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Geometric Hodge star operator with applications to the theorems of Gauss and Green

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Abstract

The classical divergence theorem for an n-dimensional domain A and a smooth vector field F in n-space

$$\int_{\partial A} F \cdot n = \int_A divF$$

requires that a normal vector field $n(p)$ be defined a.e. $p \in \partial A$. In this paper we give a new proof and extension of this theorem by replacing n with a limit $\star\partial A$ of 1-dimensional polyhedral chains taken with respect to a norm. The operator $\star$ is a geometric dual to the Hodge star operator and is defined on a large class of k-dimensional domains of integration A in n-space called *chainlets*. Chainlets include a broad range of domains, from smooth manifolds to soap bubbles and fractals. We prove as our main result the Star theorem

$$\int_{\star A} \omega = (-1)^{k(n-k)} \int_A \star\omega$$

where ω is a k-form and A is an $(n-k)$-chainlet. When combined with the general Stokes' theorem ([**H1**, **H2**])

$$\int_{\partial A} \omega = \int_A d\omega$$

this result yields optimal and concise forms of Gauss' divergence theorem

$$\int_{\star\partial A} \omega = (-1)^{(k)(n-k)} \int_A d\star\omega$$

and Green's curl theorem

$$\int_{\partial A} \omega = \int_{\star A} \star d\omega.$$

1. *Introduction*

In this paper we develop a theory of calculus on a large class of domains by taking limits of k-dimensional polyhedral chains in n-space with respect to a one parameter family of norms depending on a parameter $r \geqslant 0$. We take r to be an integer; extensions to real r are possible ([**H5**]). Elements of the Banach spaces $\mathcal{N}_k^r$ obtained on completion are called *k-chainlets* of class N^r. The norms are decreasing with r. The direct limit of the $\mathcal{N}_k^r$ is a normed linear space $\mathcal{N}_k^\infty$. The parameter r reflects the *roughness* of the chainlets. Concepts such as smooth manifolds, fractals, vector fields, differential forms, foliations and measures have counterparts in chainlet geometry. There is no geometric wedge product for all pairs of chainlets as this would lead to multiplication of distributions. However, other products and operators on differential forms do have dual geometric versions on chainlets such as the Hodge star, Laplace and Dirac operators.

Chainlets of class N^r are domains of integration for smooth differential k-forms ω of class B^r, (i.e., the $r-1$ partial derivatives of ω are bounded uniformly and satisfy Lipschitz conditions). The geometric Hodge star operator is a linear operator $\star$ from k-chainlets of class N^r to $(n-k)$-chainlets of class N^r. It applies in all dimensions and codimensions and does not require that tangents be defined anywhere. Examples include the subgraph A of the Weierstrass nowhere differentiable function f defined over a compact interval, even though ∂A has infinite length and has no tangents defined in the graph of f. See Figure 1.

THEOREM 1·1 (Star theorem). *If A is a k-chainlet of class $N^r, r \geqslant 1$, and ω is a differential k-form of class B^r defined in a neighbourhood of spt$A$, then*

$$\int_{\star A} \omega = (-1)^{k(n-k)} \int_A \star\omega.$$

THEOREM 1·2 (Generalized Stokes' theorem). *If A is a k-chainlet of class $N^r, r \geqslant 0$, and ω is a differential $(k-1)$-form of class B^{r+1} defined in a neighbourhood of spt$A$, then*

$$\int_{\partial A} \omega = \int_A d\omega.$$

This was first announced in [**H1**] and proved in [**H2**].

Extensions of the divergence and curl theorems for smooth differential forms and rough chainlets in any dimension and codimension follow immediately.

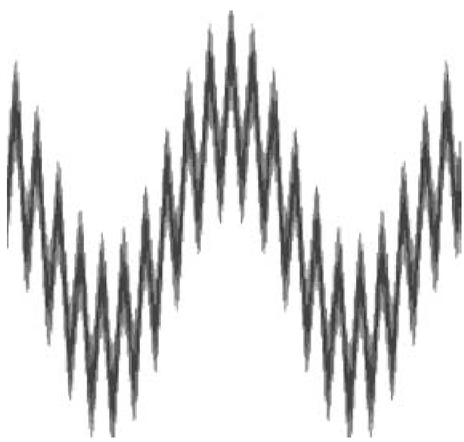

Fig. 1. The Weierstrass "nowhere differentiable" function.

Corollary 1·3 (Generalized Gauss divergence theorem). *If A is a k-chainlet of class $N^r, r \geqslant 0$, and ω is a differential $(n-k+1)$-form of class B^{r+1} defined in a neighbourhood of spt$A$, then*

$$\int_{\star\partial A} \omega = (-1)^{(k-1)(n-k+1)} \int_A d \star \omega.$$

At one extreme, the form ω must satisfy a Lipschitz condition so that $d \star \omega$ is bounded measurable. It can then be paired with a finite mass chainlet A of class N^0, e.g., a polyhedral chain, for this theorem to be satisfied. However, ∂A could have locally infinite mass as in Figure 1. At the other extreme, if ω is of class C^∞, the chainlet A is permitted to have any degree of roughness, from soap films to fractals.

Federer and de Giorgi [**F**, **deG**] proved a divergence theorem for n-dimensional currents C in $\mathbb{R}^n$ with $\mathcal{L}^n$ measurable support and a $\mathcal{H}^{n-1}$ measurable current boundary. The vector field F is assumed to be Lipschitz.

$$\int \partial C F(x) \cdot n(C,x) d\mathcal{H}^{n-1}x = \int_C div F(x) d\mathcal{L}^n x.$$

The hypotheses imply the existence a.e. of measure theoretic normals n(C, x) to the current boundary which is not required in Theorem 1·2. Our result applies to all chainlets in the Banach spaces $\mathcal{N}_k^r$ which include all currents satisfying the hypotheses of the theorem of Federer and de Giorgi. The divergence theorem was a hallmark of geometric theory. Federer wrote in the introduction to [**F**]:

> "A striking application of our theory is the Gauss-Green type theorem . . ."

and in the introduction to chapter 4:

> "Research on the problem of finding the most natural and general form of this theorem [Gauss-Green] has contributed greatly to the development of geometric measure theory."

Corollary 1·4 (Generalized Green's curl theorem). *If A is a k-chainlet of class $N^r, r \geqslant 1$, and ω is a differential $(k-1)$-form of class B^{r+1} defined in a neighbourhood of spt$A$, then*

$$\int_{\partial A} \omega = \int_{\star A} \star d\omega.$$

A geometric coboundary operator $\diamondsuit$ for chainlets is defined by

$$\diamondsuit := (-1)^{nk+n+1} \star \partial \star,$$

and a geometric Laplace operator $\square$ is defined using combinations of ∂ and $\star$:

$$\square := (\partial + \diamondsuit)^2 = \partial\diamondsuit + \diamondsuit\partial.$$

Let Δ denote the Laplace operator on differential forms.

Corollary 1·5 (Laplace operator theorem). *Let $r \geqslant 1$. If A is a k-chainlet of class N^r and ω is a differential k-form of class B^{r+2} defined in a neighbourhood of spt$A$, then*

$$\int_{\square A} \omega = (-1)^{n-1} \int_A \Delta\, \omega.$$

The main results in this paper were first announced in [**H4**]. The original draft, has been significantly improved with the use of discrete chainlet theory. The norms given below are initially defined for polyhedral k-chains in Euclidean space $\mathbb{R}^n$, and it is shown at the end how to extend the results to singular k-chains in Riemannian manifolds M^n.

2. *Norms on polyhedral chains*

A *cell* σ in R^n is the nonempty intersection of finitely many closed affine half spaces. The *dimension* of σ is k if k is the dimension of the smallest affine subspace E containing σ. The *support sptσ* of σ is the set of all points in the intersection of half spaces that determine σ.

Assume $k > 0$. An *orientation* of E is an equivalence class of ordered bases for the linear subspace parallel to E where two bases are equivalent if and only if their transformation matrix has positive determinant. An *orientation* of σ is defined to be an orientation of its subspace E. Henceforth, all k-cells are assumed to be oriented. (No orientation need be assigned to 0-cells which turn out to be single points $\{x\}$ in $\mathbb{R}^n$.) When σ is a simplex, each orientation determines an equivalence class of orderings of the set of vertices of σ.

An *algebraic k-chain* is a (formal) linear combination of oriented k-cells with coefficients in $\mathbb{F} = \mathbb{Z}$ or $\mathbb{R}$. The vector space of algebraic k-chains is the quotient of the vector space generated by oriented k-cells by the subspace generated by chains of the form $\sigma + \sigma'$ where σ' is obtained from σ by reversing its orientation. Let $P = \sum a_i \sigma_i$ be an algebraic k-chain. Following Whitney [**W**], define the function $P(x) := \sum a_i$ where the sum is taken over all i such that $x \in spt\sigma_i$. Set $P(x) := 0$ if x is not in the support of any σ_i. We say that algebraic k-chains P and Q are *equivalent* and write $P \sim Q$ if and only if the functions $P(x)$ and $Q(x)$ are equal except in a finite set of cells of dimension $< k$. For example, $(-1, 1) \sim (-1, 0) + (0, 1)$. A *polyhedral k-chain* is defined as an equivalence class of algebraic k-chains. This clever definition implies that if P' is a subdivision of the algebraic chain P, then P and P' determine the same polyhedral chain which behaves nicely for integrating forms. In particular, algebraic k-chains P and Q are equivalent iff integrals of smooth differential k-forms agree over them. (This property is sometimes taken as the definition of a polyhedral chain). If P is an algebraic chain, $[P]$ denotes the polyhedral chain of P. As an abuse of notation we usually omit the square brackets and write P instead of $[P]$. Denote the linear space of polyhedral chains by $\mathcal{P}_k$.

Remarks. Every polyhedral chain P has a nonoverlapping representative. Two cells that have the same coefficient, but opposite orientation, will cancel each other where they overlap. If they have the same orientation, their coefficients are added.

The standard boundary operator ∂ on k-cells σ produces an algebraic $(k-1)$-chain. This extends linearly to a boundary operator on algebraic k-chains. This, in turn, leads naturally to a well defined boundary operator ∂ on polyhedral k-chains for $k \geqslant 1$. For $k = 0$ we set $\partial P := 0$. Then

$$P_n \xrightarrow{\partial} P_{n-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} P_1 \xrightarrow{\partial} P_0$$

is a chain complex since $\partial \circ \partial = 0$. (We omit the proof which is standard.)

Mass of polyhedral chains

Let $M(\sigma)$ denote k-dimensional Lebesgue measure, or k-volume of a k-cell σ. Every 0-cell σ^0 takes the form $\sigma^0 = \{x\}$ and we set $M(\sigma^0) = 1$. The *mass* of P is defined by

$$M(P) \coloneqq \sum_{i=1}^{m} |a_i| M(\sigma_i)$$

where $P = \sum_{i=1}^{m} a_i \sigma_i$ and the cells σ_i are non-overlapping. We think of mass as weighted volume, or volume with multiplicity. For example, the mass of a piecewise linear curve with multiplicity two is twice its arc length. Mass is a norm on the vector space $\mathcal{P}_k$. Suppose $\sum_{i=1}^{m} a_i \sigma_i$ is a non-overlapping representative of P. The *support* of P is defined as $sptP \coloneqq \cup spt\sigma_i$.

It is worth noting to those well versed in analysis based on unions and intersections of sets that these definitions are substantially different and bring algebra of multiplicity and orientation into the mathematics at an early stage.

2·1. *The k-vector of a polyhedral chain*

([**W**, III]) The linear space of k-vectors in a vector space V is denoted $\Lambda^k(V)$. A k-vector is *simple* if it is of the form $v_1 \wedge \cdots \wedge v_k$. A simple k-vector is a *k-direction* if it has unit volume. A k-cell σ determines a unique k-dimensional subspace. This, together with its k-volume $M(\sigma)$, determines a unique simple k-vector, denoted $Vec(\sigma)$, with the same volume and direction as σ. Define the k-vector of an algebraic k-chain $A = \sum a_i \sigma_i$ by $Vec(A) \coloneqq \sum a_i Vec(\sigma_i)$. For $k = 0$, define $Vec(\sum a_i p_i) \coloneqq \sum a_i$. This definition extends to a polyhedral k-chain P since the k-vector of any chain equivalent to a k-cell is the same as the k-vector of the k-cell. The main purpose of introducing $Vec(A)$ in this paper is to show that the important k-elements defined below in Section 4 are, in fact, well defined.

PROPOSITION 2·1. *If P is a polyhedral k-chain, then $Vec(\partial P) = 0$.*

Proof. This follows since $Vec(\partial\sigma) = 0$ for every k-cell σ.

THEOREM 2·2. *Vec is a linear transformation*

$$Vec : \mathcal{P}_k \longrightarrow \Lambda^k(\mathbb{R}^n)$$

with

$$M(Vec(P)) \leqslant M(P)$$

for all $P \in \mathcal{P}_k$.

Proof. This follows since $M(Vec(\sigma)) = M(\sigma)$ for every k-cell σ.

Natural norms

For simplicity, we first define the norms in Euclidean space $\mathbb{R}^n$, and later show how to extend the definitions to Riemannian manifolds.

Difference cells

For $v \in \mathbb{R}^n$, let $|v|$ denote its norm and T_v translation through v. Let σ^0 be a k-cell in $\mathbb{R}^n$. For consistency of terminology we also call σ^0 a *difference k-cell of order zero.*

Let $v_1 \in \mathbb{R}^n$. Define the *difference k-cell of order* 1 as

$$\sigma^1 \coloneqq \sigma^0 - T_{v_1}\sigma^0.$$

This is a chain consisting of two cells, oppositely oriented. A simple example consists of the sum of the opposite faces of a cube, oppositely oriented. The chain is supported in these two faces. Given σ^0 and $v_1, \dots, v_r \in \mathbb{R}^n$, define the *difference k-cell of order j* recursively

$$\sigma^j \coloneqq \sigma^{j-1} - T_{v_j}\sigma^{j-1}.$$

A *difference k-cell chain* D^j of order j is a (formal) sum of difference k-cells of order j,

$$D^j = \sum_{i=1}^{m} a_i \sigma_i^j$$

with coefficients $a_i \in \mathbb{F}$. (See Figure 2.) The vector space of all difference k-cell chains of order j is denoted $\mathcal{D}_k^j$. The vector space of difference k-cell chains of order j is the quotient of the vector space generated by difference k-cells of order j by the subspace generated by difference chains of the form $(\sigma - T_v\sigma) + (\sigma' - T_v\sigma')$ where σ' is obtained from σ by reversing its orientation.

Difference norms

Given a difference k-cell σ^j of order j in $\mathbb{R}^n$ generated by a k-cell σ^0 and vectors $v_1, \dots, v_j$, define the *difference norms* $\|\sigma^0\|_0 \coloneqq M(\sigma^0)$ and for $j \geqslant 1$,

$$\|\sigma^j\|_j \coloneqq M(\sigma^0)|v_1||v_2| \cdots |v_j|.$$

For $D^j = \sum_{i=1}^m a_i \sigma_i^j$, possibly overlapping, define the *difference norm* as

$$\|D^j\|_j \coloneqq \sum_{i=1}^{m} |a_i| \|\sigma_i^j\|_j.$$

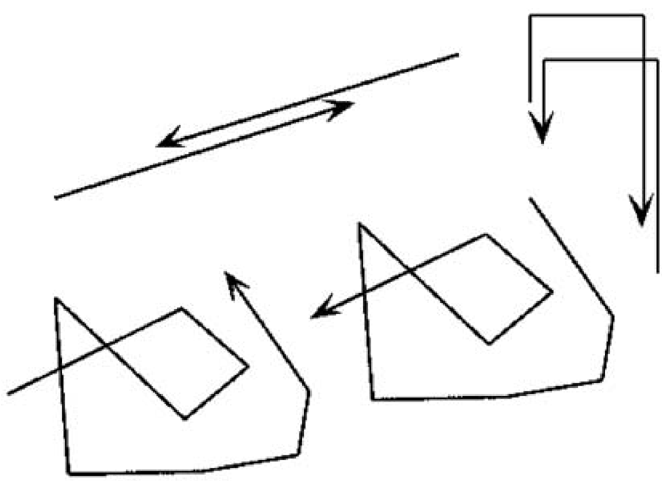

Fig. 2. A difference 1-cell chain of order 1.

r-natural norms

Let $P \in \mathcal{P}_k$ be a polyhedral k-chain. For $r = 0$ define

$$|P|^{\natural_0} := M(P).$$

For $r \geqslant 1$ define the *r-natural* norm

$$|P|^{\natural_r} := \inf\left\{\sum_{j=0}^{r} \|D^j\|_j + |C|^{\natural_{r-1}}\right\}$$

where the infimum is taken over all decompositions

$$P = \sum_{j=0}^{r} D^j + \partial C$$

where $D^j \in \mathcal{D}_k^j$ and $C \in \mathcal{P}_{k+1}$. It is clear $|\quad|^{\natural_r}$ is a semi-norm. We will prove that it is a norm.

It follows immediately from the definitions that the boundary operator on chains is bounded w.r.t. the r-natural norms.

Proposition 2·3. *If $P \in \mathcal{P}_k$, then*

$$|\partial P|^{\natural_r} \leqslant |P|^{\natural_{r-1}}.$$

3. *Isomorphisms of differential forms and cochains*

We recall two classical results from integral calculus:

Theorem 3·1 (Classical Stokes' theorem). *If P is a polyhedral k-chain and ω is a smooth k-form defined in a neighbourhood of P, then*

$$\int_{\partial P} \omega = \int_P d\omega.$$

Theorem 3·2 (Classical change of variables). *If P is a polyhedral k-chain, ω is a smooth k-form, and f is an orientation preserving diffeomorphism defined in a neighbourhood of P, then*

$$\int_{fP} \omega = \int_P f^*\omega.$$

The flat norm

Whitney's flat norm on polyhedral chains $A \in \mathcal{P}_k$ is defined as follows:

$$|A|^\flat := \inf\{M(B) + M(C) : A = B + \partial C, B \in \mathcal{P}_k, C \in \mathcal{P}_{k+1}\}.$$

Flat k-forms ([**W**, 12·4]) are characterized as all bounded measurable k-forms ω such that there exists a constant $C > 0$ with $\sup |\int_\sigma \omega| < CM(\sigma)$ for all k-cells σ and $\sup |\int_{\partial\tau} \omega| < CM(\tau)$ for all $(k+1)$-cells τ. The exterior derivative $d\omega$ of a flat form ω is defined a.e. and satisfies

$$\int_{\partial\tau} \omega = \int_\tau d\omega.$$

The *support* of a differential form is the closure of the set of all points $p \in \mathbb{R}^n$ such that $\omega(p)$ is nonzero. Let ω be a bounded measurable k-form.

Define

$$\|\omega\|_0 := \sup\left\{\frac{\int_\sigma \omega}{M(\sigma)} : \sigma \text{ is a k-cell}\right\}.$$

Recursively define

$$\|\omega\|_r := \sup\left\{\frac{\|\omega - T_v\omega\|_{r-1}}{|v|}\right\}.$$

Define

$$\|\omega\|_0' := \sup\left\{\frac{\int_{\partial\tau} \omega}{M(\tau)} : \tau \text{ is a (KH)-cell}\right\}$$

and

$$\|\omega\|_r' := \sup\left\{\frac{\|\omega - T_v\omega\|_{r-1}'}{|v|}\right\}.$$

Define

$$|\omega|_0 := \|\omega\|_0$$

and for $r \geqslant 1$,

$$|\omega|_r := \max\{\|\omega\|_o, \ldots, \|\omega\|_r, \|\omega\|_0', \ldots, \|\omega\|_{r-1}'\}.$$

We say that ω is of class B^r if $|\omega|_r < \infty$. Let $\mathcal{B}_k^r$ denote the space of differential k-forms of class B^r.

LEMMA 3·3. *If $|\omega|_1 < \infty$, then $d\omega$ is defined a.e. Furthermore,*

$$\int_{\partial\sigma} \omega = \int_\sigma d\omega.$$

Proof. If $|\omega|_1 < \infty$, then ω is a flat form. It follows from ([**W**, 12·4]) that $d\omega$ is defined a.e. and satisfies Stokes' theorem on cells.

LEMMA 3·4. *If $\omega \in \mathcal{B}_k^r$, $r \geqslant 1$, then*

$$|\omega|_r = \max\{\|\omega\|_o, \ldots, \|\omega\|_r, \|d\omega\|_0, \ldots, \|d\omega\|_{r-1}\}.$$

Therefore

$$|d\omega|_{r-1} \leqslant |\omega|_r. \tag{3·1}$$

The next result generalizes the standard integral inequality of calculus:

$$\left|\int_P \omega\right| \leqslant M(P)|\omega|_o$$

where P is polyhedral and ω is a bounded, measurable form.

THEOREM 3·5 (Fundamental integral inequality of chainlet geometry). *Let $P \in \mathcal{P}_k$, $r \in \mathbb{Z}^+$, and $\omega \in \mathcal{B}_k^r$ be defined in a neighbourhood of spt$P$. Then*

$$\left|\int_P \omega\right| \leqslant |P|^{\natural_r} |\omega|_r.$$

Proof. We first prove $\left|\int_{\sigma^j} \omega\right| \leqslant \|\sigma^j\|_j \|\omega\|_j$. Since $\|\omega\|_0 = |\omega|_0$ we know

$$\left|\int_{\sigma^0} \omega\right| \leqslant M(\sigma^0)|\omega|_0 = \|\sigma^0\|_0 \|\omega\|_0.$$

Use the change of variables formula 3·2 for the translation T_{v_j} and induction to deduce

$$\begin{aligned} \left|\int_{\sigma^j} \omega\right| = \left|\int_{\sigma^{j-1} - T_{v_j}\sigma^{j-1}} \omega\right| &= \left|\int_{\sigma^{j-1}} \omega - T^*_{v_j}\omega\right| \\ &\leqslant \|\sigma^{j-1}\|_{j-1} \|\omega - T^*_{v_j}\omega\|_{j-1} \\ &\leqslant \|\sigma^{j-1}\|_{j-1} \|\omega\|_j |v_j| \\ &= \|\sigma^j\|_j \|\omega\|_j. \end{aligned}$$

By linearity

$$\left|\int_{D^j} \omega\right| \leqslant \|D^j\|_j \|\omega\|_j$$

for all $D^j \in \mathcal{D}^j_k$.

We again use induction to prove $\left|\int_P \omega\right| \leqslant |P|^{\natural_r} |\omega|_r$. We know $\left|\int_P \omega\right| \leqslant |P|^{\natural_0}|\omega|_0$. Assume the inequality holds for $r-1$.

Let $\varepsilon > 0$. There exists $P = \sum_{j=0}^r D^j + \partial C$ such that $|P|^{\natural_r} > \sum_{j=0}^r \|D^j\|_j + |C|^{\natural_{r-1}} - \varepsilon$. By Stokes' theorem for polyhedral chains, inequality (3·1) and induction

$$\begin{aligned} \left|\int_P \omega\right| &\leqslant \textstyle\sum_{j=0}^r \left|\int_{D^j} \omega\right| + \left|\int_C d\omega\right| \\ &\leqslant \textstyle\sum_{j=0}^r \|D^j\|_j \|\omega\|_j + |C|^{\natural_{r-1}} |d\omega|_{r-1} \\ &\leqslant (\textstyle\sum_{j=0}^r \|D^j\|_j + |C|^{\natural_{r-1}})|\omega|_r \\ &\leqslant (|P|^{\natural_r} + \varepsilon)|\omega|_r. \end{aligned}$$

Since the inequality holds for all $\varepsilon > 0$ the result follows.

COROLLARY 3·6. $|P|^{\natural_r}$ *is a norm on the space of polyhedral chains* $\mathcal{P}_k$.

Proof. Suppose $P \neq 0$ is a polyhedral chain. There exists a smooth differential form ω such that $\int_P \omega \neq 0$. Then $0 < \left|\int_P \omega\right| \leqslant |P|^{\natural_r} |\omega|_r$ implies $|P|^{\natural_r} > 0$.

The Banach space of polyhedral k-chains $\mathcal{P}_k$ completed with the norm $|\quad|^{\natural_r}$ is denoted $\mathcal{N}^r_k$. The elements of $\mathcal{N}^r_k$ are called *k-chainlets of class* N^r.

It follows from Proposition 2·3 that the boundary ∂A of a k-chainlet A of class N^r is well defined as a $(k-1)$-chainlet of class N^{r+1}. If $P_i \to A$ in the r-natural norm define

$$\partial A := \lim_{i\to\infty} \partial P_i.$$

By Theorem 3·5, the integral $\int_A \omega$ is well defined for k-chainlets A of class N^r and differential k-forms of class B^r. If $P_i \to A$ in the r-natural norm define

$$\int_A \omega := \lim_{i\to\infty} \int_{P_i} \omega.$$

Examples of chainlets

(i) *The boundary of any bounded, open subset U of $\mathbb{R}^n$.* One may easily verify that the boundary of any bounded, open set $U \subset \mathbb{R}^n$, such as the Van Koch snowflake, supports a well defined chainlet $B = \partial U$ of class N^1. Suppose the frontier of U has positive Lebesgue area. Then the chainlet $B' = \partial(\mathbb{R}^n - \overline{U})$ has the same support as B, namely the frontier of U, but since $B + B'$ bounds a chainlet with positive mass it follows that B and $-B'$ are distinct chainlets. (The definition of the support of a chainlet follows Corollary 3·15 below.) We are accustomed to finding the inner boundary and outer boundary of U by approximating γ with polyhedral chains supported inside or outside U. We find two distinct chainlet boundaries B_1, B_2 of U as their difference bounds a region with nonzero area. However, the chainlet boundaries are supported in the same set, namely the frontier of U.

(ii) *Graphs of functions.* The graph of a nonnegative Riemann integrable function $f\colon K \subset \mathbb{R}^n \to \mathbb{R}$ supports a chainlet Γ_f if K is compact. This can be seen by approximating Γ_f by the polyhedral chains P_k determined by a sequence of step functions g_k approximating f. The difference $P_k - P_{k+j}$ is a difference k-cell chain of order 1. The *subgraph* of a nonnegative function f is the area between the graph of f and its domain. Since the subgraph of f has finite area, it follows that $\|P_k - P_{k+j}\|_1 \to 0$ as $j, k \to \infty$. Hence, the sequence P_k is Cauchy in the 1-natural norm. The boundary $\partial\Gamma_f$ is a chainlet that identifies the discontinuity points of f.

Characterization of cochains as differential forms

The r-natural norm of a cochain $X \in (\mathcal{N}^r)'$ is defined by

$$|X|^{\natural_r} := \sup_{P \in \mathcal{P}} \frac{|X \cdot P|}{|P|^{\natural_r}}.$$

The differential operator d on cochains is defined as the dual to the boundary operator $dX \cdot A := X \cdot \partial A$. It remains to show how cochains relate to differential forms and how the operator d given above relates to the standard exterior derivative of differential forms. If $X \in (\mathcal{N}_k^r)'$ then $dX \in (\mathcal{N}_{k+1}^{r-1})'$ by Lemma 2·3.

Cochains and differential forms

In this section we show the operator $\mathbf{\Psi}$ mapping differential forms of class B^r into the dual space of chainlets of class N^r via integration

$$\mathbf{\Psi}(\omega) \cdot A := \int_A \omega$$

is a norm preserving isomorphism of graded algebras.

It follows from Theorem 3·5 that$\mathbf{\Psi}(\omega) \in (\mathcal{N}_k^r)'$ with

$$|\mathbf{\Psi}(\omega)|^{\natural_r} \leqslant |\omega|_r.$$

THEOREM 3·7 (Extension of the theorem of de Rham to cochains and forms). *Let $r \geqslant 0$. To each cochain $X \in \left(\mathcal{N}_k^r\right)'$ there corresponds a unique differential form*

$\phi(X) \in \mathcal{B}_k^r$ such that $\psi \circ \phi = \phi \circ \psi = Id$. This correspondence is an isomorphism with

$$|X|^{\natural_r} = |\phi(X)|_r.$$

If $r \geqslant 1$, then

$$\phi(dX) = d\phi(X).$$

This is proved in [**H3**].

Corollary 3·8. *If $A, B \in \mathcal{N}_k^r$ satisfy*

$$\int_A \omega = \int_B \omega$$

for all $\omega \in \mathcal{B}_k^r$, then $A = B$.

Proof. Let $X \in (\mathcal{N}_k^r)'$. By Theorem 3·7 the form $\phi(X)$ is of class B^r. Hence

$$X \cdot (A - B) = \int_{A-B} \phi(X) = 0.$$

It follows that $A = B$.

Corollary 3·9. *If $A \in \mathcal{N}_k^r$ then*

$$|A|^{\natural_r} = \sup\left\{ \int_A \omega : \omega \in B_k^r, |\omega|_r \leqslant 1 \right\}.$$

Proof. By Theorem 3·7

$$\begin{aligned} |A|^{\natural_r} &= \sup\left\{ \tfrac{|X \cdot A|}{|X|^{\natural_r}} : X \in (\mathcal{N}_k^r)' \right\} \\ &= \sup\left\{ \tfrac{|\int_A \phi(X)|}{|\phi(X)|_r} : \phi(X) \in \mathcal{B}_k^r \right\} \\ &= \sup\left\{ \tfrac{|\int_A \omega|}{|\omega|_r} : \omega \in \mathcal{B}_k^r \right\}. \end{aligned}$$

Cup product

Given a k-cochain X and a j-cochain Y, we define their *cup product* as the $(j+k)$-cochain

$$X \cup Y := \Psi(\phi(X) \wedge \phi(Y)).$$

The next result follows directly from Theorem 3·7.

Lemma 3·10. *Given $X \in (\mathcal{N}_k^r)'$ and $Y \in (\mathcal{N}_j^r)'$ the cochain $X \cup Y \in (\mathcal{N}_{k+j}^r(R^n))'$ with*

$$|X \cup Y|^{\natural_r} = |\phi(X) \wedge \phi(Y)|_r.$$

Furthermore

$$\phi(X \cup Y) = \phi(X) \wedge \phi(Y).$$

Theorem 3·11. *If $X \in (\mathcal{N}_k^r)', Y \in (\mathcal{N}_j^r)', Z \in (\mathcal{N}_\ell^r)'$, and $f \in \mathcal{B}_0^{r+1}$, then:*

(i) $|X \cup Y|^{\natural_r} \leqslant |X|^{\natural_r} |Y|^{\natural_r}$;
(ii) $d(X \cup Y) = dX \cup Y + (-1)^{j+k} X \cup dY$;

(iii) $(X \cup Y) + (Z \cup Y) = (X + Z) \cup Y$; *and*
(iv) $a(X \cup Y) = (aX \cup Y) = (X \cup aY)$.

Proof. These follow by using the isomorphism of differential forms and cochains Theorem 3·7 and then applying corresponding results for differential forms and their wedge products.

Therefore, the isomorphism Ψ of Theorem 3·7 is one on graded algebras.

Continuity of **Vec(P)**

LEMMA 3·12. *Suppose P is a polyhedral chain and ω is a bounded, measurable differential form. If $\omega(p) = \omega_0$ for a fixed covector ω_0 and for all p, then*

$$\int_P \omega = \omega_0 \cdot Vec(P).$$

Proof. This follows from the definition of the Riemann integral.

THEOREM 3·13. *If P is a polyhedral k-chain and $r \geqslant 1$, then*

$$M(Vec(P)) \leqslant |P|^{\natural_r}.$$

If $sptP \subset B_\varepsilon(p)$ for some $p \in \mathbb{R}^n$ and $\varepsilon > 0$, then

$$|P|^{\natural_1} \leqslant M(Vec(P)) + \varepsilon M(P).$$

Proof. Set $\alpha = Vec(P)$ and let η_0 be a covector such that $|\eta_0|_0 = 1$, and $\eta_0 \cdot \alpha = M(\alpha)$. Define the k-form η by $\eta(p, \beta) := \eta_0(\beta)$. Since η is constant, it follows that $\|\eta\|_r = 0$ for all $r > 0$ and $\|d\eta\|_r = 0$ for all $r \geqslant 0$. Hence $|\eta|_r = |\eta|_0 = |\eta_0|_0 = 1$. By Lemma 3·12 and Theorem 3·5, it follows that

$$M(Vec(P)) = \eta_0 \cdot Vec(P) = \int_P \eta \leqslant |\eta|_r |P|^{\natural_r} = |P|^{\natural_r}.$$

For the second inequality, we use Corollary 3·9. It suffices to show that $|\int_P \omega|/|\omega|_1$ is less than or equal to the right-hand side for any 1-form ω of class B^1. Given such ω, define the k-form $\omega_0(q, \beta) := \omega(p, \beta)$ for all q. By Lemma 3·12

$$\begin{aligned}\left|\int_P \omega\right| &\leqslant \left|\int_P \omega_0\right| + \left|\int_P \omega - \omega_0\right| \\ &\leqslant |\omega(p) \cdot Vec(P)| + \sup_{q \in sptP} |\omega(p) - \omega(q)| M(P) \\ &\leqslant \|\omega\|_0 M(Vec(P)) + \varepsilon \|\omega\|_1 M(P) \\ &\leqslant |\omega|_1 (M(Vec(P)) + \varepsilon M(P)).\end{aligned}$$

If $A = \lim_{i\to\infty} P_i$ in the r natural norm then $\{P_i\}$ forms a Cauchy sequence in the r-natural norm. By Theorem 3·13, $\{Vec(P_i)\}$ forms a Cauchy sequence in the mass norm on $\Lambda^k(\mathbb{R}^n)$. Define

$$Vec(A) := lim_{i\to\infty} Vec(P_i).$$

This is independent of the choice of approximating P_i, again by Theorem 3·13.

COROLLARY 3·14. *The linear transformation*

$$Vec : \mathcal{N}_k^r \longrightarrow \Lambda^k(\mathbb{R}^n)$$

is well defined.

Corollary 3·15. *Suppose A is a chainlet of class N^r and ω is a differential form of class B^r. If $\omega(p) = \omega_0$ for a fixed covector ω_0 and for all p, then*

$$\int_A \omega = \omega_0 \cdot Vec(A).$$

Proof. This is merely Lemma 3·12 if A is a polyhedral chain. Theorem 3·13 lets us take limits in the r-natural norm. If $P_i \to A$ in $\mathcal{N}_k^r$, then by Corollary 3·14 $Vec(P_i) \to Vec(A)$ in the mass norm Therefore,

$$\int_A \omega = \lim_{i\to\infty} \int_{P_i} \omega = \lim_{i\to\infty} \omega_0 \cdot Vec(P_i) = \omega_0 \cdot Vec(A).$$

The supports of a cochain and of a chainlet

The support $sptX$ of a cochain X is the set of points p such that for each $\varepsilon > 0$ there is a cell $\sigma \subset U_\varepsilon(p)$ such that $X \cdot \sigma \neq 0$.

The support $sptA$ of a chainlet A of class N^r is the set of points p such that for each $\varepsilon > 0$ there is a cochain X of class N^r such that $X \cdot A \neq 0$ and $X \cdot \sigma = 0$ for each σ supported outside $U_\varepsilon(p)$. We prove that this coincides with the definition of the support of A if A is a polyhedral chain. Assume $A = \sum_{i=1}^m a_i \sigma_i$ is nonoverlapping and the a_i are nonzero. We must show that $sptA$ is the union F of the $spt\sigma_i$ using this new definition. Since $X \cdot A = \int_A \phi(X)$ it follows that $sptA \subset F$. Now suppose $x \in F$; say $x \in \sigma_i$. Let $\varepsilon > 0$. We easily find a smooth differential form ω supported in $U_\varepsilon(p)$, $\int_{\sigma_i} \omega \neq 0$, $\int_{\sigma_j} \omega = 0, j \neq i$. Let X be the cochain determined by ω via integration. Then $X \cdot A \neq 0$ and $X \cdot \sigma = 0$ for each σ supported outside $U_\varepsilon(p)$.!

Proposition 3·16. *If A is a chainlet of class N^r with $sptA = \emptyset$, then $A = 0$. If X is a cochain of class N^r with $sptX = \emptyset$ then $X = 0$.*

Proof. By Corollary 3·9, it suffices to show $X \cdot A = 0$ for any cochain X of class N^r. Each $p \in sptX$ is in some neighbourhood $U(p)$ such that $Y \cdot A = 0$ for any Y of class N^r with $\phi(Y) = 0$ outside $U(p)$. Choose a locally finite covering $\{U_i, i \geqslant 1\}$ of $sptX$. Using a partition of unity $\{\eta_i\}$ subordinate to this covering we have

$$X = \sum \eta_i X$$

and $\phi(\eta_i X) = \eta_i \phi(X) = 0$ outside U_i. Hence

$$X \cdot A = \sum (\eta_i X \cdot A) = 0.$$

For the second part it suffices to show that $X \cdot \sigma = 0$ for all simplexes σ. Each $p \in \sigma$ is in some neighbourhood $U(p)$ such that $X \cdot \tau = 0$ for all $\tau \subset U(p)$. We may find a subdivision $\sum \sigma_i$ of σ such that each σ_i is in some $U(p)$. Therefore $X \cdot \sigma = \sum X \cdot \sigma_i = 0$.

4. *Geometric star operator*

k-elements

In this section we make precise the notion of an *infinitesimal* of calculus. Imagine taking an infinitely thin card and cutting it into many pieces. Stack the pieces and repeat, taking a limit. What mathematical object do we obtain? The reader will recall Dirac monopoles which are closely related. We show the limit, called a *k-element*, exists as a well defined chainlet, and thus may be acted upon by any chainlet operator. We emphasize that these operators have geometric definitions, as opposed to the weak definitions arising from duals of differential forms.

Let $p \in \mathbb{R}^n$ and α be a k-direction in $\mathbb{R}^n$. A *unit k-element* α_p is defined as follows: for each $\ell \geqslant 0$, let $Q_\ell = Q_\ell(p,\alpha)$ be the weighted k-cube centered at p with k-direction α, edge $2^{-\ell}$ and coefficient $2^{k\ell}$. Then $M(Q_\ell) = 1$ and $Vec(Q_\ell) = \alpha$. We show that $\{Q_\ell\}$ forms a Cauchy sequence in the 1-natural norm. Let $j \geqslant 1$ and estimate $|Q_\ell - Q_{\ell+j}|^{\natural_1}$. Subdivide Q_ℓ into 2^{kj} binary cubes $Q_{\ell,i}$ and consider $Q_{\ell+j}$ as 2^{kj} copies of $(1/2^{kj})Q_{\ell+j}$. We form difference k-cells of order 1 of these subcubes of $Q_\ell - Q_{\ell+j}$ with translation distance $\leqslant 2^{-\ell}$. Since the mass of each Q_ℓ is one, it follows that

$$|Q_\ell - Q_{\ell+j}|^{\natural_1} = \left|\sum_{i=1}^{2^{kj}} \left(Q_{\ell,i} - \frac{1}{2^{kj}}Q_{\ell+j}\right)\right|^{\natural_1} \leqslant \sum_{i=1}^{2^{kj}} \left\|Q_{\ell,i} - \frac{1}{2^{kj}}Q_{\ell+j}\right\|_1 \leqslant 2^{-\ell}.$$

Thus Q_ℓ converges to a 1-natural, chain denoted α_p, with $|\alpha_p - Q_\ell|^{\natural_1} \leqslant 2^{1-\ell}$. If we let α be any simple k-vector with nonzero mass, the same process will produce a chainlet $\alpha_p \in \mathcal{N}_k^1$ depending only on α and p, whose mass is the same as that of α and supported in p. We obtain

$$\alpha_p = \lim Q_\ell \text{ and } |\alpha_p - Q_\ell|^{\natural_1} \leqslant 2^{1-\ell}M(\alpha). \tag{4·1}$$

Since $Vec(Q_\ell) = \alpha$ for all ℓ, it follows from Corollary 3·14 that $Vec(\alpha_p) = \alpha$. If ω is a form of class $\mathcal{B}^1$ defined in a neighbourhood of p, then $\int_{\alpha_p} \omega = \omega(p;\alpha)$ by Corollary 3·15.

PROPOSITION 4·1. *For each nonzero simple k-vector α and $p \in \mathbb{R}^n$, there exists a unique chainlet $\alpha_p \in \mathcal{N}_k^1$ such that $Vec(\alpha_p) = \alpha$, $spt\alpha_p = \{p\}$ and $\int_{\alpha_p} \omega = \omega(p;\alpha)$ for all forms ω of class $\mathcal{B}_k^1$.*

Proof. Let $\alpha_p = \lim Q_\ell$ be as in (4·1). It is unique by Corollary 3·8 since $\int_{\alpha_p} \omega = \omega(p;\alpha)$ for all forms ω of class $\mathcal{B}_k^1$. Since $Vec(\alpha_p) = \alpha$ we know $\alpha_p \neq 0$. Since $sptQ_\ell \subset B_p(2^{-\ell})$ then $spt\alpha_p$ is either the empty set or the set $\{p\}$. By Proposition 3·16 $sptA = \emptyset \implies A = 0$. Hence $sptA = \{p\}$.

The next proposition tells us that the particular shapes of the approximating polyhedral chains to α_p do not matter. There is nothing special about cubes.

PROPOSITION 4·2. *Let $\{P_i\}$ be a sequence of polyhedral k-chains such that*

$$M(P_i) \leqslant C, sptP_i \subset B_{\varepsilon_i}(p), Vec(P_i) \longrightarrow \alpha$$

for some $C > 0$ and $\varepsilon_i \to 0$. Then $P_i \to \alpha_p$ in the 1-natural norm.

Proof. By (4·1) $\alpha_p = \lim Q_i$ with $Vec(Q_i) = \alpha$. By Theorem 3·13 and Corollary 3·14

$$|P_i - Q_i|^{\natural_1} \leqslant M(Vec(P_i) - Vec(Q_i)) + \varepsilon_i M(P_i - Q_i) \to 0.$$

THEOREM 4·3. *Fix $p \in \mathbb{R}^n$. The linear transformation*

$$Vec : \mathcal{N}_k^r \longrightarrow \Lambda^k(\mathbb{R}^n)$$

is one-one on chainlets supported in p.

Proof. By Proposition 4·1 and Theorem 2·2 we only need to show that if $A \in \mathcal{N}_k^r$ which is supported in p and satisfies $Vec(A) = 0$, then $A = 0$. Let X be an r-natural cochain. Define X_0 by

$$\phi(X_0)(q) := \phi(X)(p) \text{ for all } q.$$

By Corollary 3·15

$$X \cdot A = X_0 \cdot A = \phi(X)(p) \cdot Vec(A) = 0$$

implying $A = 0$.

In [**H5**] we develop the discrete theory more fully. The boundary of a k-element is studied, as well as actions of other operators. The full calculus is developed starting with k-elements replacing k-dimensional tangent spaces with k-elements.

A *k-element chain* $\dot{P} = \sum_{i=1}^m b_i(\alpha_p)_i$ is a chain of k-elements $(\alpha_p)_i$ with coefficients b_i in $\mathbb{F}$. (Note that both the k-vector α and point p may vary with i.) Denote the vector space of k-element chains in $\mathbb{R}^n$ by $\mathcal{E}_k$. The next theorem is a quantization of chainlets including, for example, fractals, soap films, light cones, and manifolds.

THEOREM 4·4 (Density of element chains). *The space of k-element chains $\mathcal{E}_k$ is dense in $\mathcal{N}_k^r$.*

Proof. Let R be a unit k-cube in $\mathbb{R}^n$ centered at p with k-direction α. For each $j \geqslant 1$ subdivide R into 2^{kj} binary cubes $R_{j,i}$ with midpoint $p_{j,i}$ and edge 2^{-j}. Since $R_{j,i} = 2^{-jk}Q_j(p_{j,i}, \alpha)$ by using (4·1) it follows that

$$\begin{aligned} |R_{j,i} - 2^{-jk}\alpha_{p_{j,i}}|^{\natural_1} &\leqslant 2^{-jk}|Q_j(p_{j,i}, \alpha) - \alpha_{p_{j,i}}|^{\natural_1} \\ &\leqslant 2^{-jk}2^{-j+1} = 2^{-j+1}M(R_{j,i}). \end{aligned}$$

Let $\dot{P}_j = \sum_{i=1}^m 2^{-jk}\alpha_{p_{j,i}}$. Then

$$|R - \dot{P}_j|^{\natural_1} \leqslant 2^{-j+1} \sum M(R_{j,i}) = 2^{-j+1}M(R) = 2^{-j+1}.$$

This demonstrates that $\dot{P}_j \xrightarrow{\natural_1} R$. This readily extends to any cube with edge ε.

Use the Whitney decomposition to subdivide a k-cell τ into binary k-cubes. For each $j \geqslant 1$, consider the finite sum of these cubes with edge $\geqslant 2^{-j}$. Subdivide each of these cubes into subcubes Q_{ji} with edge 2^{-j} obtaining $\sum_i Q_{ji} \to \tau$ in the mass norm as $j \to \infty$. Let $\alpha = Vec(\tau)$ and p_{ji} the midpoint of Q_{ji}. Then

$$\left|\tau - \sum_i \alpha_{p_{ji}}\right|^{\natural_1} \leqslant \left|\tau - \sum_i Q_{ji}\right|^{\natural_1} + \sum_i |Q_{ji} - \alpha_{p_{ji}}|^{\natural_1}.$$

We have seen that the first term of the right-hand side tends to zero as $j \to \infty$. By (4·1) the second is bounded by $\sum_i M(Q_{ji})2^{-j+1} \leqslant M(\tau)2^{-j+1} \to 0$. It follows that τ

is approximated by k-element chains in the 1-natural norm. Thus k-element chains are dense in $\mathcal{P}_k$. The result follows since polyhedral chains are dense in chainlets.

Geometric Hodge star

Recall the Hodge star operator $\star$ of differential forms ω. We next define a geometric star operator on chainlets. If α is a simple k-vector in $\mathbb{R}^n$, then $\star\alpha$ is defined to be the simple $(n-k)$-vector with $(n-k)$-direction orthogonal to the k-direction of α, with complementary orientation and with $M(\alpha) = M(\star\alpha)$. The operator $\star$ extends to k-element chains $\dot{P}$ by linearity. It follows immediately that $\star\omega(p;\star\alpha) = \omega(p;\alpha)$. (Indeed, we prefer to define $\star\omega$ in this way, as dual to the geometric $\star$.) Hence $\int_{\dot{P}} \omega = \int_{\star\dot{P}} \star\omega$. According to Theorem 3·9 $|\star\dot{P}|^{\natural_r} = |\dot{P}|^{\natural_r}$. We may therefore define $\star A$ for any chainlet A of class N^r as follows: by Theorem 4·4 there exists k-element chains $\{\dot{P}_j\}$ such that $A = \lim_{j\to\infty} \dot{P}_j$ in the r-natural norm. Since $\{\dot{P}_j\}$ forms a Cauchy sequence we know $\{\star\dot{P}_j\}$ also forms a Cauchy sequence. Its limit in the r-natural norm is denoted $\star A$. This definition is independent of the choice of the sequence $\{\dot{P}_j\}$. (See Figure 3 for an example.)

THEOREM 4·5 (Star theorem). $\star : \mathcal{N}_k^r \to \mathcal{N}_{n-k}^r$ *is a norm-preserving linear operator that is dual to the Hodge star operator on forms. It satisfies* $\star\star = (-1)^{k(n-k)}I$ *and*

$$\int_{\star A} \omega = (-1)^{k(n-k)} \int_A \star\omega$$

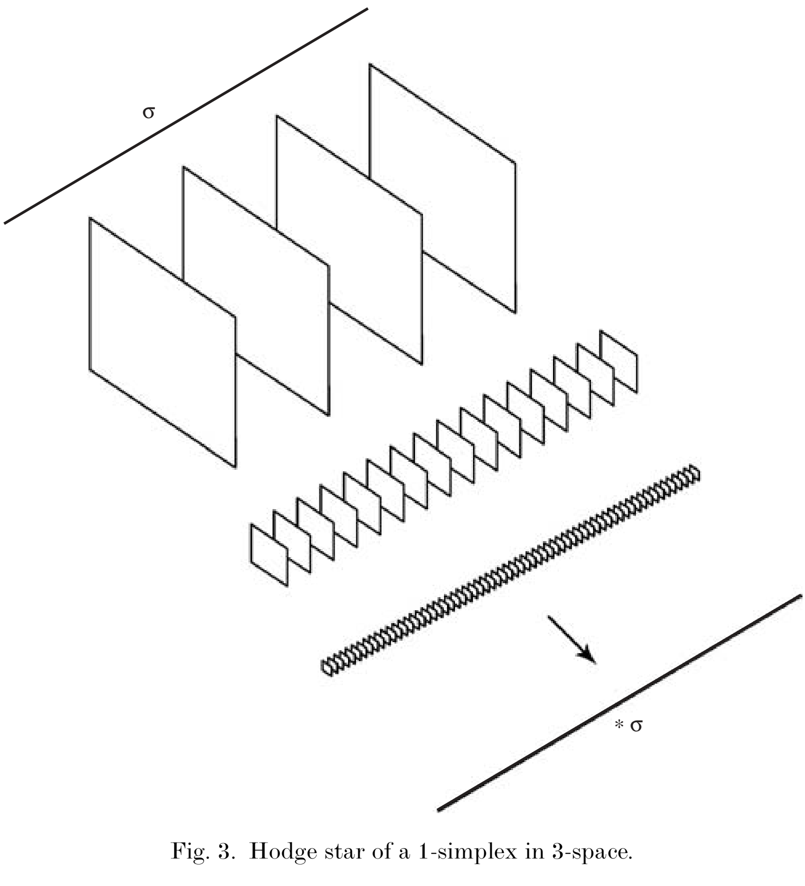

Fig. 3. Hodge star of a 1-simplex in 3-space.

for all $A \in \mathcal{N}_k^r$ and all $(n-k)$-forms ω of class $B^r, r \geqslant 1$, defined in a neighbourhood of sptA.

Proof. We first prove this for k-elements α_p. Since α_p is a 1-natural chainlet, we may integrate ω over it. Hence

$$\int_{\alpha_p} \omega = \omega(p;\alpha) = \star\omega(p;\star\alpha) = \int_{\star\alpha_p} \star\omega.$$

It follows by linearity that $\int_{\dot{P}} \omega = \int_{\star\dot{P}} \star\omega$ for any k-element chain $\dot{P}$. Let A be a chainlet of class N^r. It follows from Theorem 4·4 that A is approximated by k-element chains $A = \lim_{j\to\infty} \dot{P}_j$ in the r-natural norm. We may apply continuity of the integral (Theorem 3·5) to deduce

$$\int_A \omega = \int_{\star A} \star\omega.$$

The Hodge star operator on forms satisfies $\star\star\,\omega = (-1)^{k(n-k)}\omega$. Therefore

$$\int_A \star\omega = \int_{\star A} \star\star\,\omega = (-1)^{k(n-k)} \int_{\star A} \omega.$$

Geometric coboundary of a chainlet

Define the *geometric coboundary* operator

$$\diamondsuit : \mathcal{N}_k^r \to \mathcal{N}_{k+1}^{r+1}$$

by

$$\diamondsuit := (-1)^{nk+n+1} \star \partial \star .$$

Since $\partial^2 = 0$ and $\star\star = \pm I$ it follows that $\diamondsuit^2 = 0$.

The following theorem follows immediately from properties of boundary ∂ and star $\star$. Let $\delta := (-1)^{nk+n+1} \star d\star$ denote the coboundary operator on differential forms.

THEOREM 4·6 (Coboundary operator theorem). $\diamondsuit : \mathcal{N}_k^r \to \mathcal{N}_{k+1}^{r+1}$ *is a nilpotent linear operator satisfying*:

(i) $\int_{\diamondsuit A} \omega = (-1)^{n+1} \int_A \delta\omega$ *for all ω defined in a neighbourhood of sptA*;

(ii) $\star\partial = (-1)^{n+k^2+1} \diamondsuit\star$; *and*

(iii) $|\diamondsuit A|^{\natural_r} \leqslant |A|^{\natural_{r-1}}$ *for all chainlets $A \in \mathcal{N}_k^r$.*

Geometric interpretation of the coboundary of a chainlet

This has a geometric interpretation seen by taking approximations by polyhedral chains. For example, the coboundary of 0-chain Q_0 in $\mathbb{R}^2$ with unit 0-mass and supported in a single point $\{p\}$ is the limit of 1-chains P_k depicted in Figure 4.

The coboundary of a 1-dimensional unit cell Q_1 in $\mathbb{R}^3$ is approximated by a "paddle wheel", supported in a neighbourhood of σ.

If Q_2 is a unit 2-dimensional square in $\mathbb{R}^3$ then its coboundary $\diamondsuit Q_2$ is approximated by the sum of two weighted sums of oppositely oriented pairs of small 3-dimensional

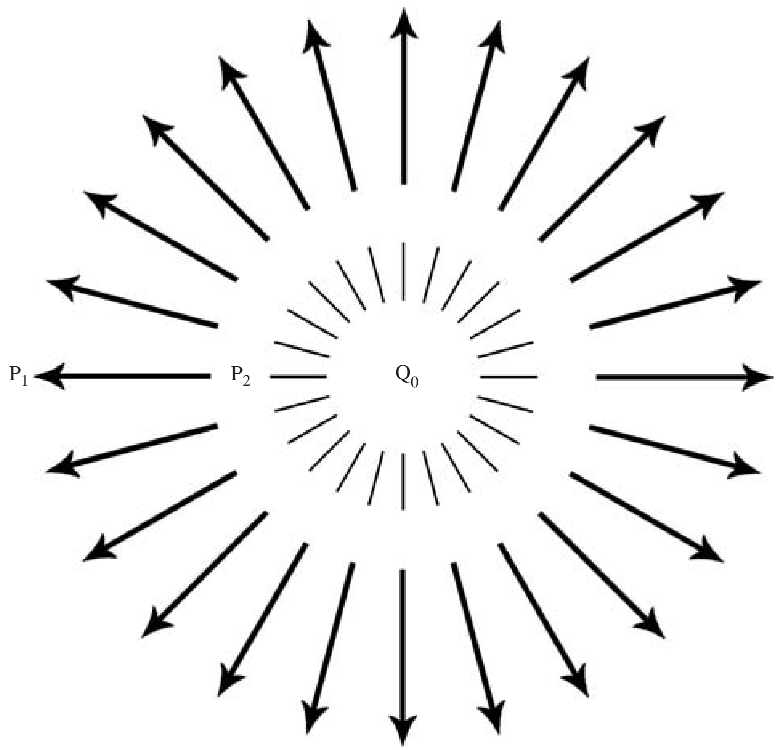

Fig. 4. Geometric coboundary of a point Q_0 as a limit of polyhedra P_k.

balls, one collection slightly above Q_2, like a mist, the other collection slightly below Q_2. A snake approaching the boundary of a lake knows when it has arrived. A bird approaching the coboundary of a lake knows when it has arrived.

Geometric Laplace operator

The geometric Laplace operator

$$\Box : \mathcal{N}_k^r \longrightarrow \mathcal{N}_k^{r+2}$$

is defined on chainlets by

$$\Box := (\partial + \diamondsuit)^2 = (\partial\diamondsuit + \diamondsuit\partial).$$

THEOREM 4·7 (Laplace operator theorem). *Suppose* $A \in \mathcal{N}_k^r$ *and* $\omega \in \mathcal{B}_k^{r+2}$ *is defined in a neighbourhood of spt*A. *Then* $\Box A \in \mathcal{N}_k^{r+2}$,

$$|\Box A|^{\natural r+2} \leqslant |A|^{\natural_r},$$

and

$$\int_{\Box A} \omega = (-1)^{n+1} \int_A \Delta\omega.$$

The geometric Laplace operator on chainlets requires at least the 2-natural norm. Multiple iterations of Δ require the r-natural norm for larger and larger r. For spectral analysis and applications to dynamical systems the normed linear space

$\mathcal{N}_k^\infty := \lim_{r\to\infty} \mathcal{N}_k^r$ with the operator

$$\Box : \mathcal{N}_k^\infty \to \mathcal{N}_k^\infty$$

should prove useful. (See [**H5**] for further discussion of the direct limit space $\mathcal{N}_k^\infty$.)

Geometric representation of differentiation of distributions

An *r-distribution* on $\mathbb{R}^1$ is a bounded linear functional on functions $f \in \mathcal{B}_0^r(\mathbb{R}^1)$ with compact support. Given a one-dimensional chainlet A of class N^r, define the r-distribution θ_A by $\theta_A(f) := \int_A f(x)dx$, for $f \in \mathcal{B}_0^r(\mathbb{R}^1)$.

THEOREM 4·8. *θ_A is linear and injective. Differentiation in the sense of distributions corresponds geometrically to the operator $\star\partial$. That is,*

$$\theta_{\star\partial A} = (\theta_A)'.$$

Proof. Suppose $\theta_A = \theta_B$. Then $\int_A f(x)dx = \int_B f(x)dx$ for all functions $f \in \mathcal{B}_0^r$. But all 1-forms $\omega \in \mathcal{B}_1^r$ can be written $\omega = fdx$. By Corollary 3·8 chainlets are determined by their integrals and thus $A = B$.

We next show that $\theta_{\star\partial A} = (\theta_A)'$. Note that $\star(f(x)dx) = f(x)$. Thus

$$\begin{aligned}\theta_{\star\partial A}(f) &= \int_{\star\partial A} f(x)dx = \int_{\partial A} f = \int_A df \\ &= \int_A f'(x)dx = \theta_A(f') = (\theta_A)'(f).\end{aligned}$$

5. *Extensions of theorems of Green and Gauss*

Curl of a vector field over a chainlet

Let S denote a smooth, oriented surface with boundary in $\mathbb{R}^3$, and F a smooth vector field defined in a neighbourhood of S. The usual way to integrate the curl of a vector field F over S is to integrate the Euclidean inner product of curl$F$ with the unit normal vector field of S obtaining $\int_S curlF \cdot ndA$. By the curl theorem this integral equals $\int_{\partial S} F \cdot d\sigma$.

We translate this into the language of chainlets and differential forms using the Euclidean inner product.

Let ω be the unique differential 1-form associated to F by way of the Euclidean dot product. The differential form version of *curlF* is $\star d\omega$. The unit normal vector field of S can be represented as the chainlet $\star S$. Thus the net curl of F over S takes the form $\int_{\star S} \star d\omega$. By the Star theorem (Theorem 4·5) and Stokes' theorem for chainlets. (Theorem 1·2), this integral equals $\int_S d\omega = \int_{\partial S} \omega$. The vector version of the right-hand integral is $\int_{\partial S} F \cdot ds$. The following extension of Green's curl theorem to chainlets of arbitrary dimension and codimension follows immediately from Stokes' theorem and the Star theorem, and is probably optimal.

THEOREM 5·1 (Generalized Green's curl theorem). *Let A be a k-chainlet of class N^r and ω a differential $(k-1)$-form of class B^r defined in a neighbourhood of $sptA$. Then*

$$\int_{\star A} \star d\omega = \int_{\partial A} \omega.$$

Proof. This is a direct consequence of Theorems 1·2 and 4·5.

It is not necessary for tangent spaces to exist for A or ∂A for this theorem to hold.

Divergence of a vector field over a chainlet

The usual way to calculate divergence of a vector field F across a boundary of a smooth surface D in $\mathbb{R}^2$ is to integrate the dot product of F with the unit normal vector field of ∂D. According to Green's Theorem, this quantity equals the integral of the divergence of F over D. That is,

$$\int_{\partial D} F \cdot n d\sigma = \int_D divF dA.$$

Translating this into the language of differential forms and chainlets, we replace the unit normal vector field over ∂D with the chainlet $\star\partial D$ and $divF$ with the differential form $d \star \omega$. We next give an extension of the Divergence theorem to k-chainlets in n-space. As before, this follows immediately from Stokes' theorem and the Star theorem, and is probably optimal.

THEOREM 5·2 (Generalized Gauss divergence theorem). *Let A be a k-chainlet of class N^r, and ω a differential $(n-k+1)$-form of class B^{r+1} defined in a neighbourhood of $sptA$. Then*

$$\int_{\star\partial A} \omega = (-1)^{(k-1)(n-k-1)} \int_A d \star \omega.$$

Proof. This is a direct consequence of Theorems 1·2 and 4·5.

As before, tangent vectors need not be defined for the theorem to be valid and it holds in all dimensions and codimensions.

Manifolds

The diffeomorphic image $\phi_* C$ in Euclidean space $\mathbb{R}^n$ of a k-cell C in $\mathbb{R}^n$ supports a unique k-chainlet for which integrals of k-forms coincide. A simple proof of this uses the implicit function theorem. The image is locally the graph of a smooth function and all such graphs naturally support chainlets. Therefore, every diffeomorphism $\phi: U \to V$ of open sets in $\mathbb{R}^n$ induces a linear map from chainlets in U to chainlets in V, commuting with the boundary and pushforward operators. If W is a coordinate domain in a smooth manifold M, it then makes sense to speak of *chainlets in W*, meaning the image of a chainlet in $\mathbb{R}^n$. Define a chainlet in M to be a finite sum of chainlets in coordinate domains. Thus to each smooth manifold corresponds a family of Banach spaces of chainlets.

Stokes' theorem follows by using partitions of unity and the theorem for $\mathbb{R}^n$. With metrics, star, curl, divergence, etc. may be introduced.

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