Applications of embedded contact homology

Michael Hutchings

Department of Mathematics
University of California, Berkeley

conference in honor of Michael Freedman
June 8, 2011

Outline

Floer homology of 3-manifolds

Throughout this talk, Y denotes a closed oriented (connected) 3-manifold.

Definition

A spin-c structure on Y is an equivalence class of oriented 2-plane fields on Y, where two oriented 2-plane fields are equivalent if they are homotopic on the complement of a ball.

The set of spin-c structures on Y is an affine space over $H^2(Y;\mathbb{Z})$.

We are interested in three particular kinds of Floer theory of a three-manifold Y with a spin-c structure $\mathfrak{s}$.

Seiberg-Witten Floer cohomology $\widehat{HM}^*(Y, \mathfrak{s})$

- Roughly speaking, this is the homology of a chain complex which is generated by $\mathbb{R}$-invariant solutions to the Seiberg-Witten equations on $\mathbb{R} \times Y$, and whose differential counts non-$\mathbb{R}$-invariant solutions. (Detailed definition by Kronheimer-Mrowka.)
- When $c_1(\mathfrak{s})$ is torsion, due to the presence of "reducible" solutions, the definition of $\widehat{HM}^*(Y, \mathfrak{s})$ is more complicated. In this case there is an alternate version $\check{HM}^*(Y, \mathfrak{s})$ of Seiberg-Witten Floer cohomology, and there is an exact triangle

$$\widehat{HM}^*(Y, \mathfrak{s}) \to \check{HM}^*(Y, \mathfrak{s}) \to \overline{HM}^*(Y, \mathfrak{s}) \to \widehat{HM}^{*+1}(Y, \mathfrak{s}) \to \cdots$$

Here $\overline{HM}^*(Y, \mathfrak{s})$ is determined by the reducibles and can be computed in terms of the triple cup product on Y.

Heegaard Floer homology $HF_*^+(Y,\mathfrak{s})$ (Ozsváth-Szabó)

- This is defined in terms of a Heegaard splitting of Y along a genus g surface Σ.
- It is a kind of Lagrangian Floer homology for two Lagrangians in $\mathrm{Sym}^g(\Sigma)$.
- Skilled practitioners can compute this in many examples, and Manolescu-Ozsváth-Thurston give a general algorithm for computing it with $\mathbb{Z}/2$ coefficients.

Contact forms

Let Y be a closed oriented 3-manifold.

Definition

A contact form on Y is a 1-form λ on Y such that $\lambda \wedge d\lambda > 0$.

A contact form λ determines:

- A Reeb vector field R, defined by $d\lambda(R, \cdot) = 0$ and $\lambda(R) = 1$.
- A contact structure, namely the oriented 2-plane field $\xi = \mathrm{Ker}(\lambda)$.

Definition

A Reeb orbit is a periodic orbit of R, i.e. a map $\gamma : \mathbb{R}/T\mathbb{Z} \to Y$ for some $T > 0$, modulo reparametrization, such that $\gamma'(t) = R(\gamma(t))$.

λ is called nondegenerate if all Reeb orbits are "cut out transversely". "Generic" contact forms have this property.

Embedded contact homology $ECH_*(Y, \lambda, \Gamma)$

Let Y be a closed oriented 3-manifold, let λ be a nondegenerate contact form on Y, and let $\Gamma \in H_1(Y)$.
One can then define the embedded contact homology $ECH_*(Y, \lambda, \Gamma)$.
This is the homology of a chain complex over $\mathbb{Z}$.
A generator of the chain complex is a finite set of pairs $\alpha = \{(\alpha_i, m_i)\}$ where:

- The α_i's are distinct embedded Reeb orbits.
- The m_i's are positive integers.
- $m_i = 1$ when α_i is hyperbolic.
- $\sum_i m_i[\alpha_i] = \Gamma \in H_1(Y)$.

To define the ECH differential, one chooses a generic almost complex structure J on $\mathbb{R} \times Y$ such that:

- $J(\partial_s) = R$, where s denotes the $\mathbb{R}$ coordinate.
- $J(\xi) = \xi$ and $d\lambda(v, Jv) \geq 0$ for $v \in \xi$.
- J is $\mathbb{R}$-invariant.

If α and β are two chain complex generators, the differential coefficient $\langle \partial\alpha, \beta \rangle$ counts certain (mostly) embedded index one J-holomorphic curves in $\mathbb{R} \times Y$, modulo the $\mathbb{R}$ action, which are asymptotic (as currents) to α as $s \to +\infty$ and to β as $s \to -\infty$.

Facts

- $\partial^2 = 0$. (H.-Taubes, 2007)
- The homology of the chain complex does not depend on J, and not really on λ either. (Follows from next slide.)

Isomorphism theorems

Fix a contact form λ with contact structure ξ. Suppose $\Gamma \in H_1(Y)$ is related to the spin-c structure $\mathfrak{s}$ by

$$\mathfrak{s} = [\xi] + \mathrm{PD}(\Gamma).$$

Theorem (Taubes, 2008)

$ECH_*(Y, \lambda, \Gamma)$ is canonically isomorphic to $\widehat{HM}^{-*}(Y, \mathfrak{s})$ as relatively graded groups.

This is a three-dimensional analogue of Taubes's "SW=Gr" theorem for closed symplectic four-manifolds.

Theorem (Kutluhan-Lee-Taubes, Colin-Ghiggini-Honda, 2010-)

Both are isomorphic to $HF^+_*(-Y, \mathfrak{s})$ as relatively graded groups.

These isomorphisms allow one to transfer information between topology and contact geometry in three dimensions.

Some additional structure on ECH

- There is a degree -2 map

$$U : ECH_*(Y, \lambda, \Gamma) \to ECH_{*-2}(Y, \lambda, \Gamma).$$

  This is induced by a chain map which is defined like ∂, but now counting index 2 holomorphic curves that pass through a base point in $\mathbb{R} \times Y$.

- There is a canonical class

$$c(\xi) \in ECH_*(Y, \lambda, 0),$$

  represented by the empty set of Reeb orbits.

These agree with analogous structures on $\widehat{HM}^*$ (and HF_*^+).

Applications of ECH

So far the main applications of ECH have been to the following:

- (Slight) extensions of the Weinstein conjecture in three dimensions.
- The Arnold chord conjecture in three dimensions.
- New, sometimes sharp obstructions to symplectic embeddings in four dimensions.

3d Weinstein conjecture

Every contact form on a closed oriented 3-manifold has a Reeb orbit.

Many partial results, e.g. Hofer, Abbas-Cieliebak-Hofer, Colin-Honda.

Theorem (Taubes, 2006)

Let λ be a contact form on a closed oriented 3-manifold Y. Let $\Gamma \in H_1(Y)$ such that $c_1(\xi) + 2\,\mathrm{PD}(\Gamma)$ is torsion in $H^2(Y;\mathbb{Z})$. Then there exists a nonempty finite set of (possibly multiply covered) Reeb orbits $\{\alpha_i\}$ with $\sum_i [\alpha_i] = \Gamma$.

Note that classes Γ as in the theorem always exist, because $c_1(\xi)$ is always divisible by 2.

The proof of the theorem is a first step in the proof of the isomorphism of ECH with $\widehat{HM}$, and the theorem also follows from this isomorphism (when λ is nondegenerate).

Deducing the Weinstein conjecture from $ECH \simeq \widehat{HM}$

Theorem (Kronheimer-Mrowka)

If $c_1(\mathfrak{s})$ is torsion then $\widehat{HM}^*(Y,\mathfrak{s})$ is infinitely generated.

Corollary

If λ is nondegenerate and if $c_1(\xi) + 2\,\mathrm{PD}(\Gamma)$ is torsion then there exists a nonempty ECH generator $\alpha = \{(\alpha_i, m_i)\}$ with $\sum_i m_i[\alpha_i] = \Gamma$.

Proof. Γ corresponds to $\mathfrak{s}$ with $c_1(\mathfrak{s})$ torsion, so by Taubes's isomorphism, $ECH_*(Y,\lambda,\Gamma)$ is infinitely generated. If there is no nonempty ECH generator in the class Γ, then

$$ECH_*(Y,\lambda,\Gamma) = \begin{cases} \mathbb{Z}, & \Gamma = 0, \\ 0, & \Gamma \neq 0. \end{cases}$$

which is a contradiction. (Here the $\mathbb{Z}$ is generated by the empty set of Reeb orbits.) □

Example: the boundary of an irrational ellipsoid

Identify $\mathbb{R}^4 = \mathbb{C}^2$ with coordinates $z_j = x_j + iy_j$ for $j = 1, 2$. Let $a, b > 0$ with a/b irrational. Consider the ellipsoid

$$E(a,b) = \left\{ (z_1, z_2) \in \mathbb{C}^2 \;\middle|\; \frac{\pi|z_1|^2}{a} + \frac{\pi|z_2|^2}{b} \le 1 \right\}.$$

Let $Y = \partial E(a,b)$. Since Y is transverse to rays from the origin,

$$\lambda = \frac{1}{2}\sum_{j=1}^{2}(x_j dy_j - y_j dx_j)$$

restricts to a contact form on Y. The Reeb vector field is given by

$$R = \frac{2\pi}{a}\frac{\partial}{\partial\theta_1} + \frac{2\pi}{b}\frac{\partial}{\partial\theta_2}.$$

The only embedded Reeb orbits are $\gamma_1 = (z_2 = 0)$ and $\gamma_2 = (z_1 = 0)$. These are elliptic, so the ECH generators are $\gamma_1^{m_1}\gamma_2^{m_2}$ where $m_1, m_2 \ge 0$. (The ECH differential vanishes here.)

I do not know if there is any example of a contact form on a closed 3-manifold with only finitely many Reeb orbits, other then the irrational ellipsoid example, and quotients of it on lens spaces.

Theorem (H.-Taubes, 2008)

Let λ be a nondegenerate contact form on a closed oriented connected three-manifold Y such that all Reeb orbits are elliptic. Then Y is S^3 or a lens space, there are exactly two embedded Reeb orbits, and these are the core circles of a genus 1 Heegaard splitting of Y.

The proof of this theorem uses the isomorphism $ECH_* \simeq \widehat{HM}^*$ much more strongly. The genus one Heegaard splitting is obtained from a foliation of $Y\setminus$ Reeb orbits by J-holomorphic curves counted by U.

Theorem (H.-Taubes, 2008)

If λ is a nondegenerate contact form on a closed 3-manifold Y, and if Y is not S^3 or a lens space, then there are at least three embedded Reeb orbits.

Additional structure on ECH

Other applications make use of two additional structures on ECH:

- The symplectic action filtration.
- Cobordism maps.

Symplectic action filtration

- If $\alpha = \{(\alpha_i, m_i)\}$ is an ECH generator, define its symplectic action by
$$\mathcal{A}(\alpha) = \sum_i m_i \int_{\alpha_i} \lambda.$$
- The differential decreases symplectic action: if $\langle \partial\alpha, \beta \rangle \neq 0$ then $\mathcal{A}(\alpha) > \mathcal{A}(\beta)$.
- Given $L \in \mathbb{R}$, define $ECH^L(Y, \lambda, \Gamma)$ to be the homology of the subcomplex generated by α with $\mathcal{A}(\alpha) < L$.
- This depends on λ (not just ξ), for example
$$ECH^{cL}(Y, c\lambda, \Gamma) = ECH^L(Y, \lambda, \Gamma)$$
for $c \in \mathbb{R}$. It has no known counterpart in $\widehat{HM}$ or HF^+.

Cobordism maps

Let (Y_+, λ_+) and (Y_-, λ_-) be closed oriented 3-manifolds with contact forms. An exact symplectic cobordism from (Y_+, λ_+) to (Y_-, λ_-) is a compact symplectic four-manifold (X, ω) with $\partial X = Y_+ - Y_-$ such that there exists a 1-form λ on X with $d\lambda = \omega$ and $\lambda|_{Y_\pm} = \lambda_\pm$.

Theorem (H.-Taubes, to appear soon)

An exact symplectic cobordism as above induces maps

$$\Phi^L(X, \omega) : ECH^L(Y_+, \lambda_+) \to ECH_*^L(Y_-, \lambda_-)$$

for each $L \in \mathbb{R}$ such that

$$\Phi(X, \omega) = \lim_{L \to \infty} \Phi^L(X, \omega) : ECH(Y_+, \lambda_+) \to ECH(Y_-, \lambda_-)$$

agrees with the induced map on $\widehat{HM}^*$.

Here we use $\mathbb{Z}/2$ coeffients and sum over all $\Gamma \in H_1(Y)$.

Cobordism maps and holomorphic curves

The cobordism map $\Phi^L(X, \omega)$ is constructed using Seiberg-Witten theory. It is presumably possible to define it by counting holomorphic curves instead, but this is technically difficult. In any case we know:

Holomorphic curves axiom

Given a suitable almost complex structure J on the "symplectization completion" $\overline{X}$ of X, the map $\Phi^L(X, \omega)$ is induced by a (noncanonical) chain map ϕ such that if $\alpha_\pm$ are ECH generators for $\lambda_\pm$ then:

- If $\langle \phi\alpha_+, \alpha_- \rangle \neq 0$ then there exists a (possibly broken) J-holomorphic curve in $\overline{X}$ from α_+ to α_-.
- If the only (possibly broken) J-holomorphic curve from α_+ to α_- is a union of "product cylinders", then $\langle \phi\alpha_+, \alpha_- \rangle = 1$.

The Arnold chord conjecture in 3 dimensions

Let Y be a closed oriented connected 3-manifold with a contact form λ.

- A Legendrian knot in (Y, λ) is a knot $K \subset Y$ such that $TK \subset \xi|_K$.
- A Reeb chord of K is a path $\gamma : [0, T] \to Y$ for some $T > 0$ such that $\gamma'(t) = R(\gamma(t))$ and $\gamma(0), \gamma(T) \in K$.

Arnold chord conjecture in 3d (proved by H.-Taubes, 2010)

Every Legendrian knot in (Y, λ) has a Reeb chord.

Outline of proof:

Let (Y', λ') be obtained from (Y, λ) by Legendrian surgery along K.

- Y' is obtained from Y by surgery along K with framing $tb(K) - 1$.
- λ' agrees with λ outside of the surgery region.
- $\exists$ exact symplectic cobordism (X, ω) from (Y', λ') to (Y, λ).

- Suppose to get a contradiction that K has no Reeb chord. Also assume that λ is nondegenerate.
- Then for any $L > 0$, one can arrange that λ' is nondegenerate and has the same Reeb orbits as λ up to action L.
- By the Holomorphic Curves axiom, $\Phi^L(X,\omega) : ECH^L(Y',\lambda') \to ECH^L(Y,\lambda)$ is induced by an upper triangular chain map, hence an isomorphism.
- Therefore $\Phi(X,\omega) : ECH(Y',\lambda') \to ECH(Y,\lambda)$ is an isomorphism.
- By Kronheimer-Mrowka, this map fits into an exact triangle

$$\cdots \to \widehat{HM}^*(Y'') \to \widehat{HM}^*(Y') \to \widehat{HM}^*(Y) \to \cdots$$

  where Y'' is a different surgery along K. ($\mathbb{Z}/2$ coefficients.)
- This contradicts the fact that $\widehat{HM}^*(Y'')$ is infinitely generated.
- Proof in degenerate case: In nondegenerate case, there exists a Reeb chord with an upper bound on the length that depends continuously on λ.

Summary of the preceding

So far we have used $ECH_* = \widehat{HM}^*$ as follows:

- Nontriviality of $\widehat{HM}^*$ implies the Weinstein conjecture.
- Additional properties of $\widehat{HM}^*$ lead to refinements of the Weinstein conjecture.
- Nonisomorphism of cobordism maps on $\widehat{HM}^*$ induced by Dehn surgeries implies the chord conjecture.

We now consider some further applications of ECH which do not use the isomorphism with $\widehat{HM}$ (except as currently needed to define ECH cobordism maps).

Four-dimensional symplectic embedding problems

Define a (4-dimensional) Liouville domain to be a compact symplectic 4-manifold (X, ω) such that ω is exact, and there exists a contact form λ on ∂X with $d\lambda = \omega|_{\partial X}$.

General question

If (X_0, ω_0) and (X_1, ω_1) are two Liouville domains, when does there exist a symplectic embedding $X_0 \to X_1$?

- Obvious necessary condition: $\mathrm{vol}(X_0, \omega_0) \le \mathrm{vol}(X_1, \omega_1)$, where $\mathrm{vol}(X, \omega) = \frac{1}{2}\int_X \omega \wedge \omega$.
- This is far from sufficient, as shown by Gromov nonsqueezing.
- The answer to the question is unknown, or only recently known, for some very simple examples.

Symplectic embeddings of four-dimensional ellipsoids

Given positive real numbers a and b, let $\mathcal{N}(a, b)$ denote the sequence of nonnegative integer linear combinations of a and b, arranged in nondecreasing order. For example.

$$\mathcal{N}(1, 2) = (0, 1, 2, 2, 3, 3, 4, 4, 4, 5, 5, 5, \ldots).$$

Theorem (McDuff, 2010)

int$(E(a, b))$ *symplectically embeds into* $E(c, d)$ *if and only if* $\mathcal{N}(a, b) \le \mathcal{N}(c, d)$.

The "only if" part of this theorem can be proved using ECH.

ECH capacities

To each Liouville domain (X, ω) we associate a sequence of real numbers

$$0 = c_0(X, \omega) \le c_1(X, \omega) \le c_2(X, \omega) \le \cdots \le \infty,$$

called ECH capacities. Below, consider ECH with $\mathbb{Z}/2$ coefficients.

Definition (when ∂X is connected)

$c_k(X, \omega)$ is the infimum of L such that there exists $\sigma \in ECH^L(\partial X, \lambda, 0)$ with $U^k \sigma = c(\xi)$.

Example

$\{c_k(E(a, b))\} = \mathcal{N}(a, b)$.

Proof. U is an isomorphism on the nonempty ECH generators and decreases symplectic action. The generator $\gamma_1^m \gamma_2^n$ has symplectic actiom $ma + nb$. □

Monotonicity of ECH capacities

Theorem

If there is a symplectic embedding $\varphi : (X_0, \omega_0) \to (X_1, \omega_1)$, then

$$c_k(X_0, \omega_0) \le c_k(X_1, \omega_1).$$

Proof. We can assume that $\varphi(X_0) \subset \mathrm{int}(X_1)$. Then $(X_1 \setminus \mathrm{int}(\varphi(X_0)), \omega_1)$ is a "weakly exact symplectic cobordism" from $(\partial X_1, \lambda_1)$ to $(\partial X_0, \lambda_0)$. Similarly to previous theorem, it induces a cobordism map

$$ECH^L(\partial X_1, \lambda_1, 0) \to ECH^L(\partial X_0, \lambda_0, 0)$$

for each $L \in \mathbb{R}$. This map respects U and $c(\xi)$ and decreases symplectic action. $\square$

More examples of ECH capacities

Proposition

$$c_k\left(\coprod_{i=1}^{n}(X_i,\omega_i)\right) = \max\left\{\sum_{i=1}^{n} c_{k_i}(X_i,\omega_i) \;\middle|\; \sum_{i=1}^{n} k_i = k\right\}.$$

McDuff has also shown that ECH capacities give a sharp obstruction to symplectically embedding a disjoint union of ellipsoids into an ellipsoid.

Example

Given $a, b > 0$, define the polydisk

$$P(a,b) = \{(z_1,z_2)\in\mathbb{C}^2 \mid \pi|z_1|^2 \le a,\ \pi|z_2|^2 \le b\}.$$

Then

$$c_k(P(a,b)) = \min\{am+bn \mid m,n\in\mathbb{N},\ (m+1)(n+1)\ge k+1\}.$$

More about ECH capacities

- Work of D. Müller implies that ECH capacities give a sharp obstruction to symplectically embedding an ellipsoid into a polydisk.
- However they do not give an sharp obstruction to symplectically embedding a polydisk into an ellipsoid, because $P(1,1)$ and $E(1,2)$ have the same ECH capacities, but $P(1,1)$ symplectically embeds into $E(a,2a)$ if and only if $a \geq 3/2$.

Conjecture (confirmed in many cases)

If $c_k(X,\omega) < \infty$ for all k, then

$$\lim_{k\to\infty} \frac{c_k(X,\omega)^2}{k} = 4\,\mathrm{vol}(X,\omega).$$

That is, asymptotically the ECH obstruction to a symplectic embedding recovers the volume constraint.

Some open questions

- Let Y be a closed oriented connected 3-manifold other than a lens space (or S^3). Does every contact form on Y have infinitely many embedded Reeb orbits?
- Is there a quantitative refinement of the Weinstein conjecture? For example, does a closed contact 3-manifold (Y, λ) always have a Reeb orbit with symplectic action at most $\sqrt{2\,\mathrm{vol}(Y, \lambda)}$?
- Can one prove that ECH depends only on the contact structure, and construct the ECH cobordism maps, by counting holomorphic curves (i.e. without using Seiberg-Witten theory)?
- Is there a direct explanation for why the ECH obstruction to symplectically embedding one ellipsoid into another is sharp? To what extent is the ECH obstruction sharp for other symplectic embedding problems?