q\gamma(S^1)$$

which does not depend on U. The eigenvalues of $d\phi_q$ do not depend on q. We say that γ is *nondegenerate* if $1 - d\phi_q$ is invertible. In this case we define the *Lefschetz sign* $(-1)^{\mu(\gamma)}$ to be the sign of $\det(1 - d\phi_q)$.

A *critical point* is a zero of α. A critical point $p \in X$ is *nondegenerate* if the graph of α in the cotangent bundle T^*X intersects the zero section transversely at p. In this case the derivative $\nabla V : T_pX \to T_pX$ is invertible and self-adjoint; the *index* of p, denoted by $\mathrm{ind}(p)$, is the number of negative eigenvalues of ∇V. The *descending manifold* $\mathscr{D}(p)$ is the set of all $x \in X$ such that the trajectory of the flow $+V$ starting at x converges to p. Similarly, the *ascending manifold* $\mathscr{A}(p)$ is the set of all $x \in X$ from which the trajectory of $-V$ converges to p. If p is nondegenerate, then $\mathscr{D}(p)$ and $\mathscr{A}(p)$ are embedded open balls of dimension $\mathrm{ind}(p)$ and $\dim(X) - \mathrm{ind}(p)$, respectively.

If p and q are critical points, a *flow line* (of $-V$) from p to q is a map $f : \mathbb{R} \to X$ such that $f'(t) = -V(f(t))$ and $\lim_{t\to-\infty} f(t) = p$ and $\lim_{t\to+\infty} f(t) = q$. We consider two flow lines to be equivalent if they differ by a translation of $\mathbb{R}$. The space of flow lines from p to q is naturally identified with $(\mathscr{D}(p) \cap \mathscr{A}(q))/\mathbb{R}$, where $\mathbb{R}$ acts by the flow. Thus, if p and q are nondegenerate, the expected dimension of the space of flow lines from p to q is $\operatorname{ind}(p) - \operatorname{ind}(q) - 1$.

We will impose the following transversality conditions.

Definition 1.1. The pair (α, g) is *admissible* if:

(a) All critical points of V are nondegenerate.

(b) The ascending and descending manifolds of critical points of V intersect transversely.

(c) All closed orbits of V are nondegenerate.

A straightforward transversality calculation (cf. [AB, Sc, H]) shows that for a fixed cohomology class $[\alpha] \in H^1(X; \mathbb{R})$, these conditions hold for generic pairs (α, g).

1.2 Coverings and Novikov rings

In the Morse theory of nonexact closed 1-forms, there may be infinitely many closed orbits and flow lines between critical points of index difference one. To enable finite counting, we consider a covering of X.

Choice 1.2. We choose a connected abelian covering $\pi : \tilde{X} \to X$ such that $\pi^*\alpha$ is exact.

Let H denote the group of covering transformations. There is a surjection $H_1(X) \to H$, whose kernel consists of homology classes of loops in X that lift to $\tilde{X}$.

Our counting will take place in the *Novikov ring* $\Lambda := \operatorname{Nov}(H; [-\alpha])$. The meaning of this notation is that if G is an abelian group and $N : G \to \mathbb{R}$ is a homomorphism, then $\operatorname{Nov}(G; N)$ consists of formal sums $\sum_{g\in G} a_g \cdot g$, with $a_g \in \mathbb{Z}$, such that for each $R \in \mathbb{R}$, there are only finitely many nonzero coefficients a_g with $N(g) < R$. This ring has the obvious addition, and the convolution product [N, HS]. There is a natural inclusion of the group ring $\mathbb{Z}[G]$ into the Novikov ring $\operatorname{Nov}(G; N)$, which is the identity if and only if $N \equiv 0$.

Example 1.3. Suppose $\alpha = df$, where $f : X \to S^1$ is not nullhomotopic. The simplest choice is to take the covering $\tilde{X}$ to be a component of the fiber product of X and $\mathbb{R}$ over S^1. Then $H \simeq \mathbb{Z}$, and the Novikov ring is $\Lambda \simeq \mathbb{Z}((t)) = \{\sum_{k=m}^{\infty} a_k t^k | m, a_k \in \mathbb{Z}\}$, the ring of integer Laurent series.

This is essentially the setup of [HL1]. (In [HL1], $\tilde{X}$ was the entire fiber product of X and $\mathbb{R}$ over S^1. As a result, t here equals t^N in that paper, where N is the number of components of the fiber product, or equivalently the divisibility of $[\alpha] \in H^1(X; \mathbb{Z})$.) For more refined invariants, one can take $\tilde{X}$ to be the universal abelian cover, as in [HL2].

1.3 Counting closed orbits

If (α, g) is admissible, we count closed orbits using the *zeta function*

$$(1.1)\qquad \zeta := \exp \sum_{\gamma \in \mathcal{O}} \frac{(-1)^{\mu(\gamma)}}{p(\gamma)} [\gamma] \in \Lambda.$$

(Cf. [Fr1, Pa3].) Here $\mathcal{O}$ denotes the set of closed orbits, and $[\gamma] \in H$ denotes the image of the homology class $\gamma_*[S^1]$ under the projection $H_1(X) \to H$.

Let us review why ζ is a well defined element of Λ, as the ideas in this argument will be important later. First, we claim that for each $R \in \mathbb{R}$, there are only finitely many closed orbits γ with $[-\alpha](\gamma) < R$. The length of such an orbit away from the critical points is bounded above by some multiple of R. An elementary compactness argument [H, Sa1] then shows that an infinite sequence of such orbits would accumulate to either (i) a degenerate closed orbit, or (ii) a "broken" closed orbit with stopovers at one or more critical points. Situation (i) would violate admissibility condition (c). In situation (ii), there would necessarily be a flow line from a critical point of index i to a critical point of index $\geq i$. This would violate admissibility condition (b), since the expected dimension of the space of such flow lines is negative.

Let Λ^+ denote the set of sums $\sum_{h \in H} a_h \cdot h \in \Lambda$ such that $a_h = 0$ whenever $[-\alpha](h) \leq 0$. Let $\Lambda^+_{\mathbb{Q}} := \Lambda^+ \otimes \mathbb{Q}$. The above paragraph shows that

$$\sum_{\gamma \in \mathcal{O}} \frac{(-1)^{\mu(\gamma)}}{p(\gamma)} [\gamma] \in \Lambda^+_{\mathbb{Q}}.$$

Now $\exp : \Lambda^+_{\mathbb{Q}} \to 1 + \Lambda^+_{\mathbb{Q}}$ is well defined by the usual power series. Therefore $\zeta \in \Lambda \otimes \mathbb{Q}$.

But in fact ζ has integer coefficients, because there is a product formula

$$(1.2)\qquad \zeta = \prod_{\gamma \in \mathscr{I}} (1 \pm [\gamma])^{\pm 1}.$$

Here $\mathscr{I}$ denotes the set of *irreducible* (period 1) closed orbits, and the two signs associated to each irreducible orbit are determined by the eigenvalues of the return map. The formula is proved by taking the logarithm of both sides, cf. [Fr2, HL1, IP, Sm].

Remark 1.4. The inverse of exp above is also well defined via the usual power series $\log(1 + x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k}$. We will use this fact in §3.4.

1.4 Counting flow lines

We count flow lines using the *Novikov complex* (CN_*, ∂), which is defined as follows. Let $\tilde{\mathscr{C}}_i$ denote the set of index i critical points of $\pi^* V$ in $\tilde{X}$. Choose $f : \tilde{X} \to \mathbb{R}$ with

$df = \pi^*\alpha$. We define CN_i to be the set of formal sums $\sum_{\tilde{p}\in\tilde{\mathscr{C}}_i} a_{\tilde{p}} \cdot \tilde{p}$ with $a_{\tilde{p}} \in \mathbb{Z}$, such that for each $R \in \mathbb{R}$, there are only finitely many nonzero coefficients $a_{\tilde{p}}$ with $f(\tilde{p}) > R$. The action of H on $\tilde{\mathscr{C}}_i$ by covering transformations makes CN_i into a module over the Novikov ring Λ. This module is free; one can obtain a basis by choosing a lift of each critical point in X to $\tilde{X}$.

We define the boundary operator $\partial : CN_i \to CN_{i-1}$ by

$$\partial\tilde{p} := \sum_{\tilde{q}\in\tilde{\mathscr{C}}_{i-1}} \langle \partial\tilde{p}, \tilde{q}\rangle \cdot \tilde{q} \tag{1.3}$$

for $\tilde{p} \in \tilde{\mathscr{C}}_i$. Here $\langle \partial\tilde{p}, \tilde{q}\rangle$ denotes the signed number of flow lines of $-\pi^* V$ from $\tilde{p}$ to $\tilde{q}$.

The signs are determined as follows [Sa2]. We choose orientations of the descending manifolds of the critical points in X, and lift them to orient the descending manifolds in $\tilde{X}$. Given a nondegenerate flow line from $\tilde{p}$ to $\tilde{q}$, let $v \in T_{\tilde{p}}\mathscr{D}(\tilde{p})$ be an outward tangent vector of the flow line. The flow near the flow line determines an isomorphism $T_{\tilde{p}}\mathscr{D}(\tilde{p})/v \to T_{\tilde{q}}\mathscr{D}(\tilde{q})$. We declare the flow line to have positive sign if the orientations on $T_{\tilde{p}}\mathscr{D}(\tilde{p})$ and $\mathbb{R}v \oplus T_{\tilde{q}}\mathscr{D}(\tilde{q})$ agree. (We do not need to assume that X is oriented.)

A compactness argument as in §1.3 and [Sa1, Po, H], using the fact that $\pi^*\alpha$ is exact, shows that ∂ is well defined if (α, g) is admissible. A standard argument [Po, Sc] then shows that $\partial^2 = 0$.

The homology of the Novikov complex depends only on the cohomology class $[\alpha]$ and the covering $\tilde{X}$. To describe it topologically, choose a smooth triangulation of X, and lift the simplices to obtain an equivariant triangulation of $\tilde{X}$. We denote the corresponding chain complex by $C_*(\tilde{X})$; this is a module over the group ring $\mathbb{Z}[H]$. There is then a natural isomorphism

$$H_i(CN_*) \simeq H_i(C_*(\tilde{X}) \otimes_{\mathbb{Z}[H]} \Lambda). \tag{1.4}$$

This was stated by Novikov [N], and proofs may be found in [Pa1, Po, HL1]. (Any cell decomposition would suffice here, but we will shortly need to restrict to triangulations, in order to define Reidemeister torsion refined by an Euler structure.)

1.5 Reidemeister torsion and the invariant *I*

The Novikov homology (1.4) often vanishes, at least after tensoring with a field, and it is then interesting to consider the Reidemeister torsion of the complexes CN_* and $C_*(\tilde{X})$.

For certain rings R, including $\mathbb{Z}[H]$ and Λ, if C_* is a finite free chain complex over R with a chosen basis b, then we can define the *Reidemeister torsion* $\tau(C_*)(b) \in Q(R)$, see Appendix A. The Novikov complex CN_* and equivariant cell-chain complex $C_*(\tilde{X})$ have natural bases consisting of lifts of the critical points or simplices of the triangulation from X to $\tilde{X}$. We denote the resulting torsion invariants by

$$\overline{T_m} \in Q(\Lambda)/\pm H, \quad \overline{T(\tilde{X})} \in Q(\mathbb{Z}[H])/\pm H.$$

We have to mod out by $\pm H$ because of the ambiguity in choosing lifts and ordering the bases. It turns out that one can resolve the former ambiguity by choosing an *Euler structure*, see Appendix B. The space $\mathscr{E}(X)$ of Euler structures is a natural affine space over $H_1(X)$ defined by Turaev. We thus obtain refined torsion invariants, which are $H_1(X)$-equivariant maps

$$T_m : \mathscr{E}(X) \to Q(\Lambda)/\pm 1,$$

$$T(\tilde{X}) : \mathscr{E}(X) \to Q(\mathbb{Z}[H])/\pm 1.$$

Results in [Tu2] show that the refined topological torsion $T(\tilde{X})$ depends only on the covering $\tilde{X} \to X$, and not on the choice of triangulation. For example, when X is the 3-manifold obtained from 0-surgery on a knot K, the invariant $T(\tilde{X})$ is related to the Alexander polynomial of K, see e.g. [Tu1, HL1]. By contrast, the Morse-theoretic torsion T_m depends on the admissible pair (α, g), if $[\alpha]$ is fixed and nonzero. (See Example 1.7.) To get a topological invariant, we must multiply by the zeta function.

Definition 1.5. [HL2] We define $\bar{I} := \overline{T_m} \cdot \zeta \in Q(\Lambda)/\pm H$, and

$$I := T_m \cdot \zeta : \mathscr{E}(X) \to Q(\Lambda)/\pm 1.$$

Remark 1.6. In the rest of this paper, in any equation involving the Reidemeister torsions $T(\tilde{X})$ and T_m or the invariant I, it is to be understood that there is an implicit '$\pm$' sign. One can use a homology orientation of X to remove the sign ambiguity in the topological torsion $T(\tilde{X})$ (see [Tu1]), and presumably in the Morse-theoretic torsion T_m as well, but we will not go into that here.

1.6 The main results and basic examples

Our main results are Theorems A and B below. These were proved in [HL2] (generalizing [HL1]) when the cohomology class of α is a real multiple of an integral class. A related result was proved by Pajitnov [Pa3] at about the same time as [HL2]; the connection of this result with our work is discussed in §5.

Theorem A. *For (α, g) admissible, the Morse theory invariant I depends only on the cohomology class $[\alpha] \in H^1(X; \mathbb{R})$ and the choice of covering $\tilde{X}$.*

(If we change the cohomology class $[\alpha]$, aside from multiplying it by a positive real number, then the Novikov ring Λ changes, and also the choice of covering $\tilde{X}$ may no longer be valid, so we generally cannot directly compare the invariants I. See §4 for an exception.)

We can identify the invariant I as follows. The natural inclusion $\mathbb{Z}[H] \to \Lambda$ induces an inclusion of quotient rings $\imath : Q(\mathbb{Z}[H]) \to Q(\Lambda)$. (To see this, one must check that the inclusion $\mathbb{Z}[H] \to \Lambda$ sends nonzerodivisors to nonzerodivisors. This follows from a "leading coefficients" argument or from Lemma A.4.) We then have:

Theorem B. *If (α, g) is admissible, then the Morse theory invariant I agrees with the topological Reidemeister torsion:*

$$\bar{I} = \imath(\overline{T(\tilde{X})}) \in Q(\Lambda)/\pm H.$$

We will also sketch a proof that the refined invariants agree, i.e. that

$$(1.5) \qquad I = \imath \circ T(\tilde{X}) : \mathscr{E}(X) \to Q(\Lambda)/\pm 1.$$

Of course (1.5) implies Theorem A, since $T(\tilde{X})$ is a topological invariant. However our strategy will be to prove Theorem A first, and then deduce Theorem B and (1.5).

Example 1.7. Suppose $X = S^1$ and $[\alpha] \neq 0$. We take $\tilde{X} = \mathbb{R}$, so that $\Lambda \simeq \mathbb{Z}((t))$. It is then not hard to check the following: If α has no critical points, then $\overline{T_m} = 1$ and $\zeta = (1-t)^{-1}$. If α has critical points, then $\overline{T_m} = (1-t)^{-1}$ and $\zeta = 1$. In any case, $\overline{T(\tilde{X})} = (1-t)^{-1}$.

Example 1.8. Suppose $\alpha = df$ with $f : X \to \mathbb{R}$ a Morse function. Then there are no closed orbits, so $\zeta = 1$. In this case it is classical that $\overline{T_m} = \overline{T(\tilde{X})}$, cf. [Mi1]. (Note that $\imath$ is the identity map in this case.)

Here is a sketch of a proof that in fact $T_m = T(\tilde{X})$ (cf. [HL2]). Choose a triangulation $\mathscr{T}$ and let $v_{\mathscr{T}}$ be the associated vector field as in Appendix B. One can apparently find a Morse function $f_{\mathscr{T}}$ and a metric $g_{\mathscr{T}}$ such that the gradient $g_{\mathscr{T}}^{-1}\, df_{\mathscr{T}}$ is a perturbation of $v_{\mathscr{T}}$, so that we have a natural isomorphism of chain complexes $CN_* \simeq C_*(\tilde{X})$, respecting the bases determined by an Euler structure. Then (1.5) holds for $(df_{\mathscr{T}}, g_{\mathscr{T}})$, and by Theorem A it holds for all exact 1-forms.

Example 1.9. Suppose $\alpha = df$ where $f : X^n \to S^1$ is a fiber bundle with connected fibers. In particular, there are no critical points. Let $\tilde{X}$ be the infinite cyclic cover as in Example 1.3, so that $\Lambda \simeq \mathbb{Z}((t))$. Let Σ be a fiber, and let $\phi : \Sigma \to \Sigma$ be the return map obtained by following the flow $-V$ from Σ through X and back to Σ. In this case the zeta function counts fixed points of ϕ and its iterates with their Lefschetz signs:

$$(1.6) \qquad \zeta = \exp \sum_{k=1}^{\infty} \#\mathrm{Fix}(\phi^k) \frac{t^k}{k}.$$

There is a canonical Euler structure $\xi_0 = i_V^{-1}(0)$ (see Appendix B), and $T_m(\xi_0) = 1$. One can also show (cf. [Fr2, HL2]) that

$$T(\tilde{X})(\xi_0) = \prod_{i=0}^{n-1} \det(1 - tH_i(\phi))^{(-1)^{i+1}}$$

where $H_i(\phi)$ is the induced map on $H_i(\Sigma; \mathbb{Q})$. So (1.5) gives here

$$\exp \sum_{k=1}^{\infty} \#\mathrm{Fix}(\phi^k) \frac{t^k}{k} = \prod_{i=0}^{n-1} \det(1 - tH_i(\phi))^{(-1)^{i+1}}.$$

This is equivalent to the Lefschetz theorem for ϕ^k, as one can see by taking the logarithmic derivative of both sides. If we choose a larger covering $\tilde{X}$, we obtain an equivariant version of the Lefschetz theorem [Fr2, H].

Remark 1.10. The relation between torsion and the zeta function in this example goes back to Milnor [Mi2], and was extended to count closed orbits of certain nonsingular flows by Fried [Fr1]. The version of the zeta function in equation (1.6) goes back to Weil [We].

Example 1.11. When X is an oriented 3-manifold with $b_1(X) > 0$, we conjectured in [HL1], based on Taubes' work [Ta1, Ta2, Ta3], that the invariant I equals a certain reparametrization of the Seiberg-Witten invariant of X. In [HL2], we combined this conjecture with Theorem B to derive a formula for the Seiberg-Witten invariant of X in terms of topological torsion, which had been conjectured by Turaev [Tu3]. This result was later independently proved by Turaev [Tu4], refining a result of Meng and Taubes [MT], and indirectly verifying the conjecture in [HL1]. For additional details and references see [HL1, HL2].

More recently, using ideas from TQFT, a paper by Donaldson [D] has appeared giving an alternate approach to the Meng-Taubes formula, and T. Mark [Ma] has given a more direct proof of most of the conjecture in [HL1].

1.7 Ideas of the proofs

The strategy for the proof of Theorem A is to analyze the effect on T_m and ζ as we deform the pair (α, g), fixing the cohomology class $[\alpha]$. As long as the pair (α, g) remains admissible, the Novikov complex and zeta function do not change. But in a generic 1-parameter family, the following types of bifurcations (failures of admissibility) may occur:

(1) A degenerate flow line from $\tilde{p} \in \tilde{\mathscr{C}}_i$ to $\tilde{q} \in \tilde{\mathscr{C}}_{i-1}$,

(2) A degenerate closed orbit,

(3) A flow line from $\tilde{p} \in \tilde{\mathscr{C}}_i$ to $\tilde{q} \in \tilde{\mathscr{C}}_i$, where $\pi(\tilde{p}) \neq \pi(\tilde{q})$,

(4) A flow line from $\tilde{p}$ to $h\tilde{p}$, where $h \in H$,

(5) Birth or death of two critical points at a degenerate critical point.

The first two bifurcations change neither the Novikov complex nor the zeta function. This follows from compactness arguments for the moduli spaces of closed orbits and flow lines. Actually bifurcation (2) includes not only simple cancellation of closed orbits of opposite sign, but also period-doubling bifurcations. Thus it is important that the closed orbits are "counted correctly" by the zeta function (1.1), see Remark 3.3.

The third bifurcation does not affect the zeta function, but it does change the Novikov complex, effectively performing a change of basis in which $\tilde{p}$ is replaced by $\tilde{p} \pm \tilde{q}$. This does not change T_m because the change of basis matrix has determinant one, cf. Proposition A.5.

In the last two bifurcations, ζ and T_m both change, due to the interaction of closed orbits with critical points. In bifurcation (4), a closed orbit homologous to h is created or destroyed, intuitively because the flow line from $\tilde{p}$ to $h\tilde{p}$ is a "broken closed orbit", which should constitute a boundary point in the one-dimensional moduli space of closed orbits as time varies. As a result, the zeta function is multiplied by a power series $1 \pm h + O(h^2)$. At the same time, our understanding of bifurcation (3) suggests that a change of basis occurs in the Novikov complex in which $\tilde{p}$ is multiplied by a power series $1 \pm h + O(h^2)$. Consequently the torsion T_m is multiplied by this power series or its inverse. We find in this way that I is unchanged "to first order".

The higher order terms are more difficult to understand, essentially because the deformation upstairs in $\tilde{X}$ is not generic, due to its H-equivariance, so that there are multiply broken closed orbits and flow lines at the time of bifurcation. It appears that ζ and T_m are simply multiplied by series of the form $(1 \pm h)^{\pm 1}$. But instead of trying to prove this, we consider a non-equivariant perturbation of the deformation in a finite cyclic cover $\hat{X} \to X$. The idea is that invariance to first order in $\hat{X}$ implies invariance to higher order in X, which we prove after working out the behavior of the invariant I with respect to finite cyclic covers. In this way we show that I is unchanged by a bifurcation of type (4).

Bifurcation (5) also has the subtlety of multiply broken flow lines and closed orbits, arising from concatenations of flow lines from the degenerate critical point to itself. Here we use direct finite dimensional analysis to show that every multiply broken closed orbit or flow line leads to an unbroken closed orbit or flow line on the side of the bifurcation time where the two critical points die, but not on the other side. Some miraculous algebra then yields that I is invariant.

A small complication in the argument outlined above is that the times at which bifurcations occur might not be isolated. But bifurcations involving "short" closed orbits or flow lines are isolated, and the long bifurcations change only "higher order" terms in I. Taking a limit in which we consider successively longer bifurcations, we conclude that I is invariant.

With Theorem A established, Theorem B can be deduced rather easily. We already saw in Example 1.8 that Theorem B holds when α is exact. For general α, we use a trick of F. Latour which allows us to "approximate" α by an exact 1-form (!). Namely, we let $f : X \to \mathbb{R}$ be a Morse function and we replace α with the cohomologous form

$$\beta = \alpha + C\,df$$

for $C \in \mathbb{R}$ large. The Novikov complex and zeta function of β are the same as those of the rescaled form

$$C^{-1}\beta = df + C^{-1}\alpha.$$

For C large, there are no closed orbits, and the Novikov complex of β coincides with that of df (under an inclusion of Novikov rings). So by Example 1.8, Theorem B holds for β, and by invariance (Theorem A) it holds for α as well.

2 Proof of invariance I: setup

In this section we make some general preparations for the proof of Theorem A by bifurcation analysis. In §3 we will undertake the analysis of specific bifurcations and complete the proof of Theorem A.

2.1 Semi-isolated bifurcations

Consider a 1-parameter family $\{(\alpha_t, g_t)\}$ of 1-forms and metrics, parametrized by $t \in [0, 1]$. A generic family may have a countably infinite set of bifurcations. In this section we set up a framework in which we only need to consider one bifurcation at a time. More precisely, Lemma 2.7 makes sense of the change in I caused by a single bifurcation, and Lemma 2.9 shows that if all these individual changes are zero, then I is invariant. Note that we always assume that the cohomology class $[\alpha_t]$ is independent of t.

Definition 2.1. A flow line between two critical points is *degenerate* if it corresponds to a nontransverse intersection of ascending and descending manifolds. A (k times) *broken flow line* from $\tilde{p}$ to $\tilde{q}$ is a concatenation of flow lines from $\tilde{p}$ to $\tilde{r}_1$ to $\tilde{r}_2$ to ... to $\tilde{r}_k$ to $\tilde{q}$, where $\tilde{r}_1, \ldots, \tilde{r}_k$ are critical points and $k \geq 1$. A *broken closed orbit* in the homology class h is a (possibly broken) flow line from $\tilde{p}$ to $h\tilde{p}$ for some critical point $\tilde{p}$.

Let $\mathcal{M}_t(\tilde{p}, \tilde{q})$ denote the space of (unbroken) flow lines from $\tilde{p}$ to $\tilde{q}$ at time t. Let $\mathcal{O}_t(h)$ denote the space of (unbroken) closed orbits homologous to h at time t. If the zeroes of α_t are nondegenerate for all $t \in [t_1, t_2]$, then there is a canonical identification of critical points $\tilde{\mathscr{C}}(t) = \tilde{\mathscr{C}}(t')$ for any $t, t' \in [t_1, t_2]$, which we make implicitly below.

The following lemma implies that our invariant does not change if there are no bifurcations, as a result of suitable compactness.

Lemma 2.2. *Let $t_1 < t_2$. Suppose (α_t, g_t) is admissible at t_1 and t_2.*

(a) *Suppose there are no degenerate critical points for $t \in [t_1, t_2]$. Let $\tilde{p} \in \tilde{\mathscr{C}}_i$ and $\tilde{q} \in \tilde{\mathscr{C}}_{i-1}$. Suppose there are no degenerate or broken flow lines from $\tilde{p}$ to $\tilde{q}$ for $t \in [t_1, t_2]$. Then*

$$\mathcal{M}_{t_1}(\tilde{p}, \tilde{q}) = \mathcal{M}_{t_2}(\tilde{p}, \tilde{q}).$$

(b) *Let $h \in H$. Suppose there are no degenerate or broken closed orbits homologous to h for $t \in [t_1, t_2]$. Then*

$$\mathcal{O}_{t_1}(h) = \mathcal{O}_{t_2}(h).$$

Moreover, the above bijections are orientation preserving.

Proof. (a) We first claim that $\bigcup_{t \in [t_1, t_2]} \mathcal{M}_t(\tilde{p}, \tilde{q})$ is compact. To see this, let $S(\tilde{p})$ be a small sphere in the descending manifold around $\tilde{p}$, and let $S(\tilde{q})$ be a small sphere in the ascending manifold around $\tilde{q}$. A flow line corresponds to a triple $(x, y, s) \in S(\tilde{p}) \times S(\tilde{q}) \times \mathbb{R}$ such that downward flow from x for time s hits y. Compactness will follow from an upper bound on s. If s is unbounded, then one can show as in §1.3 that there is a broken flow line from $\tilde{p}$ to $\tilde{q}$ at some time $t \in [t_1, t_2]$, contradicting our hypothesis.

Now (possibly after a perturbation as in §2.2), $\bigcup_{t \in [t_1, t_2]} \mathcal{M}_t(\tilde{p}, \tilde{q})$ is a compact one-manifold with boundary $\mathcal{M}_{t_2}(\tilde{p}, \tilde{q}) - \mathcal{M}_{t_1}(\tilde{p}, \tilde{q})$. Moreover there are no cancellations, since there are no degenerate flow lines from $\tilde{p}$ to $\tilde{q}$ for $t \in [t_1, t_2]$.

For part (b), we get a similar compactness for $\bigcup_{t \in [t_1, t_2]} \mathcal{O}_t(h)$, as in §1.3, since there are no broken closed orbits. Since the orbits remain nondegenerate, none are created or destroyed, and the Lefschetz signs cannot change. □

Definition 2.3. A *bifurcation* of the family $\{(\alpha_t, g_t)\}$ is a time $t_0 \in \mathbb{R}$ such that the pair (α_{t_0}, g_{t_0}) fails to be admissible.

For nonexact closed 1-forms, a generic one-parameter family may contain infinitely many bifurcations (for basically the same reason that a generic nonexact closed 1-form may have infinitely many closed orbits and flow lines between critical points of index difference one). However a generic one-parameter family will only contain finitely many bifurcations with a given upper bound on the "length".

Definition 2.4. The *length* of a bifurcation t_0 is the smallest of the following numbers:

(a) 0, if α_{t_0} has a degenerate zero.

(b) $[-\alpha_{t_0}](h)$, where h is the homology class of a degenerate or broken closed orbit.

(c) $\int_\gamma -\alpha_{t_0}$, where γ is a degenerate (downward) flow line.

Lemma 2.5. *For any family $\{(\alpha_t, g_t)\}$ and any $R \in \mathbb{R}$, the set of bifurcations $t \in [0, 1]$ of length $\leq R$ is closed.*

Proof. Let $\{t_n\}$ be a sequence of bifurcations converging to t_0. After passing to a subsequence, we may assume that each of the bifurcations t_n includes the same type of degenerate object (a critical point, a closed orbit, or a flow line between two given critical points in X) with length $\leq R$. By a compactness argument as in §1.3, these objects accumulate to an object of the same type (possibly broken) at time t_0 with length $\leq R$. Since transversality is an open condition, this object at time t_0 is also degenerate (or broken), so t_0 is a bifurcation of length $\leq R$. □

We now digress to introduce the notion of limits in Novikov rings. Given $x = \sum_g a_g \cdot g \in \mathrm{Nov}(G, N)$ and $R \in \mathbb{R}$, we write "$x = O(R)$" if $a_g = 0$ whenever $N(g) < R$. Given a sequence $\{x_n\}$ in $\mathrm{Nov}(G; N)$ and $x \in \mathrm{Nov}(G; N)$, we write "$\lim_{n\to\infty} x_n = x$" if for every $R \in \mathbb{R}$ there exists n_0 such that $x - x_n = O(R)$ for all $n > n_0$.

We can extend these definitions to the quotient ring $Q(\Lambda)$ as follows. If G is a finitely generated abelian group, then by Lemma A.4 we have a decomposition $Q(\mathrm{Nov}(G; N)) = \bigoplus F_j$ into a sum of fields. By (A.3), each field F_j can be identified with the tensor product of $\mathrm{Nov}(G/\mathrm{Ker}(N); N)$ with a certain field. The notion of "$O(R)$" is then well defined for elements of F_j. We say that an element of $Q(\mathrm{Nov}(G; N))$ is "$O(R)$" if its projection to each subfield F_j is $O(R)$, and we define limits accordingly.

Definition 2.6. A time t_0 is *good* if:

(a) t_0 is not a limit of bifurcations of bounded length.

(b) For each $\varepsilon > 0$, the intervals $(t_0 - \varepsilon, t_0)$ and $(t_0, t_0 + \varepsilon)$ both contain some times t which are not bifurcations.

Lemma 2.7. *If t_0 is good, then the limits as $t \nearrow t_0$ and $t \searrow t_0$ of ζ and (CN_*, ∂) are well defined. If moreover t_0 is not a bifurcation, then the left and right limits of ζ (resp. CN_*) are both equal to $\zeta(t_0)$ (resp. $CN_*(t_0)$).*

Proof. Consider the limit as $t \nearrow t_0$. There exists $\varepsilon > 0$ such that all critical points of α_t are nondegenerate for $t \in (t - \varepsilon, t_0)$, so that $\tilde{\mathscr{C}}(t) = \tilde{\mathscr{C}}(t')$ for $t, t' \in (t_0 - \varepsilon, t_0)$. For convergence of ∂, we must show that for $\tilde{p} \in \tilde{\mathscr{C}}_i$ and $\tilde{q} \in \tilde{\mathscr{C}}_{i-1}$, there exists $x \in \Lambda$ such that

$$\lim_{t \nearrow t_0} \sum_{h \in H} \langle \tilde{p}, h\tilde{q} \rangle \cdot h = x,$$

where t ranges over any sequence of non-bifurcation values converging to t_0 from below. We use Lemma 2.2(a). For any path γ from $\tilde{p}$ to $h\tilde{q}$, we have

$$\int_\gamma -\alpha = C + [-\alpha](h)$$

where C is a constant which is independent of h and varies continuously with t. Thus if γ is a downward gradient flow line and $[-\alpha](h)$ is bounded from above then $\int_\gamma -\alpha$ is also bounded from above. So if we are sufficiently close to t_0 then there are no degenerate or broken flow lines from $\tilde{p}$ to $h\tilde{q}$ by definition of semi-isolated, so $\langle \tilde{p}, h\tilde{q} \rangle$ cannot change by Lemma 2.2(a).

Similary, Lemma 2.2(b) implies that the zeta functions converge.

The last sentence of the lemma follows from Lemma 2.2. $\square$

Let ζ^+, ζ^-, (CN_*^+, ∂^+), (CN_*^-, ∂^-) denote these limits. An Euler structure gives a basis for the limiting complexes CN_*^+ and CN_*^-; let T_m^+, T_m^- denote their Reidemeister torsion, and let $I^\pm := T_m^\pm \cdot \zeta^\pm$.

Lemma 2.8. *If t_0 is good, then for any Euler structure ξ,*

$$\lim_{t\nearrow t_0} I(t)(\xi) = I^-(t_0)(\xi), \quad \lim_{t\searrow t_0} I(t)(\xi) = I^+(t_0)(\xi),$$

where t ranges over non-bifurcations.

Proof. Consider the limit as $t \nearrow t_0$. By definition we have $\lim_{t\nearrow t_0} \zeta(t) = \zeta^-(t_0)$. So we have to prove that $\lim_{t\nearrow t_0} T_m(t)(\xi) = T_m^-(t_0)(\xi)$.

For ε sufficiently small we can identify the critical points for different $t \in (t_0 - \varepsilon, t_0)$. Fix a basis for CN_* consisting of a lift of each critical point to $\tilde{X}$, in the equivalence class determined by ξ.

For a non-bifurcation t, recall that T_m is the sum of the torsions of $CN_* \otimes F_j$. The torsion of $CN_* \otimes F_j$ is zero if $CN_* \otimes F_j$ is not acyclic; this criterion is independent of t, by the Novikov isomorphism (1.4). Moreover, even if t_0 is a bifurcation, the limiting complex $CN_*^- \otimes F_j$ is acyclic if and only if $CN_* \otimes F_j$ is acyclic for all non-bifurcations t, because the Novikov isomorphism (1.4), as constructed in [HL1], can be extended by a limiting argument to CN_*^-.

When $CN_* \otimes F_j$ is acyclic, we compute its torsion using Proposition A.2. Choose subbases D_i and E_i as in Proposition A.2 for $CN_*^- \otimes F_j$. We can use these same subbases in the interval $(t_0 - \delta, t_0)$ for some δ, because if the determinants in Proposition A.2 are nonzero in the limiting complex, then they are nonzero near t_0. The reason is that each determinant for the limiting complex has a nonzero "leading term" involving flow lines of length $< R$ for some R, which will be unchanged near t_0 by Lemma 2.2(a).

For $a, b \in F_j$ we have $\dfrac{1}{a} - \dfrac{1}{b} = O(R)$ when the leading order of $a - b$ exceeds the leading order of a and b by at least R. This means that a high order change in the denominator of $T_m(\xi)$, as computed above, will change $T_m(\xi)$ by high order terms. We are now done by condition 2.6(a) and Lemma 2.2(a). $\square$

Lemma 2.9. *Let $\{(\alpha_t, g_t)\}$ be a family parametrized by $t \in [0,1]$, with α_t in a fixed cohomology class and (α_0, g_0) and (α_1, g_1) admissible. Suppose that every bifurcation $t_0 \in (0,1)$ is good and satisfies $I^+(t_0) = I^-(t_0)$. Then $I(0) = I(1)$.*

Proof. We first observe that every $t_0 \in [0,1]$ is good. If t_0 fails to satisfy condition 2.6(a), then t_0 is a bifurcation by Lemma 2.5, contradicting our hypothesis that all bifurcations are good. Since we also assumed that every bifurcation satisfies condition 2.6(b), it follows that the non-bifurcations are dense in $[0,1]$, so every $t_0 \in [0,1]$ satisfies condition 2.6(b).

Next, by the assumptions and Lemma 2.7, we have $I^+(t_0) = I^-(t_0)$ for each $t_0 \in [0,1]$. It follows from Lemma 2.8 that if we fix an Euler structure ξ and $R > 0$, then for all $t_0 \in [0,1]$, there exists $\varepsilon > 0$ such that

$$I(t)(\xi) = I(t')(\xi) + O(R)$$

for all non-bifurcations $t, t' \in (t_0 - \varepsilon, t_0 + \varepsilon)$. Since $[0,1]$ is compact and the non-bifurcations are dense, it follows that $I(0)(\xi) - I(1)(\xi) = O(R)$. Taking $R \to \infty$, while keeping ξ fixed, completes the proof. □

To reduce complications, we will consider good bifurcations which satisfy an additional condition.

Definition 2.10. A bifurcation t_0 is *semi-isolated* if t_0 is good, and:

(*) The pair (α_{t_0}, g_{t_0}) violates only one of the admissibility conditions in Definition 1.1, and at only one degenerate object.

2.2 Generic one-parameter families

The following lemma implies that in a generic one-parameter family, only the five types of bifurcations listed in §1.7 may occur, and are semi-isolated.

Lemma 2.11. *Let $\{(\alpha_t, g_t), t \in [0,1]\}$ be a 1-parameter family with α_t in a fixed cohomology class and (α_0, g_0) and (α_1, g_1) admissible. Then after a perturbation fixing the endpoints, we may arrange that:*

(a) *Near a degenerate critical point at time t_0, there are local coordinates $x_1, \ldots, x_n$ in which*

$$g_t^{-1}\alpha_t = (x_1^2 \pm (t - t_0), -x_2, \ldots, -x_i, x_{i+1}, \ldots, x_n). \tag{2.1}$$

(b) *Suppose that for $t \in [t_1, t_2]$, we have critical points $\tilde{p}(t), \tilde{q}(t) \in \tilde{\mathscr{C}}(\alpha_t)$ which depend continuously on t. Then $\bigcup_t \mathscr{D}(\tilde{p}(t))$ and $\bigcup_t \mathscr{A}(\tilde{q}(t))$ intersect transversely in $\tilde{X} \times [t_1, t_2]$, and the projection of the intersection to $[t_1, t_2]$ is a Morse function.*

(c) *All bifurcations are semi-isolated.*

If there are no degenerate critical points in the original family (α_t, g_t), then we may choose this perturbation to be C^k small for any k.

Proof. We will work with C^k families so that we can use the Sard-Smale theorem. After arranging (a), we will show that in the space of C^k families, there is a countable intersection of open dense sets, whose elements are families with the desired properties (b) and (c). As in [MS, Ta2], we then obtain a dense set in C^∞.

We begin by making the graph of $\bigcup_t \alpha_t$ transverse to the 0-section of $T^*X \times [0,1]$. Then $\bigcup_t \alpha_t^{-1}(0)$ is a smooth 1-dimensional submanifold of $X \times [0,1]$. We can further arrange that t is a Morse function on $\bigcup_t \alpha_t^{-1}(0)$ such that all critical points have distinct values. A critical point of t on $\bigcup_t \alpha_t^{-1}(0)$ is a pair (x,t) where α_t has a degenerate zero at x. By a lemma of Cerf [Ce] we can choose (possibly time dependent) local coordinates $x_1, \ldots, x_n$ near such a point so that

$$\alpha_t = \frac{x_1^3}{3} \pm (t - t_0)x_1 + \frac{-x_2^2 - \cdots - x_i^2 + x_{i+1}^2 + \cdots + x_n^2}{2}.$$

We now fix the metric g_t on X to be Euclidean near the origin in these coordinates. This gives (a).

By a standard transversality argument, we can obtain (b) in a countable intersection of open dense sets. Fixing the metric near the degenerate critical points does not interfere with the transversality argument because no flow line or closed orbit is completely supported near a degenerate critical point.

To obtain (c), we first arrange for the space of irreducible closed orbits in $X \times [0, 1]$ to be cut out transversely (regarded as the zero set of a section of a vector bundle as in Remark 3.3), for the projection from this space to $[0, 1]$ to be a Morse function, and for the degenerate covers with a given multiplicity of a given connected family of orbits to be isolated. We then use a compactness argument as in Lemma 2.5 to show that (i) for each R, only finitely many degenerate objects of length $< R$ occur. We can arrange that these degenerate objects occur at distinct times, by intersecting with an open dense set of deformations. So in a countable intersection of open dense sets, we can arrange that (ii) all degenerate objects occur at distinct times. Now (i) and (ii) imply (c). □

3 Proof of invariance II: bifurcation analysis

In a generic one-parameter deformation given by Lemma 2.11, only the five types of bifurcations listed in §1.7 may occur, and all bifurcations are semi-isolated. In this section we will show that $I^+ = I^-$ for each such bifurcation. By Lemma 2.9, this will complete the proof of Theorem A.

3.1 Cancellation of flow lines

Lemma 3.1. *Suppose t_0 is a semi-isolated bifurcation at which there is a degenerate flow line from $\tilde{p} \in \tilde{\mathscr{C}}_i$ to $\tilde{q} \in \tilde{\mathscr{C}}_{i-1}$. Then $\zeta^+(t_0) = \zeta^-(t_0)$ and $(CN_*^+, \partial^+) = (CN_*^-, \partial^-)$.*

Proof. By the definition of semi-isolated, we may choose $\varepsilon > 0$ such that for all t with $0 < |t - t_0| \leq \varepsilon$, there are no degenerate or broken flow lines from $\tilde{p}$ to $\tilde{q}$. As in the proof of Lemma 2.2(a), the moduli space of flow lines from $\tilde{p}$ to $\tilde{q}$ for $|t - t_0| \leq \varepsilon$ is compact, so $\langle \partial^- \tilde{p}, \tilde{q} \rangle = \langle \partial^+ \tilde{p}, \tilde{q} \rangle$, since the signed number of boundary points of a compact 1-manifold is zero.

For every $R > 0$, for every other pair of critical points $\tilde{r}, \tilde{s}$ with index difference 1 and $\int_{\tilde{s}}^{\tilde{r}} \alpha < R$, the coefficient $\langle \partial \tilde{r}, \tilde{s} \rangle$ likewise does not change for t sufficiently close to t_0.

For every $R > 0$, the coefficients in the zeta function of h with $[-\alpha](h) < R$ do not change for t sufficently close to t_0, by Lemma 2.2(b). □

3.2 Cancellation of closed orbits

Lemma 3.2. *Suppose t_0 is a semi-isolated bifurcation at which there exists a degenerate closed orbit. Then $(CN_*^+, \partial^+) = (CN_*^-, \partial^-)$ and $\zeta^+ = \zeta^-$.*

Proof. The Novikov complex is unchanged as in the proof of Lemma 3.1. To show that the zeta function does not change, the idea is that locally the zeta function looks like (1.6), and this is invariant because the signed number of fixed points of a map is invariant, assuming suitable compactness.

More precisely, at time t_0 there is an isolated irreducible closed orbit γ, with $[\gamma] = h$, such that γ or some multiple cover of it is degenerate. Choose $x \in \gamma(S^1) \subset X$, and let $D_\delta \subset X$ be a disc of radius δ transverse to γ and centered at x. Let $\phi_{\delta,t} : D_\delta \to D_\delta$ be the (partially defined) first return map for the flow $g_t^{-1}\alpha_t$. We restrict the domain of $\phi_{\delta,t}$ to a maximal connected neighborhood of x on which it is continuous. Define

$$\zeta_{\delta,t} := \exp \sum_{k=1}^{\infty} \frac{h^k}{k} \#\mathrm{Fix}(\phi_{\delta,t}^k)$$

for non-bifurcations t. We claim that

$$\frac{\zeta^+}{\zeta^-} = \lim_{\delta\to 0} \frac{\lim_{t\searrow t_0} \zeta_{\delta,t}}{\lim_{t\nearrow t_0} \zeta_{\delta,t}}. \tag{3.1}$$

To prove this, given $R > 0$, we must find $\delta > 0$ such that $\dfrac{\zeta^+}{\zeta^-} = \dfrac{\lim_{t\searrow t_0} \zeta_{\delta,t}}{\lim_{t\nearrow t_0} \zeta_{\delta,t}} + O(R)$. By the definition of semi-isolated, there exists $\varepsilon > 0$ so that for $|t - t_0| < \varepsilon$, all closed orbits γ' with $[-\alpha]([\gamma']) < R$ are nondegenerate, except for covers of γ at time $t = t_0$. By compactness as in Lemma 2.2(b), we can choose δ sufficiently small that no such closed orbit (other than covers of γ) intersects $D_{2\delta}$ at time t_0. Then for $|t - t_0|$ sufficiently small, the contribution to $\log\zeta$ from closed orbits γ' avoiding D_δ with $[-\alpha]([\gamma']) < R$ does not change, and when moreover t is not a bifurcation, the contribution to $\log\zeta$ from all other closed orbits γ' with $[-\alpha]([\gamma']) < R$ is counted by the order $< R$ terms of $\log\zeta_{\delta,t}$, as in (1.6). This proves (3.1).

Given any positive integer k, as above we can choose δ such that at time t_0, no closed orbit γ' with $[-\alpha]([\gamma']) \leq k[\alpha](h)$ (other than covers of γ) intersects $D_{2\delta}$. In particular, for $k' \leq k$, the boundary of the graph of $\phi_{\delta,t_0}^{k'}$ does not intersect the diagonal in $D_\delta \times D_\delta$. (Here we are compactifying the graph as in the proof of Lemma 3.8(b), see also [HL1].) It follows that $\#\mathrm{Fix}(\phi_{\delta,t}^{k'})$ is independent of t for non-bifurcations t close to t_0. This implies that

$$\lim_{\delta\to 0} \frac{\lim_{t\searrow t_0} \zeta_{\delta,t}}{\lim_{t\nearrow t_0} \zeta_{\delta,t}} = 1.$$

Together with (3.1), this proves the lemma. □

Remark 3.3. Here are two alternate approaches to proving this lemma, which add some insight and might generalize to Floer theory.

First, one might show that generically there is either a simple cancellation of two orbits, or a period doubling bifurcation corresponding to $(1 + h) = (1 - h^2)(1 - h)^{-1}$

in the product formula (1.2). Related analysis appears in [Ta2] for the more complicated problem of counting pseudoholomorphic tori in symplectic 4-manifolds.

Second, one might make the following formal argument rigorous. For nonzero $h \in H$, let $\mathscr{L}(h)$ denote the space of loops $\gamma : S^1 \to X$ homologous to h, modulo rotations of S^1. There is a vector bundle E over $\mathscr{L}(h) \times \mathbb{R}^+$ given by

$$E_{(\gamma,\lambda)} = \Gamma(\gamma^* TX).$$

Define a section s of E by

$$s(\gamma,\lambda)(t) = \gamma'(t) + \lambda V(\gamma(t)).$$

By definition γ is a closed orbit if and only if $s(\gamma,\lambda) = 0$ for some (unique) $\lambda > 0$. So the coefficient of h in $\log \zeta$, namely

$$(3.2) \qquad \sum_{\gamma \in \mathcal{O}, [\gamma]=h} \frac{(-1)^{\mu(\gamma)}}{p(\gamma)} \in \mathbb{Q},$$

is formally the degree of the section s. We divide by $p(\gamma)$ because $\mathscr{L}(h)$ is an orbifold with $\mathbb{Z}/p$ symmetry around loops with period p. As long as there is no interaction between closed orbits and critical points, so that λ stays bounded and the zero set of s remains compact, the coefficients (3.2) of $\log \zeta$, and hence ζ, should not change.

3.3 The slide bifurcation

A *slide bifurcation* is a semi-isolated bifurcation t_0 at which there is a downward flow line from $\tilde{p} \in \tilde{\mathscr{C}}_i$ to $\tilde{q} \in \tilde{\mathscr{C}}_i$. (For real-valued Morse functions, this bifurcation acts on the corresponding handle decomposition of X by sliding one handle over another.) By Lemma 2.11, we can (and will) assume that the flow line from $\tilde{p}$ to $\tilde{q}$ is a transverse intersection of $\bigcup_t \mathscr{D}(\tilde{p})(t)$ and $\bigcup_t \mathscr{A}(\tilde{q})(t)$.

Lemma 3.4. *For a slide bifurcation such that $\pi(\tilde{p}) \neq \pi(\tilde{q})$ in X, we have*

(a) $\zeta^+ = \zeta^-$ *and* $\partial^+ = A^{-1} \circ \partial^- \circ A$, *where* $A : CN_* \to CN_*$ *sends* $\tilde{p} \mapsto \tilde{p} \pm \tilde{q}$ *and fixes all other critical points* $\tilde{s}$ *with* $\pi(\tilde{s}) \neq \pi(\tilde{p})$.

(b) *In particular* $I^+ = I^-$.

Proof. Since the bifurcation is semi-isolated, if $\tilde{r}$ is a critical point with $\mathrm{ind}(\tilde{r}) \neq i+1$ and $\pi(\tilde{r}) \neq \pi(\tilde{p})$, then

$$(3.3) \qquad \partial_+(\tilde{r}) = \partial_-(\tilde{r}).$$

Let us denote this common value by $\partial_0(\tilde{r})$. We now claim that

$$(3.4) \qquad \partial_+(\tilde{p}) = \partial_-(\tilde{p}) \pm \partial_0(\tilde{q}).$$

To see this, let $\tilde{s}$ be a critical point of index $i-1$. We need to show that

$$(3.5) \qquad \langle \partial_+ \tilde{p}, \tilde{s} \rangle = \langle \partial_- \tilde{p}, \tilde{s} \rangle \pm \langle \partial \tilde{q}, \tilde{s} \rangle,$$

with the $\pm$ sign independent of $\tilde{s}$. (This formula makes sense because $\langle \partial \tilde{q}, \tilde{s} \rangle$ is independent of t near the bifurcation time t_0.)

One way to prove equation (3.5) (from [Lau]) is as follows. Choose $f : \tilde{X} \to \mathbb{R}$ with $df = \pi^* \alpha$, and let Σ be a level set of f just below $\tilde{q}$. Consider the "descending slices" $D(\tilde{p}) = \mathscr{D}(\tilde{p}) \cap \Sigma$ and $D(\tilde{q}) = \mathscr{D}(\tilde{q}) \cap \Sigma$, which depend on t near t_0. These can be given the structure of chains [HL1] or currents [Lau]. Let $D_+(\tilde{p})$ and $D_-(\tilde{p})$ denote the limits of $D(\tilde{p})$ as t approaches t_0 from above and below, and let $D_0(\tilde{q}) = D(\tilde{q})(t_0)$. We then have, as chains,

$$(3.6) \qquad D_+(\tilde{p}) = D_-(\tilde{p}) \pm D_0(\tilde{q}),$$

see [Lau]. Taking the intersection number with $\mathscr{A}(\tilde{s})$, we obtain (3.5).

Another approach to proving (3.5), which generalizes to infinite dimensions, is to use a gluing argument as in [Fl] to show that for ε small, we have a one-dimensional cobordism

$$\partial \left(\bigcup_{t \in [t_0 - \varepsilon, t_0 + \varepsilon]} \mathscr{M}_t(\tilde{p}, \tilde{s}) \right) = \mathscr{M}_{t_0+\varepsilon}(\tilde{p}, \tilde{s}) - \mathscr{M}_{t_0-\varepsilon}(\tilde{p}, \tilde{s}) \mp \mathscr{M}_{t_0}(\tilde{p}, \tilde{q}) \times \mathscr{M}_{t_0}(\tilde{q}, \tilde{s}).$$

(The orientations are more subtle in this approach. Related gluing arguments can also be used to prove (3.6), cf. [HL1].)

Similarly to (3.4), for each critical point $\tilde{s}$ of index $i+1$, we have

$$(3.7) \qquad \partial_+(\tilde{s}) = \partial_-(\tilde{s}) \mp \langle \partial \tilde{s}, \tilde{p} \rangle \tilde{q}.$$

Equations (3.3), (3.4) and (3.7) imply (a).

Part (b) follows by Proposition A.5, since $\det(A_i) = 1$. □

3.4 Torsion and zeta function of a finite cyclic cover

We now digress to work out the behavior of the invariant I with respect to finite cyclic covers. The answer is given in terms of the Norm map from Galois theory. This result will be needed when we use nonequivariant perturbations in the next section, and may also be of independent interest.

Suppose we have a short exact sequence of abelian groups

$$0 \to K \xrightarrow{\imath} H \xrightarrow{m} \mathbb{Z}/k \to 0.$$

Let $\rho : \hat{X} \to X$ be the k-fold cyclic covering whose monodromy is the composition $\pi_1(X) \to H \xrightarrow{m} \mathbb{Z}/k$. The covering $\tilde{X} \to X$ factors through ρ, and the covering

$\tilde{X} \to \hat{X}$ has automorphism group K. We now want to relate the invariants of X and $\hat{X}$, choosing the covering $\tilde{X}$ for both in Choice 1.2.

We need the following algebraic notation. Let $\hat{\Lambda} := \mathrm{Nov}(K; \imath^*[-\alpha])$. The map $\imath$ induces a pushforward of Novikov rings $\imath_* : \hat{\Lambda} \to \Lambda$ sending $\sum_{k \in K} a_k \cdot k \mapsto \sum_{k \in K} a_k \cdot \imath(k)$. Since $\imath$ has finite kernel, there is also a pullback $\imath^* : \Lambda \to \hat{\Lambda}$ sending $\sum_{h \in H} a_h \cdot h \mapsto \sum_{k \in K} a_{\imath(k)} \cdot k$. The pushforward $\imath_*$ makes Λ into a free module of rank k over $\hat{\Lambda}$. If $y \in \Lambda$, then multiplication by y is an endomorphism of this module, whose determinant and trace we denote by $\mathrm{Norm}(y)$ and $\mathrm{Tr}(y)$ respectively.

It will sometimes be convenient to assume that:

(3.8) m annihilates the torsion subgroup of H.

In general, the map $\imath_*$ sends nonzerodivisors to nonzerodivisors and hence induces a map on quotient rings $Q(\hat{\Lambda}) \to Q(\Lambda)$. Recall from Lemma A.4(a) that we have decompositions of $Q(\hat{\Lambda})$ and $Q(\Lambda)$ into sums of fields. Assumption (3.8) implies that $\imath_*$ respects these decompositions. We then see from (A.3) that $Q(\Lambda)$ is a free module of rank k over $Q(\hat{\Lambda})$, so Norm extends to a multiplicative map $Q(\Lambda) \to Q(\hat{\Lambda})$.

Lemma 3.5. (a) *If* $y \in \Lambda$ *then* $\mathrm{Tr}(y) = k \cdot \imath^* y$.

(b) *If* $x \in \Lambda^+$, *then* $\log \mathrm{Norm}(1 + x) = \mathrm{Tr} \log(1 + x)$.

(c) *Assuming* (3.8), *if* $y \in Q(\Lambda)$ *and* $y \neq 0$, *then* $\mathrm{Norm}(y) \neq 0$.

Proof. Let θ be a primitive k^{th} root of 1. For $0 \le i < k$, define a ring homomorphism $\sigma_i : \Lambda \otimes \mathbb{Z}[\theta] \to \Lambda \otimes \mathbb{Z}[\theta]$ by $\sigma_i(h \otimes 1) = h \otimes \theta^{i \cdot m(h)}$ for $h \in H$. By [Lan, §6.5], we have

$$(3.9) \qquad \mathrm{Tr}(y) = \sum_{i=0}^{k-1} \sigma_i(y), \quad \mathrm{Norm}(y) = \prod_{i=0}^{k-1} \sigma_i(y).$$

The first of these identities implies that for $h \in H$,

$$\mathrm{Tr}(h) = \begin{cases} kh & \text{if } m(h) = 0 \\ 0 & \text{if } m(h) \neq 0 \end{cases}$$

(which can also be seen more directly). This proves (a). To prove (b), we use (3.9) to compute that

$$\log \mathrm{Norm}(1 + x) = \sum_{i=0}^{k-1} \log \sigma_i(1 + x) = \sum_{i=0}^{k-1} \sigma_i \log(1 + x) = \mathrm{Tr} \log(1 + x).$$

Here the middle equality holds because log is defined using a power series (see Remark 1.4) and σ_i is a ring homomorphism. To prove (c), observe that assumption (3.8) implies that σ_i respects the field decomposition of $Q(\Lambda)$. Assertion (c) now follows from (3.9) and the injectivity of σ_i. □

If V is a vector field on X with nondegenerate zeroes, then the inverse image map $H_1(X, V) \to H_1(\hat{X}, \rho^* V)$ induces a natural pullback of Euler structures $\rho^* : \mathscr{E}(X) \to \mathscr{E}(\hat{X})$. It is clear that if (α, g) is admissible on X, then $(\rho^*\alpha, \rho^* g)$ is admissible on $\hat{X}$. With respect to these pullbacks, the invariant I behaves as follows.

Proposition 3.6. (a) $\zeta(\hat{X}) = \mathrm{Norm}(\zeta(X))$.

(b) *Under the assumption* (3.8), *the following diagram commutes:*

$$\begin{array}{ccc} \mathscr{E}(\hat{X}) & \xrightarrow{T_m(\hat{X})} & Q(\hat{\Lambda})/\pm 1 \\ {\scriptstyle \rho^*}\Big\uparrow & & \Big\uparrow{\scriptstyle \mathrm{Norm}} \\ \mathscr{E}(X) & \xrightarrow{T_m(X)} & Q(\Lambda)/\pm 1. \end{array}$$

Proof. (a) Every closed orbit $\hat{\gamma}$ in $\hat{X}$ is a lift of a unique closed orbit γ in X, with $[\gamma] \in K$. Conversely, if $\gamma \in \mathcal{O}(X)$ and $[\gamma] \in K$, let γ_1 denote the period one orbit underlying γ, and let l be the order of $m([\gamma_1])$ in the group $\mathbb{Z}/k$. Then γ lifts to k/l distinct closed orbits $\hat{\gamma}$, each of which has period $p(\hat{\gamma}) = p(\gamma)/l$ and Lefschetz sign $(-1)^{\mu(\hat{\gamma})} = (-1)^{\mu(\gamma)}$. Therefore

$$\log \zeta(\hat{X}) = \sum_{\hat{\gamma} \in \mathcal{O}(\hat{X})} \frac{(-1)^{\mu(\hat{\gamma})}}{p(\hat{\gamma})} [\hat{\gamma}] = \sum_{\gamma \in \mathcal{O}(X), [\gamma] \in K} \frac{k(-1)^{\mu(\gamma)}}{p(\gamma)} [\gamma] = k\imath^* \log \zeta(X).$$

By Lemma 3.5,

$$k\imath^* \log \zeta(X) = \mathrm{Tr} \log \zeta(X) = \log \mathrm{Norm}\, \zeta(X).$$

Combining the above equations and applying exp proves (a).

(b) A finite free complex C_* over Λ can be regarded as a complex $\hat{C}_*$ over $\hat{\Lambda}$ with k times as many generators. Moreover, a basis $\{\lambda_1, \ldots, \lambda_k\}$ for Λ over $\hat{\Lambda}$ determines a map $\phi : \mathscr{B}(C_*) \to \mathscr{B}(\hat{C}_*)$, and if $\chi(C_*) = 0$ then the map ϕ is independent of the choice of $\{\lambda_1, \ldots, \lambda_k\}$. Now we observe that if $\xi \in \mathscr{E}(X)$, then the Novikov complex $CN_*(\hat{X})$, with the basis determined by $\rho^*\xi$, is obtained from $CN_*(X)$ and ξ by this construction. So we need to show that $\tau(\hat{C}_*)(\phi(b)) = \mathrm{Norm}(\tau(C_*)(b))$. The assumption (3.8) implies that $\imath_*$ and Norm are compatible with the decompositions of $Q(\hat{\Lambda})$ and $Q(\Lambda)$ into sums of fields. So we can restrict attention to a complex $C_* \otimes F$ where $F \subset Q(\Lambda)$ is a field; let $\hat{F}$ denote the corresponding field in $Q(\hat{\Lambda})$. If $C_* \otimes F$ is not acyclic, then $\hat{C}_* \otimes \hat{F}$ is not acyclic either, so both torsions are zero. If $C_* \otimes F$ is acyclic, we can decompose it into a direct sum of 2-term acyclic complexes. Our claim then reduces to the fact that if ∂ is a square matrix over F and $\hat{\partial}$ is the corresponding matrix over $\hat{F}$, then $\det(\hat{\partial}) = \mathrm{Norm}(\det(\partial))$. This follows from the definition of Norm, after putting ∂ into Jordan canonical form over an algebraic closure of F. □

3.5 Sliding a critical point over itself

We now analyze bifurcation (4), in which a critical point slides over itself, following the strategy described in §1.7.

If $p \in \mathscr{C}$ and $x \in \Lambda$, let $A_p(x) : CN_* \to CN_*$ denote the Λ-module endomorphism which sends $\tilde{p} \mapsto x\tilde{p}$ for every lift $\tilde{p}$ of p and fixes all other critical points $\tilde{s}$ with $\pi(\tilde{s}) \neq \pi(\tilde{p})$.

Lemma 3.7. *Suppose $\tilde{p} \in \tilde{\mathscr{C}}_i$ slides over $h\tilde{p}$ for some $h \in H$, and let $p = \pi(\tilde{p})$. Then*

(a) *There is a power series $x = 1 + \sum_{n=1}^{\infty} a_n h^n$, with $a_n \in \mathbb{Z}$, such that*

$$(3.10) \quad \partial^+ = A_p(x)^{-1} \circ \partial^- \circ A_p(x).$$

(b) *In particular $T_m^+ = x^{(-1)^i} \cdot T_m^-$.*

(c) *The coefficient $a_1 = \pm 1$.*

Proof. (a) Let d denote the divisibility of h in H. (This is defined because h is not a torsion class.) Let k be a positive integer relatively prime to d, and let $m : H \to \mathbb{Z}/k$ be a homomorphism sending $h \mapsto 1$. Let $\rho : \hat{X} \to X$ be the k-fold cyclic cover with monodromy m. Then the critical points $\tilde{p}, h\tilde{p}, \ldots, h^{k-1}\tilde{p}$ project to distinct points in $\hat{X}$.

Let $R = [-\alpha](kh)$. By semi-isolatedness, we can find $\varepsilon > 0$ such that no bifurcation of length $< R$ occurs between time $t_0 - \varepsilon$ and $t_0 + \varepsilon$, other than the slide of $\tilde{p}$ over $h\tilde{p}$. Choose a smaller ε if necessary so that the pairs $(\alpha_{t_0 \pm \varepsilon}, g_{t_0 \pm \varepsilon})$ are admissible. Perturb the pulled back family $\{\rho^*(\alpha_t, g_t) | t \in [t_0 - \varepsilon, t_0 + \varepsilon]\}$, fixing the endpoints, to satisfy the genericity conditions of Lemma 2.11.

By a compactness argument (as in the proof of Lemma 2.5), we can choose the perturbation small enough that no bifurcations of length $< R$ occur other than slides of $h^i\tilde{p}$ over $h^j\tilde{p}$. Then iterating Lemma 3.4(a) and using Lemma 2.2(a), we find a power series $x_k = 1 + \sum_{n=1}^{k-1} a_{n,k} h^n$ such that

$$(3.11) \quad \partial^+ = A(x_k)^{-1} \circ \partial^- \circ A(x_k) + O(R).$$

(Here "$O(R)$" indicates a term involving flow lines γ with $\int_\gamma -\alpha \geq R$.)

Without loss of generality, $\partial^- \tilde{p} \neq 0$ or $\langle \partial^- \tilde{s}, \tilde{p} \rangle \neq 0$ for some s (since otherwise equation (3.10) is vacuously true for any x). Then equation (3.11) implies that for n fixed, $a_{n,k}$ is constant for large k. If we define $a_n = \lim_{k \to \infty} a_{n,k}$, then equation (3.10) follows.

Assertion (b) follows from (a) and Proposition A.5.

Now recall that the slide of $\tilde{p}$ over $h\tilde{p}$ comes from a single transverse crossing of ascending and descending manifolds. Under a sufficiently small perturbation of the deformation in $\hat{X}$, this crossing will persist, and no other such crossing will appear, by a compactness argument. So for a sufficiently small perturbation, $a_{1,k} = \pm 1$, and hence $a_1 = \pm 1$. This proves (c). $\square$

Lemma 3.8. *Suppose $\tilde{p} \in \tilde{\mathscr{C}}_i$ slides over $h\tilde{p}$. Then*

(a) *There is a power series $y = 1 + \sum_{n=1}^{\infty} b_n h^n$ such that $\zeta^+ = y \cdot \zeta^-$.*

(b) $b_1 = (-1)^{i+1} a_1$. *(See Lemma 3.7.)*

Proof. (a) By Lemma 2.2(b), a closed orbit can be created or destroyed in the bifurcation only if it is homologous to kh for some k. So $\log(\zeta^+) - \log(\zeta^-)$ is a power series in h. Thus ζ^+/ζ^- is a power series in h. (A priori the coefficients b_n are rational; it's not important here, but we actually know that $b_n \in \mathbb{Z}$, due to the product formula (1.2) for the zeta function.)

(b) Let $Z \subset X$ be a compact tubular neighborhood of the flow line γ from p to itself at t_0. There is a function $f : Z \to \mathbb{R}/\mathbb{Z}$ such that $\alpha|_Z = \lambda\, df$ for some $\lambda \in \mathbb{R}$. Let $\Sigma \subset Z$ be a level set for f away from p. The flow $-V$ induces a partially defined return map $\phi : \Sigma \to \Sigma$. Closed orbits homologous to h near γ are in one to one correspondence with fixed points of ϕ. A fixed point of ϕ is an intersection of the diagonal $\Delta \subset \Sigma \times \Sigma$ with the graph $\Gamma(\phi)$, and the Lefschetz sign of the closed orbit equals the sign of the intersection. The graph $\Gamma(\phi)$ has a natural compactification (see [HL1]) to a manifold with corners $\overline{\Gamma}$ whose codimension one stratum is

$$\partial\overline{\Gamma} = (A(p) \times D(p)) \cup Y.$$

Here $D(p)$ and $A(p)$ are the "first" intersections of the descending and ascending manifolds of p with Σ, and Y is a component arising from trajectories that escape the neighborhood Z. The number of closed orbits near γ changes whenever $D(p) \times A(p)$ crosses Δ. This is happening at time t_0 at a single point, transversely, and an orientation check shows that the sign is $(-1)^{i+1} a_1$. No other closed orbits homologous to h can be created or destroyed, as in Lemma 2.2(b). □

Remark 3.9. It should also be possible to prove (b) using a Floer-theoretic gluing argument to show that in the homology class h, a single closed orbit is created or destroyed.

Lemma 3.10. *Suppose $\tilde{p}$ slides over $h\tilde{p}$. Then $I^+ = I^-$.*

Proof. By Lemmas 3.7 and 3.8, we can write

$$I^+ = \left(\exp \sum_{n=2}^{\infty} c_n h^n\right) I^-. \tag{3.12}$$

for some $c_2, c_3, \ldots \in \mathbb{Q}$. We need to show that each coefficient c_k vanishes.

Let d denote the divisibility of h in H. Let $m : H \to \mathbb{Z}/dk$ be a homomorphism which sends $h \mapsto d$ and annihilates the torsion subgroup of H. Let $\rho : \hat{X} \to X$ be the corresponding finite cyclic cover. By Proposition 3.6 and Lemma 3.5,

$$I^+(\hat{X}) = \mathrm{Norm}\left(\exp\sum_{n=2}^{\infty} c_n h^n\right) I^-(\hat{X})$$

$$= \exp\left(dk\sum_{n=1}^{\infty} c_{kn} h^{kn}\right) I^-(\hat{X}).$$

As in Lemma 2.8, we can choose R sufficiently large that a bifurcation of length $> R$ in $\hat{X}$ near t_0 will not affect terms of order $[-\alpha](dkh)$ in $T_m(\hat{X})$ or $\zeta(\hat{X})$. Now perturb the deformation in $\hat{X}$ as in the proof of Lemma 3.7, so that modulo bifurcations of length $> R$, there are only slides of $h^i\tilde{p}$ over $h^j\tilde{p}$. When k does not divide $j - i$, we know by Lemma 3.4 that the torsion and zeta function in $\hat{X}$ do not change in such a slide. When $j - i$ divides k, we apply the analogue of (3.12) in the covering $\hat{X}$, to conclude that $I(\hat{X})$ gets multiplied by $1 + O(h^{2k})$.

It follows that $c_k = 0$, as long as we know that $I^-(\hat{X}) \neq 0$. If $CN_* \otimes F$ is acyclic for at least one of the subfields F of Λ, then $I^{\pm}(X) \neq 0$, and it follows from Lemma 3.5(c) and Proposition 3.6(b) that $I^{\pm}(\hat{X}) \neq 0$, completing the proof. If $CN_* \otimes F$ is not acyclic for any F, then $I^{\pm}(X) = 0$ and we have nothing to prove. □

Remark 3.11. The last paragraph of the above proof could be avoided by working with the *relative* torsion of the chain homotopy equivalence between CN_*^- and CN_*^+, cf. §5.

Remark 3.12. A theorem of Shil'nikov [A] asserts that in a generic bifurcation of this type, a unique irreducible closed orbit is created or destroyed. By the product formula (1.2), ζ gets multiplied by $(1 \pm h)^{\pm 1}$. By Lemma 3.10, we see *a posteriori* that T_m is also multiplied by such an expression. A possible direct explanation for this is that a flow line from $\tilde{p}$ to $h^n\tilde{p}$ is either created for all n or destroyed for all n.

3.6 Death of two critical points

We now analyze a semi-isolated *death bifurcation* given by the local model

(3.13) $\quad V = (x_1^2 + t - t_0, -x_2, \ldots, -x_i, x_{i+1}, \ldots, x_n)$

in some neighborhood U of the origin in $\mathbb{R}^n$. (*Birth* is obtained from death by reversing time. Hence there is no loss of generality in restricting attention to death. However we will see below in Proposition 3.13 that out of the death of two critical points comes an abundance of new life.)

At time t_0 there is a single degenerate critical point r. At time $t_0 + \varepsilon$, there are no critical points in U. At time $t_0 - \varepsilon$, there are two critical points $p = (-\sqrt{\varepsilon}, 0, \ldots, 0)$ and $q = (\sqrt{\varepsilon}, 0, \ldots, 0)$ of indices i and $i - 1$ respectively. Also there is a single downward gradient flow line in U from p to q in the positive x_1 direction, whose sign we denote by $(-1)^{\mu}$.

If $x, y \in X$ are critical points of index difference one, let $\mathcal{M}^-(x, y)$ denote the moduli space of flow lines from x to y immediately before the bifurcation. If in addition x, y are disjoint from p, q, let $\mathscr{M}^+(x, y)$ denote the moduli space of flow lines from x to y immediately after the bifurcation. These moduli spaces are well defined by the arguments in §2.1. Let $\mathscr{M}^0(r)$ denote the moduli space of flow lines from r to itself at the time of the bifurcation. Let $\mathscr{O}^-$ and $\mathscr{O}^+$ denote the moduli spaces of closed orbits before and after the bifurcation.

The following proposition says that for every (possibly multiply) broken flow line or closed orbit at time t_0, a new flow line or closed orbit is created after the two critical points die.

Proposition 3.13. (a) *There is an orientation preserving bijection*

$$\mathscr{O}^+ = \mathscr{O}^- \bigcup \left(\bigcup_{k=1}^{\infty} (-1)^{\mu k + k + i + 1} (\mathscr{M}^0(r))^{\times k} / (\mathbb{Z}/k) \right),$$

which preserves total homology classes of orbits. Here $\mathbb{Z}/k$ acts by cyclic permutations.

(b) *If x, y are critical points of index difference one which are disjoint from p, q, then there is an orientation preserving bijection*

$$\mathscr{M}^+(x, y) = \mathscr{M}^-(x, y) \cup \mathscr{M}^-(x, q) \times \bigcup_{k=0}^{\infty} (-1)^{(\mu+1)(k+1)} (\mathscr{M}^0(r))^{\times k} \times \mathscr{M}^-(p, y)$$

which preserves homology classes of flow lines.

Proof. In the calculations below, we will omit all orientations.

We first note that if x, y are disjoint from p, q, then no flow lines from x to y are destroyed, i.e. there is a natural inclusion $\mathscr{M}^-(x, y) \to \mathscr{M}^+(x, y)$. To see this, suppose to the contrary that a flow line is destroyed. Then by compactness as in Lemma 2.2(a), there is a sequence of flow lines from x to y before the bifurcation converging to a broken or degenerate flow line from x to y at time t_0. There are no degenerate flow lines at t_0 (by the definition of semi-isolated), so the limit flow line is broken, and the only place it can be broken is at r. In the neighborhood U, the broken flow line approaches r in the half space $(x_1 > 0)$ and leaves r in the half space $(x_1 < 0)$. But such a broken flow cannot be the limit as $\varepsilon \to 0$ of unbroken flow lines at time $t_0 - \varepsilon$, because there is a "barrier": At time $t_0 - \varepsilon$, a downward flow line cannot cross from $(x_1 > \sqrt{\varepsilon})$ to $(x_1 < \sqrt{\varepsilon})$ within the neighborhood U, since the downward gradient flow is in the positive x_1 direction for $|x_1| < \sqrt{\varepsilon}$.

Likewise, there is a natural inclusion $\mathscr{O}^- \to \mathscr{O}^+$.

To analyze what gets created, choose a small $\delta > 0$ and let $\Sigma_\pm := (x_1 = \pm\delta) \subset U$. Let $D := \Sigma_- \cap \mathscr{D}(r)$ and $A := \Sigma_+ \cap \mathscr{A}(r)$. For ε small, let $f_\varepsilon : \Sigma_+ \to \Sigma_-$ denote the partially defined map given by downward gradient flow at time $t_0 + \varepsilon$.

Consider a broken closed orbit obtained by concatenating flow lines $\gamma_1, \ldots, \gamma_k$ (in downward order) from r to itself. Choose δ small enough so that each γ_i crosses Σ_- immediately after leaving r and crosses Σ_+ immediately before returning. Let

$y_i \in D \subset \Sigma_-$ and $x_i \in A \subset \Sigma_+$ denote the corresponding intersections of γ_i with $\Sigma_\pm$. The downward flow defines a return map r_i from a neighborhood of y_i in Σ_- to a neighborhood of x_i in Σ_+.

A new closed orbit approximating the broken one gets created for each fixed point of the partially defined map

$$(3.14)\quad r_k \circ f_\varepsilon \circ \cdots \circ r_1 \circ f_\varepsilon : \Sigma_+ \to \Sigma_+$$

near x_k. We will prove below that the graph of (3.14) satisfies

$$(3.15)\quad \lim_{\varepsilon \to 0} \Gamma(r_k \circ f_\varepsilon \circ \cdots \circ r_1 \circ f_\varepsilon) = A \times r_k(D).$$

It follows that for ε small, the graph of (3.14) intersects the diagonal once near $x_k \times x_k$ transversely, because A intersects $r_k(D)$ once transversely at x_k. This proves (a). (Note that no additional closed orbits can be created, because by compactness as in Lemma 2.2(b), a closed orbit can be created only out of a broken closed orbit as above.)

To prove (b), suppose we have a broken flow line from x to y at time t_0 consisting of a flow line γ_0 from x to r, followed by the concatenation of flow lines $\gamma_1, \ldots, \gamma_k$ from r to itself as above, and finally a flow line γ_{k+1} from r to y. Let $D' \subset \Sigma_+$ and $A' \subset \Sigma_-$ denote the corresponding intersections with Σ_+ and Σ_- of the descending manifold of x and the ascending manifold of y. Let $\{x_0\} := \gamma_0 \cap D'$ and $\{y_{k+1}\} := \gamma_{k+1} \cap A'$. A new flow line is created for each intersection of the graph of the partially defined map

$$(3.16)\quad f_\varepsilon \circ r_k \circ f_\varepsilon \circ \cdots \circ r_1 \circ f_\varepsilon : \Sigma_+ \to \Sigma_-$$

with $D' \times A'$ near $x_0 \times y_{k+1}$. We will prove below that

$$(3.17)\quad \lim_{\varepsilon \to 0} \Gamma(f_\varepsilon \circ r_k \circ f_\varepsilon \circ \cdots \circ r_1 \circ f_\varepsilon) = A \times D.$$

It follows that for ε small, the graph of (3.16) intersects $D' \times A'$ once transversely near $x_0 \times y_{k+1}$, because A intersects D' transversely at x_0, and D intersects A' transversely at y_{k+1}. This proves (b).

We now prove equations (3.15) and (3.17). We first note that by the local model (3.13), we have

$$(3.18)\quad \lim_{\varepsilon \to 0} \Gamma(f_\varepsilon) = A \times D \subset \Sigma_+ \times \Sigma_-.$$

In general, if Y_1, Y_2, Y_3 are manifolds and $\phi_1 : Y_1 \to Y_2$ and $\phi_2 : Y_2 \to Y_3$ are any smooth maps, then $\Gamma(\phi_1) \times Y_3$ intersects $Y_1 \times \Gamma(\phi_2)$ transversely in $Y_1 \times Y_2 \times Y_3$ and

$$(3.19)\quad \Gamma(\phi_2 \circ \phi_1) = \pi_{1,3}((\Gamma(\phi_1) \times Y_3) \cap (Y_1 \times \Gamma(\phi_2))),$$

where $\pi_{1,3} : Y_1 \times Y_2 \times Y_3 \to Y_1 \times Y_3$ is the projection. Using (3.18) and (3.19) one proves (3.15) and (3.17) together by induction on k. $\square$

Let us now work out the algebraic consequences of the above lemma. Choose lifts $\tilde{p}$ and $\tilde{q}$ of p and q which coalesce at time t_0. Choose a basis for CN_*^- so that $\tilde{p}$ and $\tilde{q}$ are two of the basis elements. For CN_*^+, we can use the same basis with $\tilde{p}$ and $\tilde{q}$ deleted. Note that these bases correspond to the same Euler structure, by Definition B.1.

In the former basis, we can write the matrix for $\partial_i^- : CN_i^- \to CN_{i-1}^-$ in block form as

$$(3.20)\quad \partial_i^- = \begin{pmatrix} (-1)^\mu + \eta & v \\ w & N \end{pmatrix}.$$

Here w is a column vector corresponding to $\tilde{p}$, and v is a row vector corresponding to $\tilde{q}$. The power series η counts the flow lines in $\mathscr{M}^0(r)$ with their homology classes. Note that $\eta \in \Lambda^+$, so $(-1)^\mu + \eta$ is invertible.

We then have:

Corollary 3.14. (a) $T_m^- = ((-1)^\mu + \eta)^{(-1)^i} T_m^+$.
(b) $\zeta^+/\zeta^- = (1 + (-1)^\mu \eta)^{(-1)^i}$.

Proof. By Proposition 3.13(b), we have $\partial_j^+ = \partial_j^-$ for $j \neq i$, and

$$\partial_i^+ = N + \sum_{k=0}^{\infty} (-1)^{(\mu+1)(k+1)} w \eta^k v.$$

We can rewrite this as

$$(3.21)\quad \partial_i^+ = N - w((-1)^\mu + \eta)^{-1} v.$$

Now let F be a subfield of $Q(\Lambda)$, as in Lemma A.4. Choose decompositions $CN_*^+ \otimes F = D_*^+ \oplus E_*^+$ as in Proposition A.2. We can then get subbases for $CN_*^- \otimes F$ satisfying the conditions of Proposition A.2 by taking $D_i^- = D_i^+ \oplus \langle \tilde{p} \rangle$ and $E_{i-1}^- = E_{i-1}^+ \oplus \langle \tilde{q} \rangle$, and keeping the other subbases fixed. Let $N_s, v_s, w_s, \partial_s^\pm$ denote the restrictions and/or projections of the F components of $N, v, w, \partial_i^\pm$ to the appropriate subbases. Using (3.20) and (3.21), we compute

$$\begin{aligned} \det(\partial_s^- : D_i^- \to E_{i-1}^-) &= \det \begin{pmatrix} (-1)^\mu + \eta & v_s \\ w_s & N_s \end{pmatrix} \\ &= ((-1)^\mu + \eta) \det(N_s - w_s((-1)^\mu + \eta)^{-1} v_s) \\ &= ((-1)^\mu + \eta) \det(\partial_s^+ : D_i^+ \to E_{i-1}^+). \end{aligned}$$

Putting this into Proposition A.2 and summing over subfields F, we obtain (a). To prove (b), let us write

$$\eta = \sum_{m=1}^{\infty} x_m \in \Lambda^+$$

where there is one $x_m \in \pm H$ for each flow line from $\tilde{r}$ to $h\tilde{r}$ at time t_0. Then

$$\begin{aligned}\frac{\zeta^+}{\zeta^-} &= \exp \sum_{k=1}^{\infty} \sum_{m_1,\dots,m_k=1}^{\infty} \frac{(-1)^{\mu k+k+i+1}}{k} x_{m_1} \cdots x_{m_k} \\ &= \left(1 + (-1)^{\mu} \sum_m x_m\right)^{(-1)^i}.\end{aligned}$$

The first equality is a consequence of Proposition 3.13(a); the denominator k arises because summing over k-cycles and dividing by the period is equivalent to summing over k-tuples and dividing by k. The second equality can be verified by taking the logarithm of both sides. This proves (b). $\square$

Remark 3.15. In the above calculation, we used the fact that the determinant of a 2×2 block matrix is given by

$$\det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \det(\alpha)\det(\delta - \gamma\alpha^{-1}\beta),$$

provided that α is invertible. This identity played a key role in [HL2], in a different argument.

It follows from Corollary 3.14 that I is unchanged under the death bifurcation, and this completes the proof of Theorem A.

4 Proof of Theorem B (comparison)

Let (α, g) be admissible. We will now prove Theorem B, identifying our invariant $I(\alpha, g)$ with topological Reidemeister torsion.

We can reduce to the easier case of an exact one-form using the following trick, which we learned from a paper of Pajitnov [Pa1], who attributes it to F. Latour and J. Sikorav. Choose $f : X \to \mathbb{R}$ such that (df, g) is admissible, let $C \in \mathbb{R}$, and define

$$\beta := \alpha + C\,df.$$

Lemma 4.1. *If C is sufficiently large, then (β, g) is admissible, $g^{-1}\beta$ has no closed orbits, and there is a canonical isomorphism of chain complexes*

(4.1) $\quad CN_*(\beta) = CN_*(df) \otimes \Lambda$

respecting the bases determined by an Euler structure.

Proof. Since the Novikov complex is invariant under scaling, it makes no difference if we take $\beta = df + \varepsilon\alpha$ where ε is small.

Suppose γ is a closed orbit of $g^{-1}\beta$. The homology class of γ must be nonzero, since the cohomology class $[\alpha]$ pairs nontrivially with it. We can then put a lower bound on the length of γ away from the critical points. Since there is a positive lower bound on $|df|$ away from the critical points, we deduce a lower bound on $\int_\gamma (df + \varepsilon\alpha)$. If ε is sufficiently small, then the closed orbit γ cannot exist, or else we would get a positive lower bound on $\int_\gamma df$, contradicting the fact that $\int_\gamma df = 0$.

Transversality and intersection number are invariant under small perturbations, so if ε is sufficiently small, then the critical points of β will be small perturbations of the critical points of f and remain nondegenerate, and the ascending and descending manifolds will still intersect transversely with the same intersection numbers. This implies admissibility and (4.1). □

To prove Theorem B, choose a constant C sufficiently large for the conclusions of Lemma 4.1 to hold. By Theorem A and Lemma 4.1,

(4.2) $\quad I(\alpha, g) = I(\beta, g) = T_m(\beta, g).$

We now use (4.1) to relate $T_m(\beta, g)$ to $T_m(df, g)$. Note that the Novikov ring for df is the group ring $\mathbb{Z}[H]$. By Lemma A.4 we have decompositions

$$Q(\mathbb{Z}[H]) = \bigoplus_{j=1}^{m} F_j,$$

$$Q(\Lambda) = \bigoplus_{j=1}^{m} F_j'$$

into sums of fields such that $\imath(F_j) \subset F_j'$, where $\imath : Q(\mathbb{Z}[H]) \to Q(\Lambda)$ is the natural inclusion. By Proposition A.2 we see that $CN_*(df) \otimes F_j$ is acyclic if and only if $CN_*(df) \otimes F_j'$ is, and by (4.1),

(4.3) $\quad T_m(\beta, g) = \imath \circ T_m(df, g) : \mathscr{E}(X) \to \dfrac{Q(\Lambda)}{\pm 1}.$

By Example 1.8,

(4.4) $\quad \overline{T_m}(df, g) = \overline{T(\tilde{X})}.$

Equations (4.2), (4.3), and (4.4) prove Theorem B.

Remark 4.2. D. Salamon points out that instead of Lemma 4.1, one can use a lemma of Pozniak [Po] asserting that for any cohomology class $a \in H^1(X;\mathbb{R})$, there are admissible pairs (g_1, α) and (g_2, df), where $[\alpha] = a$, with identical vector fields $g_1^{-1}\alpha = g_2^{-1}\,df$.

Remark 4.3. A rigorous justification of the sketch in Example 1.8 would allow us to remove the bars from (4.4), in which case the above argument implies the refinement of Theorem B using Euler structures (1.5).

5 Conclusion

There are several directions in which the results of this paper might be generalized.

Algebraic refinements. There are sharper notions of torsion which are defined less often. The sharpest is Whitehead torsion [Co, Mi1], which is only defined for an acyclic complex over a ring R, and lives in the ring $\overline{K_1}(R)$. One can also define the relative Whitehead torsion of a chain homotopy equivalence between two complexes which need not be acyclic. A homotopy $\{(\alpha_t, g_t)\}$ between admissible pairs (α_0, g_0) and (α_1, g_1), with the cohomology class $[\alpha_t]$ fixed, induces a chain homotopy equivalence between the two Novikov complexes, via "continuation" (cf. [Po, Sc]). It should be possible to upgrade the algebra in Theorem A to show that *the Whitehead torsion of the continuation map equals the ratio of the two zeta functions*. Modulo Euler structures, and under slightly stronger genericity assumptions, this follows *a posteriori* from the paper of Pajitnov [Pa3], if one can show that the chain homotopy equivalence in [Pa3] commutes with continuation.

One might also generalize our results to nonabelian covers. We believe that if such a generalization exists, then the bifurcation analysis in this paper should suffice to prove it. The difficulty seems to be to formulate a result. In this direction, several earlier works, including [Si, Lat, Pa1, Pa2], investigated the Novikov complex for the universal cover and its Whitehead torsion; zeta functions for the universal cover were introduced in [GN].

Infinite dimensions. Floer theory considers finite dimensional moduli spaces of flow lines of closed 1-forms on certain infinite dimensional manifolds. Several people have suggested to us that for any such setup, one can at least formally define an analogue of our invariant I. Theorem A might generalize to prove that such a construct is invariant under exact deformations. (Whitehead torsion in Floer theory, without the zeta function, is studied in [Fu, Su].)

To give one example, consider the Floer theory of a symplectomorphism $f : X \to X$ of a symplectic manifold X. Let $M_f := X \times [0,1]/(x,1) \sim (f(x),0)$ denote the mapping torus of f. One defines a complex $CF_*(X,f)$ whose chains are fixed points of f and whose boundary operator counts pseudoholomorphic cylinders in $M_f \times \mathbb{R}$ which converge at either end to loops coming from fixed points. One can define the algebraic Reidemeister torsion of this complex just as in the finite dimensional case. Furthermore the analogue of the zeta function should count certain pseudoholomorphic tori in $M_f \times S^1(\lambda)$, where $S^1(\lambda)$ is the circle of radius λ. The signs of the tori can be defined using spectral flow, cf. [Ta2]. Due to the S^1 action, to get a moduli space of expected dimension zero, we must allow λ to vary. During a deformation, tori may disappear if $\lambda \to \infty$. However the energy of a long torus will be small on most of it, so part of the torus should be approaching a critical point, in which case we expect the loss of the torus to be reflected in a change in torsion as in bifurcations (4) and (5) on the list in §1.7.

We have tried to write the proof of Theorem A in such a way that it can be easily generalized to Floer theory. However a better understanding is needed of the gluing of multiply broken flow lines, which arises in bifurcations (4) and (5). The "non-equivariant perturbation" trick, which we used to evade this issue in bifurcation (4), does not appear to work for bifurcation (5), where we resorted in this paper to purely finite-dimensional methods.

We remark that Floer proved invariance of Floer homology by directly constructing a chain homotopy equivalence, without using bifurcation analysis. It seems however that bifurcation analysis is necessary to prove the invariance of torsion; roughly, one needs to see that the chain homotopy equivalence is composed out of a restricted set of matrix operations.

Other vector fields. The fact that our vector field V is dual to a *closed* 1-form is used mainly to give uniform bounds on the numbers of closed orbits and flow lines in a given homology class, so that finite counting is possible. Fried [Fr1] relates zeta functions to Reidemeister torsion for a rather different kind of vector field, assuming that there are no critical points. We do not know to what class of vector fields our results can be generalized. In the setting of combinatorial Morse theory, a statement resembling Theorem B was recently proved by Forman [Fo].

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A The algebra of Reidemeister torsion

In this appendix we review the algebra that underlies the definitions of topological and Morse-theoretic Reidemeister torsion, and which is needed starting in §1.5.

We call a complex (C_i, ∂) over a ring R *free* if each C_i is a free R-module, and *finite* if $\sum_i \mathrm{rk}(C_i) < \infty$. A *basis* b of a finite free complex consists of an ordered basis b_i for each C_i. We declare two bases b, b' to be equivalent if

$$\prod_i [b_{2i}, b'_{2i}] = \prod_i [b_{2i+1}, b'_{2i+1}] \in R,$$

where $[b_i, b'_i] \in R$ denotes the determinant of the change of basis matrix from b_i to b'_i. (We assume that the bases b_i and b'_i have the same cardinality, which could fail for pathological rings R.) We denote the set of equivalence classes by $\mathscr{B}(C_*)$.

If (C_*, ∂) is a finite complex over a field F, we define the Reidemeister torsion

$$\tau(C_*, \partial) : \mathscr{B}(C_*) \to F$$

as follows. The standard short exact sequences $0 \to Z_i \to C_i \xrightarrow{\partial} B_{i-1} \to 0$ and $0 \to B_i \to Z_i \to H_i \to 0$ give rise to isomorphisms

$$\det(C_i) \to \det(Z_i) \otimes \det(B_{i-1}),$$

$$\det(Z_i) \to \det(B_i) \otimes \det(H_i),$$

where 'det' denotes top exterior power. Putting the second isomorphism into the first gives an isomorphism

$$\det(C_i) \to \det(H_i) \otimes \det(B_i) \otimes \det(B_{i-1}).$$

When we take the alternating product of these isomorphisms over i, the B's cancel and we obtain an isomorphism

$$\text{(A.1)} \quad \mathscr{B}(C_*) = \bigotimes_i \det(C_i)^{\otimes(-1)^i} \to \bigotimes_i \det(H_i)^{\otimes(-1)^i}.$$

Definition A.1. If (C_*, ∂) is acyclic, then $\bigotimes_i \det(H_i)^{\otimes(-1)^i} = F$, and we define the *Reidemeister torsion* $\tau(C_*, \partial)$ to be the map (A.1). If (C_*, ∂) is not acyclic, we define $\tau(C_*, \partial) := 0$.

In practice, one can compute torsion as an alternating product of determinants of square submatrices of ∂. More precisely:

Proposition A.2. *Let (C_i, ∂) be a finite acyclic complex over a field F with a fixed basis b. We can find decompositions $C_i = D_i \oplus E_i$ such that:*

(i) *D_i and E_i are spanned by subbases of b_i, and*

(ii) *The map $\partial_s := \pi_{E_{i-1}} \circ \partial|_{D_i} : D_i \to E_{i-1}$ is an isomorphism.*

We then have

$$\tau(C_*, \partial)(b) = \pm \prod_i \det(\partial_s : D_i \to E_{i-1})^{(-1)^i}$$

where the determinants are computed using the subbases of b. □

Suppose now that C_* is a finite free complex over a ring R, such that the total quotient ring $Q(R)$ is a finite direct sum of fields,

$$Q(R) = \bigoplus_j F_j. \tag{A.2}$$

Definition A.3. [Tu3] Under the above assumption, we define

$$\tau(C_*, \partial) : \mathscr{B}(C_*) \to Q(R),$$
$$b \mapsto \sum_j \tau(C_* \otimes_R F_j, \partial \otimes 1)(b \otimes 1).$$

This depends only on R, i.e. the decomposition (A.2) is unique, because the fields F_j are characterized as the minimal ideals in $Q(R)$.

This definition applies to the complexes of interest in this paper, by:

Lemma A.4. *Let G be a finitely generated abelian group. Then:*

(a) *The total quotient rings of $\mathbb{Z}[G]$ and $\mathrm{Nov}(G;N)$ are finite sums of fields.*

(b) *These decompositions are compatible with the inclusion $\mathbb{Z}[G] \to \mathrm{Nov}(G;N)$.*

Proof. (cf. [Tu3]) Choose a splitting $G = K \oplus F$ where K is finite and F is free. Then $\mathbb{Z}[G] = \mathbb{Z}[K] \otimes \mathbb{Z}[F]$ and $\mathrm{Nov}(G;N) = \mathbb{Z}[K] \otimes \mathrm{Nov}(F;N)$. The total quotient ring of $\mathbb{Z}[K]$ is a finite sum of (cyclotomic) fields, $Q(\mathbb{Z}[K]) = \bigoplus_j L_j$. We then have

$$\begin{aligned} Q(\mathbb{Z}[G]) &= \bigoplus_j L_j \otimes Q(\mathbb{Z}[F]), \\ Q(\mathrm{Nov}(G;N)) &= \bigoplus_j L_j \otimes Q(\mathrm{Nov}(F;N)). \end{aligned} \tag{A.3}$$

A "leading coefficients" argument shows that $\mathbb{Z}[F]$ and $\mathrm{Nov}(F;N)$ are integral domains, so $L_j \otimes Q(\mathbb{Z}[F])$ and $L_j \otimes Q(\mathrm{Nov}(F;N))$ are fields. Thus equation (A.3) proves (a) and (b). □

The following "change of basis" formula is important in §3.

Proposition A.5. *Let (C_*, ∂) be a finite free complex over R, where $Q(R)$ is a finite sum of fields. If $A_* \in \mathrm{Aut}(C_*)$ preserves the grading, then*

$$\tau(C_*, A^{-1}\partial A) = \tau(C_*, \partial) \cdot \prod_i \det(A_i)^{(-1)^i}. \qquad \square$$

B Euler structures

In this appendix we explain how to resolve the H ambiguity in topological and Morse-theoretic Reidemeister torsion (cf. §1.5), using Turaev's Euler structures.

We begin with a definition of Euler structures which is slightly different from Turaev's. If v is a smooth vector field on X with nondegenerate zeroes, let $\mathscr{E}(X,v)$ denote the set of homology classes of 1-chains γ with $\partial\gamma = v^{-1}(0)$, where $v^{-1}(0)$ is oriented in the standard way. The set $\mathscr{E}(X,v)$ is a subset of the relative homology $H_1(X, v^{-1}(0))$, and it is an affine space modelled on $H_1(X)$. The set $\mathscr{E}(X,v)$ is non-empty because we are assuming $\chi(X) = 0$.

If v_0, v_1 are two such vector fields, define

$$\phi_{v_1,v_0} : \mathscr{E}(X,v_0) \to \mathscr{E}(X,v_1)$$

as follows. Let w be a section of $TX \to X \times [0,1]$ such that $v_i = w|_{X\times\{i\}}$ and $w^{-1}(0)$ is cut out transversely. The orientation convention gives $\partial w^{-1}(0) = v_1^{-1}(0) - v_0^{-1}(0)$. Suppose $\gamma \in \mathscr{E}(X,v_0)$. Since $H_1(X\times[0,1], X\times\{1\}) = 0$, there is a 2-chain $\Sigma \subset X \times [0,1]$ with $\partial\Sigma = -w^{-1}(0) - \gamma$ (rel $X \times \{1\}$). We define $\phi_{v_1,v_0}(\gamma) := \partial\Sigma + w^{-1}(0) + \gamma$.

Definition B.1. One can check that (a) ϕ_{v_1,v_0} is independent of w and Σ, (b) $\phi_{v,v} = \mathrm{id}$, and (c) $\phi_{v_2,v_0} = \phi_{v_2,v_1}\phi_{v_1,v_0}$. This implies that all the spaces $\mathscr{E}(X,v)$ are canonically isomorphic to a single affine space over $H_1(X)$. We denote this space by $\mathscr{E}(X)$ and call an element of it an *Euler structure*. We let $i_v : \mathscr{E}(X) \to \mathscr{E}(X,v)$ denote the canonical isomorphism.

It should be emphasized that the affine space $\mathscr{E}(X)$ is not canonically isomorphic to $H_1(X)$. For example, when v_0, v_1 have no zeroes, the map ϕ_{v_1,v_0} does not necessarily respect the identifications $\mathscr{E}(X,v_i) \simeq H_1(X)$.

Remark B.2. When $\dim(X) > 1$, Turaev [Tu2] defines a (smooth) Euler structure to be a nonsingular continuous vector field, modulo homotopy through vector fields which remain nonsingular in the complement of a ball during the homotopy. To go from our definition to Turaev's, represent $\gamma \in \mathscr{E}(X,v)$ by disjoint paths connecting the zeroes of v, and cancel the zeroes of v in a neighborhood of γ.

We now explain how Euler structures determine (equivalence classes of) bases for the Novikov complex.

Definition B.3. We define a map

$$\mathscr{E}(X) \to \mathscr{B}(CN_*)/\pm 1 \tag{B.1}$$

as follows. If there are no critical points, then $CN_i = \{0\}$, so $\mathscr{B}(CN_*) = H_1(X)$. In this case we define the map (B.1) to be the composition $\mathscr{E}(X) \xrightarrow{i_V} \mathscr{E}(X,V) = H_1(X)$.

If $V^{-1}(0) \neq \emptyset$, then given $\xi \in \mathscr{E}(X)$, we can represent $i_V(\xi) \in \mathscr{E}(X,V)$ by a chain

γ consisting only of paths connecting the zeroes of V, such that each critical point is in one component of γ. Choose a lift $\tilde{\gamma}$ of γ to $\tilde{X}$. The induced lifts of the zeroes of V to the endpoints of $\tilde{\gamma}$ determine a basis for CN_*.

The equivalence class of this basis does not depend on the choice of lift $\tilde{\gamma}$, because the boundary of each component of γ consists of two critical points whose indices have opposite sign. It is also independent of γ.

Given a one-parameter family $\{(\alpha_t, g_t)\}$, if there are no degenerate critical points for $t \in [t_1, t_2]$, then the canonical identification of critical points $\tilde{\mathscr{C}}(t_1) = \tilde{\mathscr{C}}(t_2)$ respects the bases determined by an Euler structure.

We now consider bases of the equivariant cell complex, along the lines of [Tu2]. There is a standard vector field v_i on the standard i-simplex with a sink at the center of the simplex, with no other zeroes in the interior, which restricts to v_j on each j-dimensional face, and which points inward near the boundary [Tu2]. Putting the vector fields v_i onto the simplices of our triangulation $\mathscr{T}$, we obtain a continuous vector field $v_{\mathscr{T}}$ on X. We can perturb this to a smooth vector field v with a non-degenerate zero of sign $(-1)^i$ in the center of each i-simplex.

Definition B.4. We define a map

$$\mathscr{E}(X) \to \mathscr{B}(C_*(\tilde{X}))/\pm 1$$

as follows. Given $\xi \in \mathscr{E}(X)$, represent $i_v(\xi) \in \mathscr{E}(X, v)$ by a chain γ consisting only of paths connecting the centers of the simplices in pairs. Choose a lift $\tilde{\gamma}$ of γ to $\tilde{X}$. Each simplex σ in X now has a unique lift in $\tilde{X}$ such that the center of σ is lifted to one of the points of $\partial\tilde{\gamma}$. These simplices in $\tilde{X}$ give a basis for $C_*(\tilde{X})$.

The equivalence class of this basis does not depend on the perturbation v, the path γ, or the lift $\tilde{\gamma}$.

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