# Some dynamical questions in symplectic geometry

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# Hamiltonian mechanics

- Symplectic geometry is the mathematics underlying Hamiltonian mechanics.
- In the simplest version of Hamiltonian mechanics, the state of a physical system is described by a point in $\mathbb{R}^{2n}$ with “position” variables $x_1, \ldots, x_n$ and “momentum” variables $y_1, \ldots, y_n$.
- Given a “Hamiltonian” or “energy” function

$$H : \mathbb{R}^{2n} \longrightarrow \mathbb{R},$$

the time evolution of the system is given by Hamilton's equations

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial y_i}, \quad \frac{dy_i}{dt} = -\frac{\partial H}{\partial x_i}.$$

That is, we follow trajectories of the Hamiltonian vector field

$$X_H = \sum_{i=1}^{n} \left( \frac{\partial H}{\partial y_i} \frac{\partial}{\partial x_i} - \frac{\partial H}{\partial x_i} \frac{\partial}{\partial y_i} \right).$$

# Periodic orbits

- It follows from the previous equations that $dH/dt = 0$ (conservation of energy). This means that we stay on a level set

$$Y = \left\{ z \in \mathbb{R}^{2n} \mid H(z) = k \right\}.$$

- We would like to understand the dynamics of the vector field $X_H$ on $Y$.
- A basic dynamical question is to understand the periodic orbits (loops in $Y$ which are trajectories of $X_H$). These correspond to physical behavior which repeats in time.
- If $k$ is a regular value of $H$, then up to scaling, the vector field $X_H$ on $Y$ does not depend on the Hamiltonian $H$ having $Y$ as a regular level set.
- It follows that up to reparametrization, periodic orbits depend only on $Y$. These are called closed characteristics of $Y$.

# Existence of closed characteristics

Let $Y$ be a compact hypersurface in $\mathbb{R}^{2n}$ which is “star-shaped” (transverse to the radial vector field).

**Theorem (Rabinowitz, 1970s)**

Every such hypersurface has a closed characteristic.

**Old conjecture**

Every such hypersurface has at least $n$ closed characteristics.

- Trivial for $n = 1$. In this case $Y$ is a circle which is itself a closed characteristic.
- $n = 2$ case proved by myself and (at the time) graduate student Dan Cristofaro-Gardiner in 2012.
- Open for $n > 2$ (various partial results).
- An irrational ellipsoid has exactly $n$ closed characteristics.
- We expect that a generic $Y$ has infinitely many closed characteristics.

# More general context

Let $Y$ be a $2n-1$ dimensional smooth manifold.

**Definition**

A contact form on $Y$ is a 1-form $\lambda$ on $Y$ such that $\lambda \wedge (d\lambda)^{n-1} \neq 0$.

A contact form $\lambda$ determines a Reeb vector field $R$ characterized by $d\lambda(R,\cdot) = 0$ and $\lambda(R) = 1$.

**Example**

If $Y$ is a star-shaped hypersurface in $R^{2n}$, then

$$\lambda = \frac{1}{2}\sum_{i=1}^{n}(x_i\,dy_i - y_i\,dx_i)$$

restricts to a contact form on $Y$. The Reeb vector field $R$ agrees with the Hamiltonian vector field $X_H$ up to rescaling.

## Example

If $(M, g)$ is an $n$ dimensional Riemannian manifold, then the unit cotangent bundle has a canonical contact form, and the Reeb vector field corresponds to geodesic flow.

## Weinstein conjecture

For every contact form on a closed $2n-1$ dimensional manifold $Y$, the Reeb vector field has at least one periodic orbit.

- Trivial when $n = 1$.
- Proved when $n = 2$ by Taubes in 2006.
- Open when $n > 2$.
- Further results in three dimensions: at least two periodic orbits, generically two or infinitely many periodic orbits, characterization of examples with two periodic orbits, ...

# Detecting periodic orbits

A basic tool for detecting periodic orbits is contact homology. There are different versions of this, but the basic idea is as follows.

- One defines a chain complex which is generated by (some) Reeb orbits (or combinations thereof).
- The differential counts some kind of pseudoholomorphic curves in $\mathbb{R} \times Y$.
- One wants to show that the homology of the chain complex has some topological invariance.
- Then nontriviality of this homology implies existence of periodic orbits!

For more information, see the "FAQ for prospective graduate students" at math.berkeley.edu/$\sim$hutching, and/or come to my virtual office hours from 11-12.