On Plateau's Problem for Soap Films with a Bound on Energy

Jenny Harrison

Abstract. We prove existence and a.e. regularity of an area minimizing soap film with a bound on energy spanning a given Jordan curve in R^3.

1. Introduction

Given a simple, closed curve in three-space is there a surface with minimal area spanning it? Solutions to the problem of Plateau depend, of course, on the class of spanning surfaces permitted. Douglas [D] won the first Fields' medal for his proof of the existence of an area minimizing mapping of the 2-disk whose image spans the Jordan curve. Regularity took many years to establish [O] and some aspects are still unresolved. Federer and Fleming's solutions [FF] are area minimizing in the class of integral currents. Two years later Fleming [Fl2] proved any such solution is an embedded, orientable surface, smooth away from its boundary. None of these solutions consider soap films that arise in nature such as Moebius strips or films with triple branching. Almgren [A] invented varifolds to treat soap films, but the lack of a natural boundary operator slowed progress and regularity was never proved.

Plateau observed that soap films have only two possible kinds of branching: (1) three sheets of surface meeting at 120° angles along a curve and (2) four such curves meeting at approximately 109° angles at a point [P]. In [H] the author provides models for surfaces called *flat dipolyhedra* that model all such films, orientable or nonorientable, as well as the surfaces considered in [D] and [FF]. Flat dipolyhedra take advantage of the fact that soap films are actually two films essentially occupying the same space, but are not cancelling. There is a natural boundary operator of flat k-dimensional dipolyhedra into flat $(k-1)$-dimensional dipolyhedra which relies on cohesion of the soap film structure supported by a geometric version of Cartan's magic formula.

The *energy* of a flat dipolyhedron is defined to be the length of the singular branched set of A plus the surface area of A. In this paper we prove the existence

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of a surface spanning a Lipschitz Jordan curve that is area minimizing in the space of flat dipolyhedra A with energy bounded by a fixed constant.

Complete soap film regularity remains an open question, although we prove here that the solution is a smooth surface away from its branched set which is a union of Lipschitz Jordan curves of finite total length.

2. Preliminaries

Chainlets

A *k-cell* σ in $\mathbb{R}^k$ is defined to be a finite intersection of k-dimensional half spaces. A *k-cell in* $\mathbb{R}^n$ is a k-cell in a k-dimensional subspace of $\mathbb{R}^n$. Each k-cell is assumed to be oriented. The support of a k-cell σ is denoted by $|\sigma|$. An *integral cellular k-chain* is a formal sum $\sum a_i\sigma_i$ where $a_i \in \mathbb{Z}$ and the σ_i are k-cells. Equate two cellular chains $S_1 \sim S_2$ if and only if they have a common cellular subdivision which is nonoverlapping. A polyhedral k-chain is an equivalence class of cellular k-chains. The *mass* of a k-cell σ is its k-dimensional Hausdorff measure $M(\sigma) = \mathcal{H}_k(|\sigma|)$. If P is a polyhedral k-chain represented by a nonoverlapping k-chain $\sum a_i\sigma_i$, its mass is defined by $M(P) = \sum |a_i|M(\sigma_i)$.

Let G be an abelian group with a translation invariant metric making it a complete metric space. Let $|g|$ denote the distance between $g \in G$ and the group identity. If H is a closed subgroup of G, we use the quotient metric $|\bar{g}| = \inf\{|g| : g \in \bar{g}\}$. If $G = Z$ then $|g|$ denotes the absolute value. The group $Z_p = \mathbb{Z}/p\mathbb{Z}$ is of special interest and we give it the quotient metric. Let $\mathbf{P}_k(G) = G \otimes \mathbf{P}_k(\mathbb{Z})$. This is the group of polyhedral k-chains with coefficients in G. If $P \in \mathbf{P}_k(G)$ then $P = \sum g_i\sigma_i$ where the σ_i are nonoverlapping k-cells. If $P \in \mathbf{P}_k(G)$, define $M(P) = \sum |g_i|M(\sigma_i)$. Whitney's flat norm on polyhedra is defined by

$$M_\flat(P) = \inf\{M(Q) + M(R) : D = Q + \partial R, Q \in \mathbf{P}_k(G), R \in \mathbf{P}_{k+1}(G)\}.$$

The completion of the group $\mathbf{P}_k(G)$ with the norm $M_\flat$ is denoted $\mathcal{P}_k(G)$. Its elements are called *flat chains* with coefficients in G. The *support* of a flat chain A is well defined (see [W]) and is denoted $|A|$. Define subgroups

$$\begin{aligned}
\mathcal{M}_k(G) &= \{A \in \mathcal{P}_k(G) : M(A) < \infty\}\\
\mathcal{N}_k(G) &= \{A \in \mathcal{P}_k(G) : M(A) + M(\partial A) < \infty\} \text{ and}\\
\mathcal{N}_k^0(G) &= \{A \in \mathcal{N}_k(G) : |A| \text{ is compact}\}.
\end{aligned}$$

If σ is a cell and v a vector in $\mathbb{R}^n$ then $T_v\sigma$ denotes the translation of σ through v. A 1-*multicell* is a cellular chain of the form $\sigma^1 = \sigma^0 - T_{v_1}\sigma^0$ where σ^0 is a cell and v_1 is a vector. Given a vector v_j and a 2^{j-1}-multicell σ^{j-1}, define the *2^j-multicell* σ^j as the cellular chain $\sigma^j = \sigma^{j-1} - T_{v_j}\sigma^{j-1}$. Thus σ^j is generated by vectors $v_1, \dots, v_j$ and a cell σ^0. An *integral 2^j-multicellular chain in* $\mathbb{R}^n$ is a formal sum of 2^j-multicells, $S^j = \sum_{i=1}^n a_i\sigma_i^j$ with coefficients $a_i \in \mathbb{Z}$. Let $j \geq 1$. Given a 2^j-multicell σ^j generated by a cell σ^0 and vectors $v_1, \cdots, v_j$, define $\|\sigma^j\|_j = M(\sigma^0)|v_1||v_2|\cdots|v_j|$ where $|v|$ denotes the norm of a vector $v \in \mathbb{R}^n$. For consistency of notation define $\|\sigma^0\|_0 = M(\sigma^0)$. For $S^j = \sum a_i \sigma_i^j$ define $\|S^j\|_j = \sum_{i=1}^n |a_i| \|\sigma_i^j\|_j$.

Let $\mathbf{P}_k(\mathbb{Z})$ denote polyhedral k-chains in $\mathbb{R}^n$. Suppose $P \in \mathbf{P}_k$ and $r \in \mathbb{Z}^+$. For $r = 0$ define $|P|^{\natural_0} = M(P)$. For $r \geq 1$ define the r-*natural* norm

$$|P|^{\natural_r} = \inf \left\{ \sum_{s=0}^{r} \|S^j\|_j + |C|^{\natural_{r-1}} \right\}$$

where the infimum is taken over all decompositions $P = \sum_{s=0}^r [S^j] + \partial C$ with S^j a 2^j multicellular k-chain and C a polyhedral $(k+1)$-chain. The group of polyhedral k-chains $\mathbf{P}_k(\mathbb{Z})$ completed with the norm $| \quad |^{\natural_r}$ is denoted $\mathcal{N}_k^{\natural_r}(\mathbb{Z})$. Elements of $\mathcal{N}_k^{\natural_r}(\mathbb{Z})$ are called *k-dimensional chainlets of class N^r.* The boundary operator

$$\partial : \mathcal{N}_k^{\natural_r}(\mathbb{Z}) \to \mathcal{N}_{k-1}^{\natural_{r+1}}(\mathbb{Z})$$

is naturally defined and continuous.

Let σ be a k-cell supported in $\mathbb{R}^n$ and set $v = (0, 0, \cdots, 0, 1) \in \mathbb{R}^{n+1}$. In [H] *mass cells* are defined as limits in the 1-natural norm $\mu\sigma = \lim_{h \to 0} \frac{\sigma \times hv}{|h|}$. Integral mass chains are finite sums of mass cells with integer coefficients. Although the support of a mass cell is k-dimensional, as a current it is $(k+1)$-dimensional. *Dipole cells* are defined by $\delta\sigma = \partial\mu\sigma + \mu\partial\sigma$. Dipole chains S are finite sums of dipole cells. The *support* of a mass chain $T = \sum a_i \mu\sigma_i$ is defined by $|T| = \cup|\sigma_i|$ and the *support* of a dipole chain $S = \sum a_i \delta\sigma_i$ is defined to be $|S| = \cup|\sigma_i|$. Define the *mass* of a mass chain T by $M(T) = \sum |a_i| M(\sigma_i)$ and the *weight* of a dipole chain S by $W(S) = \sum |a_i| M(\sigma_i)$ where the $\{\sigma_i\}$ are nonoverlapping. Spaces of mass and dipole chains are denoted $\mathbf{T}_k(\mathbb{Z})$ and $\mathbf{S}_k(\mathbb{Z})$, resp. Let G be an abelian group. By taking the tensor product with G, we may define spaces of mass and dipole chains with coefficients in G and denote them by $\mathbf{T}_k(G)$ and $\mathbf{S}_k(G)$, resp. Define the space of *dipolyhedra* as the direct sum $\mathbf{D}_k(G) = \mathbf{S}_k(G) \oplus \mathbf{T}_k(G)$. If $D = S_D + T_D = S + T \in \mathbf{D}_k(G)$ define the *energy* of D by $E(D) = W(S) + M(T)$. Finally define the $E_\flat$ norm on the space of k-dimensional dipolyhedra by

$$E_\flat(D) = \inf\{E(Q) + E(R) : D = Q + \partial R, Q \in \mathbf{D}_k(G), R \in \mathbf{D}_{k+1}(G)\}.$$

The completion of the space of dipolyhedra $\mathbf{D}_k(G)$ with the $E_\flat$ norm is an abelian group denoted $\mathcal{D}_k(G)$. The boundary operator is continuous in the $E_\flat$ norm and satisfies $E_\flat(\partial A) \leq E_\flat(A)$. An element $A \in \mathcal{D}_k(G)$ is called a *flat dipolyhedron* with coefficients in G. In [H] it is shown that weight, mass and energy are well defined and lower semi-continuous in $\mathcal{D}_k(G)$. The operators δ and μ satisfy $E_\flat(\delta A) \leq M_\flat(A)$ and $E_\flat(\mu A) \leq 2M_\flat(A)$ for every flat chain A ([H] 5.2)).

Henceforth we set $G = \mathbb{Z}_2$ and $n = 3$ for our application to soap films in three-space. In this case the weight of a k-dipolyhedron coincides with its Hausdorff k-measure. Thus, for $k = 2$, the quantity $W(D)$ can be thought of as the *area* of D and for $k = 1$ it is the *length* of D.

3. Structure of dipolyhedra

Splittings

The orthogonal projection of a dipole k-chain S into $\mathbb{R}^3$ is a mod two k-polyhedron denoted $\overline{S}$. Then $S = \delta\overline{S}$. Similarly, the orthogonal projection of a mass k-chain T into $\mathbb{R}^3$ is a mod two $(k-1)$-polyhedron denoted $\overline{T}$ and satisfies $T = \mu\overline{T}$.

Proposition 3.1. *Suppose $D = S + T \in \mathbf{D}_k(\mathbb{Z}_2)$. Then $M_\flat(\overline{S}) \le E_\flat(D); M_\flat(\overline{T}) \le E_\flat(D)$.*

Proof. Given $\epsilon > 0$ there exist $Q \in \mathbf{D}_k(\mathbb{Z}_2), R \in \mathbf{D}_{k+1}(\mathbb{Z}_2)$ with $D = Q + \partial R$ and $E_\flat(D) > E(Q) + E(R) - \epsilon = W(S_Q) + W(S_R) + M(T_Q) + M(T_R) - \epsilon$. Note that $S_D = S_Q + S_{\partial R}$. Since $R = S_R + T_R$ then $\partial R = \partial S_R + \partial T_R$. Since the boundary of a dipole chain is also a dipole chain $S_{\partial R} = \partial S_R + S_{\partial T_R}$ and therefore $\overline{S} = \overline{S_D} = \overline{S_Q} + \overline{\partial S_R} + \overline{S_{\partial T_R}} = \overline{S_Q} + \partial\overline{S_R} + \overline{T_R}$. It follows that

$$M_\flat(\overline{S}) \le M(\overline{S_Q}) + M(\overline{S_R}) + M(\overline{T_R}) = W(S_Q) + W(S_R) + M(T_R) < E_\flat(D) + \epsilon.$$

For the second inequality, note that ∂S_R is a dipole chain and ∂T_R is a sum of a dipole chain and a mass chain $T_{\partial R}$. As before, $\partial R = \partial S_R + \partial T_R$. Therefore $\overline{T_{\partial R}} = \partial\overline{T_R}$. Since $\overline{T} = \overline{T_D} = \overline{T_Q} + \overline{T_{\partial R}}$ it follows that

$$M_\flat(\overline{T}) \le M(\overline{T_Q}) + M(\overline{T_R}) = M(T_Q) + M(T_R) < E_\flat(D) + \epsilon.$$

Since these inequalities hold for all $\epsilon > 0$, the proposition follows.

□

Theorem 3.2. *If A is a flat k-dipolyhedron then there exist a unique flat k-chain B and a unique flat $(k-1)$-chain C such that $A = \delta B + \mu C$ and $E(A) = M(B) + M(C)$. If $D_i = \delta B_i + \mu C_i \stackrel{E_\flat}{\to} A$ with $E(D_i) \to E(A)$ then $B_i \stackrel{M_\flat}{\to} B, C_i \stackrel{M_\flat}{\to} C$, $M(B_i) \to M(B)$ and $M(C_i) \to M(C)$.*

Proof. Suppose $D_i \stackrel{E_\flat}{\to} A$ where $D_i = S_i + T_i$. By Proposition 3.1 $B_i = \overline{S_i}$ is a Cauchy sequence in $M_\flat$. Let B denote its flat chain limit. By 5.2 of [H] $S_i = \delta B_i \stackrel{E_\flat}{\to} \delta B$. Similarly $C_i = \overline{T_i}$ converges to a flat chain C in $M_\flat$ and $T_i = \mu C_i \stackrel{E_\flat}{\to} \mu C$. It follows that $A = \delta B + \mu C$.

We next prove uniqueness: Suppose $A = \delta B + \mu C = 0$. Then $E_\flat(A) = 0$. By Propotision 3.1

$$M_\flat(B) = \lim M_\flat(B_i) \le \lim E_\flat(D_i) = E_\flat(A)$$

and

$$M_\flat(C) = \lim M_\flat(C_i) \le \lim E_\flat(D_i) = E_\flat(A).$$

It follows that $B = C = 0$ since $M_\flat$ is a norm. Uniqueness of B and C follows.

Suppose $E(D_i) \to E(A)$. By lower semicontinuity of mass in the flat norm [W], $E(A) = M(B) + M(C) \le \liminf M(B_i) + \liminf M(C_i) \le \liminf E(D_i) = E(A)$. Thus $E(A) = M(B)+M(C) = \liminf M(B_i)+\liminf M(C_i)$. Since $M(B) \le$

$\liminf M(B_i)$ and $M(C) \leq \liminf M(C_i)$ and all terms are nonnegative the result follows.

□

If $A = \delta B + \mu C$ we say that A *splits* into δB and μC. Since the splitting is unique we may define $W(A) = M(B)$ and $M(A) = M(C)$.

Lemma 3.3. *If $A = \delta B + \mu C$ is a flat dipolyhedron then $\partial A = \delta(\partial B + C) - \mu(\partial C)$.*

Proof. Acccording to ([H], 3.2) $\partial\delta = \delta\partial$ and $\delta = \partial\mu + \mu\partial$. The result follows . □

Corollary 3.4. *If γ is a flat $(k-1)$-chain and $A = \delta B + \mu C$ is a flat k-dipolyhedron satisfying $\partial A = \delta\gamma$ then $\partial C = 0$ and $\partial B + C = \gamma$.*

Proof. It follows from Lemma 3.3 that $\partial A = \delta(\partial B + C) - \mu\partial C = \delta\gamma$. Since the splitting is unique (Theorem 3.2) it follows that $\partial B + C = \gamma$ and $\partial C = 0$. □

The support of a flat dipolyhedron

Let B be a flat chain with finite mass. According to ([Fl1], §4) there exists a Borel measure ρ_B and for every Borel set $X \subset \mathbb{R}^3$ there exists a flat chain $B \cap X$ such that $\rho_B(X) = M(B \cap X)$. Moreover, if $P_i \overset{M_\flat}{\to} B$ with $M(P_i) \to M(B)$ then $P_i \cap X \overset{M_\flat}{\to} B \cap X$ and $M(P_i \cap X) \to M(B \cap X)$ for every X such that $\rho_B(frX) = 0$. ([Fl1] §4) Define $(\delta B) \cap X = \delta(B \cap X)$ and $(\mu B) \cap X = \mu(B \cap X)$ the *part* of δB in X and the *part* of μB in X, respectively.

Suppose $A \in \mathcal{D}_k$ is a flat dipolyhedron with bounded energy $E(A) \leq \lambda$. From Theorem 3.2 we know $A = \delta B + \mu C$ where B and C are flat chains with $M(B) + M(C) \leq \lambda$. Define

$$A \cap X = \delta(B \cap X) + \mu(C \cap X).$$

We call $A \cap X$ the *part of* A in X. Define Borel measures $\omega_A(X) = \rho_B(X)$, $\mu_A(X) = \rho_C(X)$, and $\nu_A(X) = \omega_A(X) + \mu_A(X)$. The next proposition follows directly.

Proposition 3.5. *If $A = \delta B + \mu C$ is a flat dipolyhedron with finite energy and X is a Borel set there exists a unique flat dipolyhedron $A \cap X$ such that $W(\delta B \cap X) = \omega_A(X)$, $W(\delta B - \delta B \cap X) = \omega_A(X^c)$, $M(\mu C \cap X) = \mu_A(X)$, $M(\mu C - \mu C \cap X) = \mu_A(X^c)$, $E(A \cap X) = \nu_A(X)$ and $E(A - A \cap X) = \nu_A(X^c)$. If $D_i \overset{E_\flat}{\to} A$ with $E(D_i) \to E(A)$ then $D_i \cap X \overset{E_\flat}{\to} A \cap X$ and $E(D_i \cap X) \to E(A \cap X)$ for all X such that $\nu_A(frX) = 0$.*

The *support* of a Borel measure ν is the smallest closed set X whose complement is ν-null. Say that ν is a *measure on* Y if Y contains the support of ν.

We say a closed set F *supports* a flat dipolyhedron A if for every open set U containing F there is a sequence $\{D_i\}$ of dipolyhedra tending to A in $E_\flat$ such that $|D_i| \subset U$ for each j. If there is a smallest set F which supports A then F is called the *support* of A and denoted $|A|$. The next theorem shows that every flat dipolyhedron with finite energy has a well defined support.

Theorem 3.6. *If $A \in \mathcal{D}_k$ with $E(A) < \infty$ then $|A| = |\nu_A|$.*

Proof. By Theorem 3.2 we know $A = \delta B + \mu C$ where B and C are flat chains with $M(B) < \infty$ and $M(C) < \infty$. Apply ([Fl1], 4.3) to deduce $|A| = |B| \cup |C| = |\rho_B| \cup |\rho_C| = |\nu_A|$. □

4. Cones, pushforwards and projections

Cones over dipolyhedra

Let σ be a k-cell in supported in $Q(p, r)$, the 3-cube in $\mathbb{R}^3$ centered at p with side length r. The cone $p\sigma$ is also a cell found by intersecting the cones over the half-spaces forming σ. Its boundary satisfies $\partial p\sigma = \sigma - p\partial\sigma$ and $M(p\sigma) \le \frac{r\sqrt{3}}{k+1} M(\sigma)$. Define $p\delta\sigma = \delta p\sigma$ and $p\mu\sigma = \mu p\sigma$.

It follows that $p\delta\sigma$ is a well defined dipole cell with $W(p\delta\sigma) \le \frac{r\sqrt{3}}{k+1} W(\delta\sigma)$, and $p\mu\sigma$ is a well defined mass cell with $M(p\mu\sigma) \le \frac{r\sqrt{3}}{k+1} M(\mu\sigma)$. Extend the definition by linearity to define dipole chains pS and mass chains pT taken over dipole chains S and mass chains T, respectively. Next extend the definition to dipolyhedra $pD = pS + pT$. Observe $W(pS) \le \frac{r\sqrt{3}}{k+1} W(S)$ and $M(pT) \le \frac{r\sqrt{3}}{k+1} M(T)$.

Proposition 4.1. *If $D \in \mathbf{D}_k$ then $pD \in \mathbf{D}_k$ and*

$$D = \partial(pD) + p(\partial D).$$

If D is supported in $Q(p, r)$ then so is pD and

$$E(pD) \le \frac{r\sqrt{3}}{k+1} E(D).$$

Proof. The first part reduces to showing $\delta\sigma = \partial(p\delta\sigma) + p\partial\delta\sigma$ and $\mu\sigma = \partial(p\mu\sigma) + p\partial\mu\sigma$. These follow directly from the definitions and linear relations. Suppose $D = S + T$. Then $pD = pS + pT$ where pS is a dipole chain and pT is a mass chain. Thus

$$E(pD) = W(pS) + M(pT) \le \frac{r\sqrt{3}}{k+1}(W(S) + M(T)) = \frac{r\sqrt{3}}{k+1} E(D).$$

□

Theorem 4.2. *If $D \in \mathbf{D}_k$ is supported in $Q(p, r)$ then*

$$E_\flat(pD) \le \left(1 + \frac{r\sqrt{3}}{k+1}\right) E_\flat(D).$$

Proof. Let $\epsilon > 0$. There exist dipolyhedra Q and R such that $D = Q + \partial R$ and $E_\flat(D) > E(Q) + E(R) - \epsilon$.

By Proposition 4.1 $p\partial R = R + \partial(pR)$ for all dipolyhedra R. Then

$$E_\flat(p\partial R) \le E(R) + E(pR) \le \left(1 + \frac{r\sqrt{3}}{k+1}\right) E(R).$$

Since $pD = pQ + p\partial R$ it follows that

$$\begin{aligned} E_\flat(pD) &\le E(pQ) + E_\flat(p\partial R) \\ &\le \left(1 + \tfrac{r\sqrt{3}}{k+1}\right)(E(Q) + E(R)) \\ &\le \left(1 + \tfrac{r\sqrt{3}}{k+1}\right)(E_\flat(D) + \epsilon). \end{aligned}$$

The result follows since this holds for all $\epsilon > 0$. □

It follows that if A is a flat dipolyhedron then the cone pA has unique definition as a flat dipolyhedron as follows: if $D_i \stackrel{E_\flat}{\to} A$ then $\{pD_i\}$ forms a Cauchy sequence. Denote its limit by pA.

Proposition 4.3. *If $A \in \mathcal{D}_k(\mathbb{Z}_2)$ then*

$$A = \partial(pA) + p(\partial A).$$

If A is supported in $Q(p, r)$ then

$$E_\flat(pA) \le \left(1 + \frac{r\sqrt{3}}{k+1}\right) E_\flat(A)$$

and

$$E(pA) \le \frac{r\sqrt{3}}{k+1} E(A).$$

Proof. The first two relations follow from Proposition 4.1, Theorem 4.2 and continuity of the boundary operator. Since energy is lower semicontinuous, there exists $D_i \stackrel{E_\flat}{\to} A$ such that $E(D_i) \to E(A)$. By Theorem 4.2 $pD_i \stackrel{E_\flat}{\to} pA$ and hence

$$E(pA) \le \liminf E(pD_i) \le \liminf \frac{r\sqrt{3}}{k+1} E(D_i) = \frac{r\sqrt{3}}{k+1} E(A).$$

□

Lipschitz pushfoward

Let $f : U \subset \mathbb{R}^3 \to \mathbb{R}^3$ be a Lipschitz mapping. Extend f to $\mathbb{R}^4$ by $f(x,t) = (f(x), t)$. If B is a flat k-chain supported in an open set $U \subset \mathbb{R}^3$ then the push-forward f_*B is well defined as a flat k-chain and satisfies $M_\flat(f_*B) \le |f|_{Lip}^k M_\flat(B)$ and $M(f_*B) \le |f|_{Lip}^k M(B)$ ([W]). It is called a *Lipschitz chain.* A chainlet $A = \delta B + \mu C$ is called a *Lipschitz dipolyhedron* if B and C are Lipschitz chains. If $D = \delta B + \mu C$ define $f_*D = \delta f_*B + \mu f_*C$.

Proposition 4.4. *If $D = \delta B + \mu C$ is a k-dipolyhedron and $f : U \subset \mathbb{R}^3 \to \mathbb{R}^3$ is a Lipschitz mapping with $|D| \subset U$ then f_*D is a k-dipolyhedron with $\partial f_*D = f_*\partial D$, $M(f_*D) \le |f|^k_{Lip} M(D), W(f_*D) \le |f|^k_{Lip} W(D)$ and $E_\flat(f_*D) \le |f|^k_{Lip} E_\flat(D)$.*

Proof. By Lemma 3.3 we know that $\partial f_*D = \delta(\partial f_*B + f_*C) - \mu(\partial f_*C)$. Since $\partial D = \delta(\partial B + C) - \mu(\partial D)$ we have $f_*(\partial D) = \delta(f_*\partial B + f_*C) - \mu(f_*\partial D)$. Since f_* is a chain map on flat chains we conclude $f_*(\partial D) = \partial f_*D$.

By Theorem 3.2 and Proposition 4.4 it follows that $E(f_*D) = M(f_*B) + M(f_*C) \le |f|^k_{Lip}(M(B) + M(C)) = |f|^k_{Lip}(E(D))$.

Let $\epsilon > 0$. There exists $D = Q + \partial R$ with $E_\flat(D) > E(Q) + E(R) - \epsilon$. Since $f_*D = f_*Q + f_*\partial R = f_*Q + \partial f_*R$ it follows that $E_\flat(f_*D) \le E(f_*Q) + E(f_*R) \le |f|^k_{Lip}(E(Q) + E(R)) \le |f|^k_{Lip}(E_\flat(D) + \epsilon)$.

Finally, $W(f_*D) = M(f_*S) \le |f|^k_{Lip} M(S) = |f|^k_{Lip} W(D)$. ☐

Let A be a flat dipolyhedron. By lower semicontinuity of energy we may choose $D_i \to A$ such that $E(D_i) \to E(A)$. It follows that f_*A is a well defined flat dipolyhedron with $E(f_*A) \le |f|^k_{Lip} E(A)$, and $E_\flat(f_*A) \le |f|^k_{Lip} E_\flat(A)$,

Projection into a cube

For $x = (x^1, x^2, x^3) \in \mathbb{R}^3$, define $\|x\| = \max\{|x^1|, |x^2|, |x^3|\}$. For $r > 0$, define

$$f_r(x) = \begin{cases} x, & \|x\| \le r, \\ rx/\|x\|, & \|x\| > r. \end{cases}$$

Observe that f has Lipschitz constant ≤ 1.

Denote $D(r) = f_{r*}D$, the projection of a dipolyhedron D. Since $\partial(D(r)) = (\partial D)(r)$ we can write $\partial D(r)$ with ambiguity.

Let $Q_r = Q(0, r)$. Then $f_{r*}(\mathbb{R}^3) = Q_r$. It follows from Proposition 4.4 projections $A(r)$ are uniquely defined for all flat dipolyhedra A with $E_\flat(A(r)) \le E_\flat(A)$ and $E(A(r)) \le E(A)$.

Define

$$\mathcal{B}_k(\mathbb{Z}_2) = \{A \in \mathcal{D}_k(\mathbb{Z}_2) : E(A) + E(\partial A) < \infty\} \text{ and}$$

$$\mathcal{B}^0_k(\mathbb{Z}_2) = \{A \in \mathcal{D}_k(\mathbb{Z}_2) : E(A) + E(\partial A) < \infty, |A| \text{ is compact }\}.$$

5. A deformation theorem for flat dipolyhedra

The next result is the deformation theorem, first proved for integer coefficients by Federer and Fleming [FF]. It was extended to Z_2 coefficients in [Z] and to abelian groups in ([Fl1], 7.3).

Let χ be an ϵ-cubical grid of $\mathbb{R}^3$. A k-polyhedron P is a *polyhedron of* χ if P is supported in the k-skeleton of χ and ∂P is supported in its $(k-1)$-skeleton. A k-dipolyhedron $D = \delta B + \mu C$ is a dipolyhedron *of* χ if B and C are both polyhedra of χ.

Theorem 5.1. *There exists a positive number $c = c(k,n)$ with the following property. Given $A \in \mathcal{N}_k^0(\mathbb{Z}_2)$ and $\epsilon > 0$ there exist an ϵ-cubical grid χ, a polyhedral k-chain P of χ, $Q \in \mathcal{N}_k^0(\mathbb{Z}_2)$ and $R \in \mathcal{N}_{k+1}^0(\mathbb{Z}_2)$ such that:*

1. $A - P = Q + \partial R$;
2. $M(P) \leq c(M(A) + \epsilon M(\partial A))$,
   $M(\partial P) \leq cM(\partial A)$,
   $M(Q) \leq c\epsilon M(\partial A)$,
   $M(R) \leq c\epsilon M(A)$;
3. $|P| \cup |R| \subset 6\epsilon$*-neighborhood of* $|A|$,
   $|\partial P| \cup |Q| \subset 6\epsilon$*-neighborhood of* $|\partial A|$.
4. *If A is a polyhedral (Lipschitz) chain then Q and R are polyhedral (Lipschitz chains).*

The proof makes use of successive radial projections from well chosen points in the interior of each k-cube of χ onto its lower dimensional skeleton which minimize distortion. (See [FF].) We refer to the *projection path* of a flat chain A as the rays traced by the projections of $|A|$.

Corollary 5.2. *Let γ be a Lipschitz Jordan curve in $\mathbb{R}^3$. There exists a positive number $c = c(k,n)$ with the following property. Given $A \in \mathcal{B}_k^0(\mathbb{Z}_2)$ with $\partial A = \delta\gamma$ and $\epsilon > 0$ there exist an ϵ-cubical grid χ, a dipolyhedral k-chain D of χ, $Q \in \mathcal{B}_k^0(\mathbb{Z}_2)$ and $R \in \mathcal{B}_{k+1}^0(\mathbb{Z}_2)$ such that*

1. $A - D = Q + \partial R$;
2. $E(D) \leq 2c(E(A) + \epsilon M(\partial B))$,
   $E(\partial D) \leq cM(\gamma), E(Q) \leq c\epsilon M(\gamma)$,
   $E(R) \leq c\epsilon E(A)$;
3. $|D| \cup |R| \subset 6\epsilon-$*neighborhood of* $|A|$,
   $|\partial D| \cup |Q| \subset 6\epsilon-$*neighborhood of* $|\partial A|$.
4. *If A is a (Lipschitz) dipolyhedron then Q and R are (Lipschitz) dipolyhedra.*

Proof. Suppose $A = \delta B + \mu C$ with $\partial A = \delta\gamma$. By Corollary 3.4 $\partial C = 0$ and $\partial B + C = \gamma$. Apply Theorem 5.1 to B and C to find $B - P_B = Q_B + \partial R_B$ and $C - P_C = Q_C + \partial R_C$ satisfying properties (1)-(4) of Theorem 5.1. Therefore $Q_C = 0$ and $\partial P_C = 0$. Since $\gamma = \partial B + C$ it follows that

$$\gamma - (\partial P_B + P_C) = \partial(Q_B + R_C).$$

Now the polyhedra P, Q and R are found by projecting B and C onto the k-skeleton of χ. Thus Q_B is the projection path of ∂B, R_C is the projection path of C. Hence $Q_B + R_C$ is the projection path of $\partial B + C$ which is the same as the projection path of γ. Hence

$$M(Q_B + R_C) \leq c\epsilon M(\gamma).$$

Similarly, P_C is the projection of C, ∂P_B is the projection of ∂B; hence $P_C + \partial P_B$ is the projection of $C + \partial B$. Hence

$$M(\partial P_B + P_C) \le cM(\gamma).$$

Let $D = \delta P_B + \mu P_C$. Then $A - D = Q + \partial R$ where $Q = \delta(Q_B + R_C)$ and $R = \delta R_B - \mu R_C$. This establishes (1).

By Theorems 3.2 and 5.1 (2)

$$\begin{aligned} E(D) = M(P_B) + M(P_C) \quad &\le c(M(B) + M(C) + \epsilon(M(\partial B) + M(\partial C))) \\ &\le c(E(A) + \epsilon M(\partial B)). \end{aligned}$$

Now $\partial D = \delta(\partial P_B + P_C)$ since $\partial P_C = 0$. Then

$$E(\partial D) = M(\partial P_B + P_C) \le cM(\gamma).$$

and

$$E(Q) = M(Q_B + R_C) \le c\epsilon M(\gamma).$$

Finally,

$$E(R) = M(R_B) + M(R_C) \le c\epsilon(M(B) + M(C)) = c\epsilon E(A).$$

This completes the proof of (2).

The first part of (3) follows from the flat chain analogue since $|D| = |P_B| \cup |P_C|$ and $|R| = |R_B| \cup |R_C|$. For the second part, Q is the projection path of ∂A. Thus $|\partial Q| \subset 6\epsilon-$nbd of $|\partial A|$.

Part (4) is an easy consequence of the flat chain analogue and the definitions of A, Q and R.

□

We say that a flat dipolyhedron A *spans* $\delta\gamma$ if $\partial A = \delta\gamma$ and the following condition holds: if X is a 2-dimensional subspace of $\mathbb{R}^3$ and $\Pi : \mathbb{R}^3 \to X$ is an orthogonal projection which is an immersion of γ, then $\partial \Pi_* A = \Pi_* \partial A$.

Lemma 5.3. *If $D_i \to A$ and D_i spans $\delta\gamma$ then A spans $\delta\gamma$.*

Proof. By continuity of the boundary and pushforward operators $\partial D_i \to \partial A = \delta\gamma$ and $\partial \Pi_* A = \lim \partial \Pi_* D_i = \lim \Pi_* \partial D_i = \Pi_* \partial A$. □

Lemma 5.4. *There exists $\epsilon > 0$ such that if A spans $\delta\gamma$ then $W(A) > \epsilon$.*

Proof. Choose a projection Π so that $\Pi_*\gamma$ is a Jordan curve in X. Let K denote the chain whose support is the inside of $\Pi_*\gamma$ in X so that $\partial K = \Pi_*\gamma$. Let $\epsilon = M(K)$. The condition that $\partial \Pi_* A = \Pi_* \partial A = \Pi_* \gamma$ implies that $\Pi_* A = K$. The result follows since $W(A) \ge W(\Pi_* A) \ge \epsilon$. □

Choose λ sufficiently large so that $\gamma \subset Q_{\lambda'}$ where $\lambda' = \lambda(k+1)/M(\gamma)$.

Define

$$\Gamma(\lambda) = \{A \in \mathcal{B}_k^0(\mathbb{Z}_2) : E(A) \le \lambda, |A| \subset Q_{\lambda'}, A \text{ spans } \delta\gamma\}.$$

This collection of supports $|A|$ of dipolyhedra in $\Gamma(\lambda)$ contains solutions to other Plateau type problems including the supports of

- mappings of the two disk B, $f : B \to \mathbb{R}^3$, $f(\partial B) = \gamma$ and f is smooth away from ∂B.
- area minimizing integral currents
- area minimizing soap films S, as observed by Plateau: The set S is smooth away from its branch set B. Consider the connected components X_i complementary to B. These are embedded and smooth mod two surfaces. Let $X = \sum X_i$ and $D = \delta X + \mu B$. Then $|D| = |X| = S$ and $\partial D = \delta\gamma$.

Theorem 5.5. $\Gamma(\lambda)$ *is compact and nonempty in the* $E_\flat$ *norm.*

Proof. Suppose $A_i \stackrel{E_\flat}{\to} A$ where $A_i \in \Gamma(\lambda)$. We know $E(A) \le \lambda$ by lower semicontinuity of energy, and $|A| \subset Q_{\lambda'}$ since each $|A_i| \subset Q_{\lambda'}$. Since each A_i spans $\delta\gamma$ it follows that A spans $\delta\gamma$. Thus $\Gamma(\lambda)$ is closed. Use Corollary 5.2 to show it is totally bounded. Given $\epsilon > 0$ there exists a dipolyhedron D such that $A - D = Q + \partial R$ with

$$E_\flat(A - D) \le E(Q) + E(R) \le c\epsilon(E(\partial A) + E(A)) \le c\epsilon(M(\gamma) + \lambda).$$

It follows that $\Gamma(\lambda)$ is totally bounded and thus compact.

We show that $\delta 0\gamma \in \Gamma(\lambda)$. Since 0γ is a cone over a polyhedron it is a flat chain with $E(\delta 0\gamma) = M(0\gamma) \le \frac{\lambda'}{k+1} M(\gamma) = \lambda$ ([Fl1] §6). By Proposition 4.1 $\partial\delta 0\gamma = \delta\gamma$ and we know $\delta 0\gamma$ spans its boundary. Since $|\delta 0\gamma| \subset Q_{\lambda'}$ the result follows □

Let $m = \inf\{W(A) : A \in \Gamma(\lambda)\}$. There exist $A_i \in \Gamma(\lambda)$ such that $W(A_i) \to m$. By compactness (Theorem 5.5) the sequence A_i has a subsequential limit $A \in \Gamma(\lambda)$ with $W(A) \le \liminf W(A_i) \le m$. Then $W(A) = m$. According to Lemma 5.4 it follows that $m > 0$.

FIGURE 1. A dipolyhedron that does not span its boundary. Drawing by Harrison Pugh

It may be that there is another A with smaller area spanning γ with $E(A) \leq \lambda$ that is not supported in Q_λ. Let f_λ denote the projection into Q_λ. Since A spans its boundary so does $A_\lambda = f_\lambda A$. From Proposition 4.4 we conclude $A_\lambda \in \Gamma(\lambda)$. By Proposition 4.4 $W(A_\lambda) \leq W(A)$ and $E(A_\lambda) \leq E(A) \leq \lambda$, so we may replace A with A_λ.

The flat dipolyhedron A is our solution to Plateau's problem for soap films with energy bounded by λ.

Theorem 5.6 (Almost everywhere regularity)**.** *The set* $|A| \setminus |\partial A|$ *is a smooth surface except on a union of Lipschitz Jordan curves with finite total length.*

Proof. By Theorem 3.2 $A = \delta B + \mu C$ where B and C are flat chains (mod two) with finite mass and finite boundary mass. Decompose B into its indecomposable parts $B = \sum B_i$.) Since B_i is area minimizing, we may apply mod two regularity ([Fl2]) to deduce each surface $|B_i|$ is smoothly embedded away from its boundary. In his thesis, Ziemer proved that mod two boundaries are integral currents ([Z], 6.5). Fleming proved that integral 1-cycles with finite mass are sums of closed curves each of which is Lipschitz. Since $\partial C = 0$ and $M(C) < \lambda$ then C is a sum of Lipschitz Jordan curves with finite total length. (See [Fe], 4.2.25) ☐

References

[A] F.J. Almgren, Jr., Plateau's Problem, Benjamin, New York, 1966.

[D] Jesse Douglas, Solution of the problem of Plateau, Trans. Amer. Math. Soc. 33 (1931), 263-321.

[Fe] Federer, Herbert, Geometric Measure Theory, Springer-Verlag, New York, 1969

[FF] Herbert Federer and Wendell H. Fleming, Normal and integral currents, Ann. of Math. 72 (1960), 458-520.

[Fl1] Wendell H. Fleming, Flat chains over a finite coefficient group, Trans. AMS, Jan-Feb 1966 v 121 pp160-186

[Fl2] Wendell H. Fleming, On the oriented Plateau problem, Rend. Circ. Mat. Palermo (II) 11 (1962), 1-22

[H] Harrison, Jenny, Cartan's magic formula and soap film structures, to appear, Journal of Geometric Analysis

[O] Robert Osserman, A Proof of the Regularity Everywhere of the Classical Solution to Plateau's Problem, Ann. of Math. 91, (1970), 550-569.

[P] J.A.F. Plateau, Statique Experimentale et Theorique des Diquides Soumis aux Seules Forces Moleculaires, Paris, Gauthier-Villars, 1873.

[W] Hassler Whitney, Geometric Integration Theory, Princeton University Press, 1957.

[Z] William P. Ziemer, Integral currents mod 2, Trans of the AMS, Vol 105, Issue 3, (1962), 496-524

Department of Mathematics, University of California, Berkeley, Berkeley, CA, 94705, USA
E-mail address: `harrison@math.berkeley.edu`