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THE DOUBLE BUBBLE CONJECTURE

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ABSTRACT. The classical isoperimetric inequality states that the surface of smallest area enclosing a given volume in R^3 is a sphere. We show that the least area surface enclosing two equal volumes is a double bubble, a surface made of two pieces of round spheres separated by a flat disk, meeting along a single circle at an angle of $2\pi/3$.

1. INTRODUCTION

Double, double, toil and trouble,
Fire burn and cauldron bubble.
Macbeth Act 4, Scene 1, Line 10

The double bubble is the surface in R^3 obtained by taking two pieces of round spheres separated by a flat disk, meeting along a single circle at an angle of $2\pi/3$. It has long been thought that the double bubble minimizes area among all piecewise-smooth surfaces enclosing two equal volumes.

Experimental evidence towards this conjecture can be obtained by blowing soap bubbles and observing the resulting shapes. If one blows two soap bubbles of equal size and pushes them together until they conglomerate to form a compound bubble, one obtains a double bubble. Such experiments were carried out by the Belgian physicist J. Plateau in the middle of the 19th century. Plateau established experimentally that a soap bubble cluster is a piecewise-smooth surface having only two types of singularities. The first type of singularity occurs when three smooth surfaces come together along a smooth triple curve at an angle of 120^o. The second type of singularity occurs when six smooth surfaces and four triple curves converge at a point, with all angles equal. The angles are equal to those of the cone over the 1-skeleton of a regular tetrahedron.

C.V. Boys, discussing the work of Plateau in his famous book on soap bubbles [5] writes,

"When however the bubble is not single, say two have been blown in real contact with one another, again the bubbles must together take such a form that the total surface of the two spherical segments and of the part common to both, which I

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shall call the interface, is the smallest possible surface which will contain the two volumes of air and keep them separate."

We have obtained a proof of this conjecture for the case of two equal volumes.

Theorem 1. [9] *The double bubble uniquely minimizes area among all surfaces in R^3 enclosing two equal volumes.*

We remark that a planar analogue has recently been solved in [1].

Our result can also be viewed as an isoperimetric inequality.

Corollary 2. *For any surface in R^3 enclosing two regions, each having volume V, the area A satisfies*

$$A^3 \geq 243\pi V^2$$

with equality if and only if it is isomorphic to the standard symmetric double bubble enclosing two regions of volume V by an isometry of R^3.

This result gives the first explicit example of closed minimizing surfaces in R^3 which exhibits any of the singularities predicted by Plateau.

2. An outline of the proof

Existence and regularity of a minimizer were established by F. Almgren and J. Taylor. Almgren showed in [2] that there exists an area minimizing surface in R^3 among the set of surfaces enclosing a given pair of volumes. Here *surface* refers to a generalized notion used in geometric measure theory, which includes piecewise-smooth surfaces. Almgren showed that the solution is a smooth surface almost everywhere. Taylor obtained additional information on the nature of the singularities [14]. She showed that a minimizer is a piecewise-smooth surface whose singularities consist of smooth triple curves along which three smooth surfaces come together at an angle of 120^o, and isolated points where pieces of surface converge. At these isolated points the asymptotic cone is the cone over the 1-skeleton of a regular tetrahedron.

Our proof that the double bubble minimizes is by a direct computational attack on the space of surfaces. The space of surfaces enclosing two equal volumes is infinite dimensional. By a series of analytic and geometric arguments this space is reduced first to a union of finite dimensional sets, then a compact two-dimensional set, and ultimately the conjecture is reduced to a finite number of numeric computations.

It is a classical result that any surface minimizing area while enclosing a given volume has constant mean curvature on each smooth piece. The second ingredient in our proof is a general theorem about symmetry in soap bubble clusters.

Theorem 3. *An area minimizing enclosure of m volumes in R^n, for $m < n$, is rotationally symmetric about an $(m-1)$-dimensional plane.*

The ideas behind this theorem are due to Brian White and Frank Morgan, and versions of it are written in [8], [11] and [10]. It implies the classical isoperimetric theorem, and also tells us that a minimizing bubble enclosing two given regions in R^3 is a surface of revolution. Constant mean curvature surfaces of revolution in R^3 were classified by Delaunay [6], and they form the pieces of our minimizing bubble.

Almgren's theorem provides no information about the topological complexity of the possible solutions. The existence theorem allows for the strange possibility that the volumes enclosed may be disconnected. Even the exterior region may be disconnected, in which case there are "empty regions" which do not contribute to either of the two volumes we are enclosing. The main tool in controlling the topology is:

Theorem 4. [10] *If $A(V_1, V_2)$ is the minimum area for surfaces in R^3 enclosing volumes V_1 and V_2, then A is concave as a function of V_1 and V_2.*

The basic ingredient in the proofs of the symmetry theorem and the concavity theorem is the idea of "symmetrization". Suppose we have a soap bubble cluster and a hyperplane. This hyperplane divides the cluster into two halves. We can replace one half with the reflection of the other half across the hyperplane. The area (respectively volume) of the original cluster is the average of the area (respectively volume) of the two different symmetrizations.

In the special case when the hyperplane bisects both enclosed volumes and the cluster is area minimizing, both symmetrizations are minimizers, as otherwise one would have too little area. It follows that the cluster is orthogonal to the hyperplane, since otherwise the symmetrizations would have corners which could be smoothed to decrease area. Now the Borsuk-Ulam Theorem provides hyperplanes bisecting the volumes of a cluster, and one can find the axis of symmetry as an intersection of such hyperplanes.

If we could find hyperplanes dividing the volumes of a minimal cluster into other proportions, this would immediately imply concavity, since the areas of the symmetrizations must be greater than or equal to the areas of the minimizers for those volumes. The Borsuk-Ulam Theorem only allows us to bisect the volumes. However, another topological argument shows that if concavity fails, then there are extra hyperplanes of symmetry. In particular we find that a minimal bubble enclosing two volumes would have to be a union of concentric spheres, which is clearly not area minimizing.

To illustrate how concavity applies to connectedness, we can quickly deduce that there are no empty chambers. Concavity, together with the fact that $A(V_1, V_2) \to \infty$ as $V_1 \to \infty$ (by the isoperimetric theorem, since an enclosure of volumes V_1 and V_2 is also an enclosure of volume V_1), implies that $A(V_1, V_2)$ is a strictly increasing function of V_1 for V_2 fixed. Now, if an area-minimizing enclosure of volumes V_1 and V_2 has an empty chamber of volume E, then this is also an enclosure of volumes $V_1 + E$ and V_2. Then $A(V_1 + E, V_2) \leq A(V_1, V_2)$, a contradiction.

From Theorem 4 we deduce that the volumes are connected, and hence the minimizer has either the topology of the double bubble or of one other possible configuration. A *torus bubble* is a surface of revolution constructed by taking two circular arcs of the same radius, facing each other, each with one endpoint and center on the x-axis, and connecting the other endpoints with Delaunay curves meeting at 120 degrees. We then get a bubble surrounding two components, one homeomorphic to a torus and one homeomorphic to a ball. It is possible, though not immediately clear, to make such a construction so that the curves meet at 120^o angles, so that torus bubbles do indeed exist.

The possible torus bubbles may be parameterized as follows: choose a radius $r > 0$ for the arcs, angles θ_1 and θ_2 subtending the arcs, and mean curvature H_i for the inner Delaunay surface. Then the spherical pieces have mean curvature $2/r$ and the outer Delaunay surface must have mean curvature $H_o = 2/r - H_i \geq 0$.

The Delaunay curves are then determined by an ordinary differential equation and the initial conditions at either endpoint.

Geometric arguments show that given r, θ_1, and H_i, there are at most two values of θ_2 for which the curves can meet at the required 120^o angles, and these can be obtained algebraically by solving a quadratic equation. Perturbation arguments restrict the values for θ_2 that can occur in a minimizing bubble. One of these values is equal to θ_1 which gives rise to a symmetric torus bubble. We show that such symmetric bubbles are always unstable. Thus, the torus bubble is determined by r, θ_1, and H_i, and we can assume by scaling and reflection that $r = 1$ and $\theta_1 < \theta_2$.

We next do a computation to show that torus bubbles cannot be minimizers. The idea is to make an exhaustive search of all possible θ_1, H_i, where $0 \leq \theta_1 \leq \pi$ and $H_i \leq 2$. In each case, we show that either θ_2 does not exist in the appropriate range, or that the two Delaunay surfaces forming the boundary of the torus region do not match up when integrated, or that the two regions in the torus bubble have unequal volumes. It turns out that there are one-parameter families of torus bubbles which are critical points of the area function, but that if they enclose equal volumes then there is always a perturbation that will decrease their area while preserving both volumes.

The computation involves thousands of numerical integrations to get precise information about Delaunay surfaces. We use IEEE double precision arithmetic and interval arithmetic to derive strict bounds for all the estimates and calculations [4],[12]. A detailed proof appears in [9].

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