Embedded contact homology and its applications

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Floer homology of 3-manifolds

Throughout this talk, Y denotes a closed oriented (connected) 3-manifold.

Definition

A spin-c structure on Y is an equivalence class of oriented 2-plane fields on Y, where two 2-plane fields are equivalent if they are homotopic on the complement of a ball.

The set of spin-c structures on Y is an affine space over $H^2(Y;\mathbb{Z})$.

Three isomorphic Floer theories for spin-c 3-manifolds

1. Seiberg-Witten Floer cohomology $\widehat{HM}^*(Y, \mathfrak{s})$ (Kronheimer-Mrowka), defined using solutions to the Seiberg-Witten equations on $\mathbb{R} \times Y$
2. Heegaard Floer homology $HF_*^+(Y, \mathfrak{s})$ (Ozsváth-Szabó), defined using a Heegaard splitting of Y
3. Embedded contact homology $ECH_*(Y, \mathfrak{s})$, defined using a contact form on Y.

Theorem (Taubes, 2008)

$\widehat{HM}^*$ is isomorphic to ECH_*.

Theorem (Kutluhan-Lee-Taubes,Colin-Ghiggini-Honda, 2010)

$\widehat{HM}^*(Y, \mathfrak{s})$ and $ECH_*(Y, \mathfrak{s})$ are isomorphic to $HF_*^+(-Y, \mathfrak{s})$.

Applications of ECH use these isomorphisms to transfer information between topology and contact geometry in three dimensions.

Contact geometry in 3 dimensions

A contact form on a closed oriented 3-manifold Y is a 1-form λ on Y such that $\lambda \wedge d\lambda > 0$ everywhere. A contact form λ determines:

- the contact structure $\xi = \mathrm{Ker}(\lambda)$ (an oriented 2-plane field),
- the Reeb vector field R satisfying $d\lambda(R, \cdot) = 0$ and $\lambda(R) = 1$.

Definition

A Reeb orbit is a closed orbit of R, i.e. a map $\gamma : \mathbb{R}/T\mathbb{Z} \to Y$ for some $T > 0$ such that $\gamma'(t) = R(\gamma(t))$ (modulo reparametrization).

λ is nondegenerate if all Reeb orbits are "cut out transversely". Generic contact forms have this property.

General question

How does the dynamics of the Reeb vector field R relate to the topology of the 3-manifold Y (and the contact structure ξ)?

For example, given (Y, ξ), what is the minimum number of (embedded) Reeb orbits of a contact form λ with $\mathrm{Ker}(\lambda) = \xi$?

3-dimensional Weinstein conjecture

For every contact form on a closed oriented 3-manifold, there exists a Reeb orbit.

Many partial results: Hofer, Abbas-Cieliebak-Hofer, Colin-Honda,...

Theorem (Taubes, 2006)

If $\Gamma \in H_1(Y)$ is such that $c_1(\xi) + 2\,\mathrm{PD}(\Gamma)$ is torsion in $H^2(Y;\mathbb{Z})$, then there exists a nonempty finite set of Reeb orbits $\{\alpha_i\}$ with $\sum_i [\alpha_i] = \Gamma$.

Slight improvement on Weinstein conjecture (H.-Taubes, 2008)

If λ is a nondegenerate contact form on a closed oriented 3-manifold Y, and if Y is not a lens space (or S^3), then there are at least three embedded Reeb orbits.

Definition of embedded contact homology

We now define the embedded contact homology $ECH_*(Y, \xi, \Gamma)$, where:

- Y is a closed oriented 3-manifold.
- ξ is a contact structure on Y.
- $\Gamma \in H_1(Y)$.

Homology of a chain complex depending also on:

- a nondegenerate contact form λ with $\operatorname{Ker}(\lambda) = \xi$.
- A generic almost complex structure J on $\mathbb{R} \times Y$ such that:
  - J is $\mathbb{R}$-invariant.
  - $J(\partial_s) = R$, where s denotes the $\mathbb{R}$ coordinate.
  - $J(\xi) = \xi$, and $d\lambda(v, Jv) \geq 0$ for $v \in \xi$.

Definition of the chain complex

The chain complex $ECC_*(Y, \lambda, \Gamma, J)$ is freely generated over $\mathbb{Z}$.
A generator is a finite set of pairs $\{(\alpha_i, m_i)\}$, where:

- The α_i's are distinct embedded Reeb orbits.
- The m_i's are positive integers.
- $\sum_i m_i[\alpha_i] = \Gamma \in H_1(Y)$.
- If α_i is hyperbolic (i.e. if its linearized return map has real eigenvalues) then $m_i = 1$.

If $\beta = \{(\beta_j, n_j)\}$ is another generator, then the differential coefficient $\langle \partial\alpha, \beta\rangle$ is a signed count of (possibly disconnected) J-holomorphic curves C in $\mathbb{R} \times Y$ (modulo $\mathbb{R}$ translation) such that:

- C has positive ends at covers of α_i with total multiplicity m_i.
- C has negative ends at covers of β_j with total multiplicity n_j.
- C has "ECH index" $I(C) = 1$.

Theorem (H.-Taubes, 2007)

$\partial^2 = 0$.

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Definition of the ECH index

$$I(C) = c_\tau(C) + Q_\tau(C) + \sum_i \sum_{k=1}^{m_i} \mathrm{CZ}_\tau(\alpha_i^k) - \sum_j \sum_{k=1}^{n_j} \mathrm{CZ}_\tau(\beta_j^k).$$

Here:

- τ is a trivialiation of ξ over the Reeb orbits α_i and β_j.
- $c_\tau(C)$ denotes the relative first Chern class of ξ over C with respect to τ.
- Q_τ is a "relative intersection form".
- CZ_τ denotes the Conley-Zehnder index with respect to τ.

Proposition

If $I(C) = 1$ then $C = C_0 \sqcup C_1$ where:

- *C_0 is a union of (covers of) $\mathbb{R}$-invariant cylinders.*
- *C_1 is embedded and lives in a 1-dimensional moduli space.*

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Taubes's isomorphism

There is a canonical isomorphism of relatively $\mathbb{Z}/d$-graded $\mathbb{Z}$-modules

$$ECH_*(Y, \xi, \Gamma) \simeq \widehat{HM}^{-*}(Y, [\xi] + \mathrm{PD}(\Gamma)).$$

Here d denotes the divisibility of $c_1(\xi) + 2\,\mathrm{PD}(\Gamma) = c_1([\xi] + \mathrm{PD}(\Gamma))$ in $H^2(Y; \mathbb{Z}) / \mathrm{Torsion}$.

Analogy (motivation for the definition of ECH)

Taubes's "Seiberg-Witten = Gromov" theorem relates the Seiberg-Witten invariants of a closed symplectic 4-manifold (X, ω) to a count of holomorphic curves C in X with $c_1(C) + C \cdot C = 0$.
The above isomorphism is an analogue of this for $X = \mathbb{R} \times Y$.

The Weinstein conjecture

The Weinstein conjecture in 3 dimensions follows immediately from Taubes's isomorphism, together with:

Theorem (Kronheimer-Mrowka)

If $\mathfrak{s}$ is a torsion spin-c structure, i.e. $c_1(\mathfrak{s})$ is torsion in $H^2(Y;\mathbb{Z})$, then $\widehat{HM}^*(Y,\mathfrak{s})$ is infinitely generated.

If there were no Reeb orbit, then we would have

$$ECH_*(Y,\xi,\Gamma) = \begin{cases} \mathbb{Z}, & \Gamma = 0, \\ 0, & \Gamma \neq 0. \end{cases}$$

Here the $\mathbb{Z}$ is generated by the empty set of Reeb orbits.

Simplest example: the boundary of an ellipsoid

Consider $\mathbb{R}^4 = \mathbb{C}^2$ with coordinates $z_j = x_j + iy_j$ for $j = 1, 2$.
If Y is the boundary of a star-shaped subset of $\mathbb{R}^4$, then

$$\lambda = \frac{1}{2}\sum_{j=1}^{2}(x_j dy_j - y_j dx_j)$$

restricts to a contact form on Y.

If $a, b > 0$, define the ellipsoid

$$E(a,b) = \left\{ (z_1, z_2) \in \mathbb{C}^2 \;\middle|\; \frac{\pi|z_1|^2}{a} + \frac{\pi|z_2|^2}{b} \leq 1 \right\}.$$

Compute $ECH(\partial E(a,b), \lambda, 0)\ldots$

- Reeb vector field
$$R = \frac{2\pi}{a}\frac{\partial}{\partial\theta_1} + \frac{2\pi}{b}\frac{\partial}{\partial\theta_2}.$$
If a/b is irrational, the embedded Reeb orbits are $\gamma_1 = (z_2 = 0)$ and $\gamma_2 = (z_1 = 0)$, which are elliptic. Chain complex generators are $\gamma_1^{m_1}\gamma_2^{m_2}$ where $m_1, m_2 \geq 0$.
- $\mathbb{Z}$-grading: normalize $I(\emptyset) = 0$, then
$$I\left(\gamma_1^{m_1}\gamma_2^{m_2}\right) = 2\left(m_1 + m_2 + m_1 m_2 + \sum_{k=1}^{m_1}\lfloor ka/b\rfloor + \sum_{k=1}^{m_2}\lfloor kb/a\rfloor\right).$$
- One generator of each even nonnegative integer grading, so
$$ECH_*(\partial E(a,b), \lambda, 0) = \begin{cases} \mathbb{Z}, & * = 0, 2, 4, \ldots, \\ 0, & \text{otherwise.} \end{cases}$$

Some additional structure on ECH

- When Y is connected, there is a map

$$U : ECH_*(Y, \xi, \Gamma) \to ECH_{*-2}(Y, \xi, \Gamma),$$

  induced by a chain map which counts $I = 2$ holomorphic curves in $\mathbb{R} \times Y$ passing through a chosen point in $\mathbb{R} \times Y$.

- There is a canonical element

$$c(\xi) \in ECH_*(Y, \xi, 0),$$

  called the contact invariant, represented by the empty set of Reeb orbits. Although ECH depends only on Y, the contact invariant $c(\xi)$ depends on ξ, and vanishes when ξ is overtwisted.

Both U and $c(\xi)$ agree with analogous structures on $\widehat{HM}^*$ and HF_*^+.

The symplectic action filtration

- If $\alpha = \{(\alpha_i, m_i)\}$ is an ECH generator, define its symplectic action
$$\mathcal{A}(\alpha) = \sum_i m_i \int_{\alpha_i} \lambda.$$
- The differential decreases symplectic action: if $\langle \partial\alpha, \beta\rangle \neq 0$ then $\mathcal{A}(\alpha) > \mathcal{A}(\beta)$.
- If $L \in \mathbb{R}$, define $ECH^L(Y, \lambda, \Gamma)$ to be the homology of the subcomplex spanned by generators with action less than L.
This is independent of J, but heavily dependent on λ.
No obvious counterpart in $\widehat{HM}$ or HF^+.
- Write
$$ECH^L(Y, \lambda) = \bigoplus_\Gamma ECH^L(Y, \lambda, \Gamma).$$

Cobordism maps

Let (Y_+, λ_+) and (Y_-, λ_-) be closed oriented (connected) 3-manifolds with nondegenerate contact forms. An exact symplectic cobordism from (Y_+, λ_+) to (Y_-, λ_-) is a compact symplectic four-manifold (X, ω) such that $\partial X = Y_+ - Y_-$ and there is a 1-form λ on X with $d\lambda = \omega$ and $\lambda|_{Y_\pm} = \lambda_\pm$.

Theorem (H.-Taubes, 2010)

An exact symplectic cobordism as above determines maps

$$\Phi^L(X, \omega) : ECH^L(Y_+, \lambda_+) \to ECH^L(Y_-, \lambda_-)$$

for each L, such that

$$\Phi(X, \omega) = \lim_{L \to \infty} \Phi^L(X, \omega) : ECH(Y_+, \lambda_+) \to ECH(Y_-, \lambda_-)$$

agrees with the induced map on $\widehat{HM}^*$. *Here we use* $\mathbb{Z}/2$ *coefficients.*

Proof uses Seiberg-Witten theory.

Comparison with symplectic field theory

SFT (defined by Eliashberg-Givental-Hofer) associates various chain complexes to a contact manifold (Y, λ) where the generators involve Reeb orbits and the differential counts holomorphic curves in $\mathbb{R} \times Y$.

- SFT is defined in $2n + 1$ dimensions, while ECH is only defined in 3 dimensions.
- SFT has more generators than ECH, and counts some non-embedded holomorphic curves.
- SFT depends on the contact structure ξ, and is trivial when ξ is overtwisted. ECH does not depend on ξ and does not vanish when ξ is overtwisted. (But the ECH contact invariant $c(\xi)$ does.)

Applications of ECH

1. Extensions of the three-dimensional Weinstein conjecture
2. The Arnold chord conjecture in three dimensions
3. Obstructions to symplectic embeddings in four dimensions

By Taubes's isomorphism $ECH_* \simeq \widehat{HM}^*$, and results of Kronheimer-Mrowka about $\widehat{HM}^*$, we know that:

- $ECH_*(Y, \xi, \Gamma)$ is finitely generated for each value of the grading $*$.
- $ECH_*(Y, \xi, \Gamma)$ is nonzero for only finitely many Γ.
- If $c_1(\xi) + 2\,\mathrm{PD}(\Gamma)$ is torsion in $H^2(Y; \mathbb{Z})$, then:
  - $ECH_*(Y, \xi, \Gamma)$ is infinitely generated.
  - $ECH_*(Y, \xi, \Gamma) = 0$ when $*$ is sufficiently small.
  - $U : ECH_*(Y, \xi, \Gamma) \to ECH_{*-2}(Y, \xi, \Gamma)$ is an isomorphism when $*$ is sufficiently large.

These facts have implications for contact geometry.

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Extensions of the Weinstein conjecture

Theorem (H.-Taubes, 2008)

Let λ be a nondegenerate contact form on a closed oriented connected 3-manifold Y such that all Reeb orbits are elliptic. Then there are exactly two embedded Reeb orbits and Y is a lens space (or S^3).

Outline of proof:

- Since all Reeb orbits are elliptic, all generators have even grading, so $\partial = 0$.
- Since $ECH_*(Y, \xi, \Gamma) \neq 0$ for only finitely many Γ, all Reeb orbits represent torsion homology classes, and $c_1(\xi)$ is torsion.
- Linear growth of $\mathrm{rk}(\bigoplus_{*\le k} ECH_*(Y, \xi))$ as a function of k implies that there are exactly two embedded Reeb orbits γ_1, γ_2.
- The holomorphic curves contributing to U include a cylinder which projects to an embedded cylinder in Y that generates a foliation of $Y \setminus (\gamma_1 \cup \gamma_2)$ by cylinders.
- This foliation determines a genus 1 Heegaard splitting of Y.

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- This foliation determines a genus 1 Heegaard splitting of Y.

- Similar arguments show that if λ is nondegenerate and if Y is not a lens space (or S^3), then there are at least three embedded Reeb orbits.
- This theorem can be used to extend the Weinstein conjecture to “stable Hamiltonian structures” (a generalization of contact forms) on 3-manifolds that are not T^2-bundles over S^1.

The Arnold chord conjecture in 3 dimensions

Let Y be a closed oriented connected 3-manifold with a contact form λ.

- A Legendrian knot in (Y, λ) is a knot $K \subset Y$ such that $TK \subset \xi|_K$.
- A Reeb chord of K is a path $\gamma : [0, T] \to Y$ for some $T > 0$ such that $\gamma'(t) = R(\gamma(t))$ and $\gamma(0), \gamma(T) \in K$.

Arnold chord conjecture in 3d (proved by H.-Taubes, 2010)

Every Legendrian knot in (Y, λ) has a Reeb chord.

Outline of proof:

Let (Y', λ') be obtained from (Y, λ) by Legendrian surgery along K.

- Y' is obtained from Y by surgery along K with framing $tb(K) - 1$.
- λ' agrees with λ outside of the surgery region.
- $\exists$ exact symplectic cobordism (X, ω) from (Y', λ') to (Y, λ).

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- $\exists$ exact symplectic cobordism (X, ω) from (Y', λ') to (Y, λ).

- Suppose to get a contradiction that K has no Reeb chord. Also assume that λ is nondegenerate.
- Then for any $L > 0$, one can arrange that λ' is nondegenerate and has the same Reeb orbits as λ up to action L.
- Then $\Phi^L(X, \omega) : ECH^L(Y', \lambda') \to ECH^L(Y, \lambda)$ is an isomorphism (induced by an upper triangular chain map).
- Therefore $\Phi(X, \omega) : ECH(Y', \lambda') \to ECH(Y, \lambda)$ is an isomorphism.
- By Kronheimer-Mrowka, this map fits into an exact triangle

$$\cdots \to \widehat{HM}^*(Y'') \to \widehat{HM}^*(Y') \to \widehat{HM}^*(Y) \to \cdots$$

  where Y'' is a different surgery along K.
- This contradicts the fact that $\widehat{HM}^*(Y'')$ is infinitely generated.
- Proof in degenerate case: In nondegenerate case, there exists a Reeb chord with an upper bound on the length that depends continuously on λ.

Obstructions to 4-dimensional symplectic embeddings

Define a (4-dimensional) Liouville domain to be a compact symplectic 4-manifold (X, ω) such that ω is exact, and there exists a 1-form λ on ∂X with $d\lambda = \omega|_{\partial X}$.

General question

If (X_0, ω_0) and (X_1, ω_1) are two Liouville domains, when does there exist a symplectic embedding $X_0 \to X_1$?

- Obvious necessary condition: $\mathrm{vol}(X_0, \omega_0) \leq \mathrm{vol}(X_1, \omega_1)$, where $\mathrm{vol}(X, \omega) = \frac{1}{2}\int_X \omega \wedge \omega$.
- This is far from sufficient, as shown by Gromov nonsqueezing.
- The answer to the question is unknown for some very simple examples.

ECH capacities

To each Liouville domain (X, ω) we associate a sequence of real numbers

$$0 = c_0(X, \omega) \leq c_1(X, \omega) \leq c_2(X, \omega) \leq \cdots \leq \infty,$$

called ECH capacities.

Definition (when ∂X is connected)

$c_k(X, \omega)$ is the least symplectic action needed to represent a class $\sigma \in ECH(\partial X, \lambda, 0)$ with $U^k \sigma = c(\xi)$.

Theorem

If there is a symplectic embedding $(X_0, \omega_0) \to (X_1, \omega_1)$*, then*

$$c_k(X_0, \omega_0) \leq c_k(X_1, \omega_1).$$

Proof: ECH cobordism map induced by $X_1 \setminus X_0$ respects U and $c(\xi)$ and decreases symplectic action.

Basic examples of ECH capacities

Proposition

$c_k(E(a,b)) = (a,b)_{k+1}$, where $(a,b)_k$ denotes the k^{th} smallest entry in the array $(am+bn)_{m,n\geq 0}$.

Proposition

$$c_k\left(\coprod_{i=1}^n (X_i,\omega_i)\right) = \max\left\{\sum_{i=1}^n c_{k_i}(X_i,\omega_i) \;\middle|\; \sum_{i=1}^n k_i = k\right\}.$$

Theorem (McDuff, 2010)

$\operatorname{int}(E(a,b))$ *symplectically embeds into* $\operatorname{int}(E(c,d))$ *if and only if* $c_k(E(a,b)) \leq c_k(E(c,d))$ *for all* k.

Fact (follows from work of Biran)

The ECH obstruction to symplectically embedding a disjoint union of balls into a ball is likewise sharp.

ECH capacities and volume

Conjecture (confirmed in many cases)

If $c_k(X,\omega) < \infty$ for all k, then

$$\lim_{k\to\infty} \frac{c_k(X,\omega)^2}{k} = 4\,\mathrm{vol}(X,\omega).$$

That is, asymptotically the ECH obstruction to a symplectic embedding recovers the volume constraint.

Some open questions

- Let Y be a closed oriented connected 3-manifold other than a lens space (or S^3). Does every contact form on Y have infinitely many embedded Reeb orbits?
- Is there a quantitative refinement of the Weinstein conjecture? For example, does a closed contact 3-manifold (Y, λ) always have a Reeb orbit with symplectic action at most $\sqrt{2\,\mathrm{vol}(Y, \lambda)}$?
- Can one prove that ECH depends only on the contact structure, and construct the ECH cobordism maps, by counting holomorphic curves (i.e. without using Seiberg-Witten theory)?
- Is there a direct explanation for why the ECH obstruction to symplectically embedding one ellipsoid into another is sharp? To what extent is the ECH obstruction sharp for other symplectic embeddings? (It's not always sharp for embedding a polydisk into an ellipsoid.)