OPENING CLOSED LEAVES OF FOLIATIONS

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Novikov proved that every C^1 codimension one foliation of S^3 has a closed leaf ([**5**, Theorems 6.1 and 7.1]). In higher codimension, the situation is quite different. According to Schweitzer, if a manifold M has a C^r foliation of codimension $q \geqslant 3$ with $0 \leqslant r \leqslant \infty$, then it possesses a foliation with no closed leaves ([**6**, Theorem D]). To get the same result in codimension $q = 2$, Schweitzer uses the celebrated Denjoy C^1-toroidal flow containing a proper minimal set with no closed trajectories (see [**1**]). Since that phenomenon cannot occur in a C^2 flow on a surface (see [**1**]) his methods give only C^1 results when $q = 2$.

In [**3**] the author topologically embeds the Denjoy C^1 vector field in a C^2 vector field defined on a punctured, thickened torus, $N = (T^2 \setminus D^2) \times I$ to obtain a C^2 "flow plug". Much as in Schweitzer, but with an alternate exposition, this flow plug can be modified and used to open closed leaves of C^2 foliations.

Let M be a C^∞ smooth, paracompact manifold without boundary of dimension $k \geqslant 3$, and $\mathscr{F}$ a C^r foliation of M. A leaf of $\mathscr{F}$ is *closed*, if it is closed as a subset of M.

THEOREM A. *If there exists a C^r foliation $\mathscr{F}_0$ of M of codimension two, $r = 0$, 1 or 2, then there exists such a foliation $\mathscr{F}_1$ with no closed leaves.*

In order to prove Theorem A, we reduce the problem to the case where the closed leaves of $\mathscr{F}_0$ have a locally finite family of disjoint neighborhoods in M. To do this, we use the following lemma and corollary.

LEMMA 1 (Fuller [**2**]). *There exists a C^∞ non-singular vector field X_1 defined on a neighborhood of the closed unit cube I^3 in $\mathbb{R}^3$ satisfying*

(i) $X_1(p) = -\partial/\partial z$ *for p in a neighborhood of* ∂I^3;

(ii) X_1 *has exactly one periodic trajectory;*

(iii) *every trajectory of X_1 starting in some open subset of the top face of I^3 enters* Int(I^3) *and never exits.*

Sketch proof. Let Y_1 be a vector field on the annulus $A = S^1 \times I^1$ such that $S^1 \times \{\frac{1}{2}\}$ is its only periodic trajectory. Let $Y = Y_1 \times 0$ be the trivial product vector field on the thickened annulus $A \times I$. Smoothly embed $A \times I$ in Int(I^3) so that $A \times \{t\} \subset I^2 \times \{\frac{1}{2}t\}$. Let $Z = -\partial/\partial z$ and suitably average Y and Z to obtain X_1 satisfying (i)–(iii).

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COROLLARY 1. *There exists a C^∞ codimension two foliation $\mathscr{G}_1$ defined on a neighborhood of the closed unit cube I^k in $\mathbb{R}^k$ $(k > 3)$ satisfying*

(i) *near ∂I^k, the first two coordinates of a leaf of $\mathscr{G}_1$ are constant;*

(ii) *$\mathscr{G}_1$ has exactly one compact leaf;*

(iii) *there exists an open subset W_k of ∂I^k such that if a leaf L of $\mathscr{G}_1$ meets W_k then $L \cap I^k$ is not closed.*

Proof. We use the vector field X_1 of Lemma 1 to construct a foliation with corresponding properties. Let $n = k-2$, let D^n denote the unit disk in $\mathbb{R}^{k-2}$ and let $D^n_0 = D^n - \{0\}$. Let $\rho : D^n \to I$ be the Euclidean norm $\rho(x) = \|x\|$. Then

$$Id \times \rho_0 : I^2 \times D^n_0 \to I^2 \times (0, 1]$$

is a submersion since $\rho_0 = \rho \mid D^n_0$ is. Let $\mathscr{H}_1$ be the C^2 foliation of I^3 by trajectories of the vector field X_1. It follows from results of Wilson [7] that $\mathscr{H}_1$ induces a C^2 foliation $(Id \times \rho_0)^{-1}\mathscr{H}_1$ of $I^2 \times D^n_0$. Near $I^2 \times \{0\}$ the leaves have the same form as $x \times D^n_0, x \in I^2$ (see Lemma 2 (ii)) and thus the foliation extends uniquely to a C^2 foliation $\mathscr{G}'_1$ of $I^2 \times D^n \subset I^k$ and then trivially to a C^2 foliation $\mathscr{G}_1$ of I^k.

The leaves of $\mathscr{G}'_1$ have the form $(Id \times \rho)^{-1}(L)$ where L is a leaf of $\mathscr{H}_1$. Thus the desired properties (i)–(iii) of the leaves of $\mathscr{G}_1$ follow from the corresponding properties of X_1, and therefore of $\mathscr{H}_1$ given by Lemma 1.

PROPOSITION 1 (Wilson). *Every C^r $(r \geqslant 0)$ codimension two foliation $\mathscr{F}_0$ is homotopic to a C^r foliation $\mathscr{F}'_0$ whose closed leaves have a locally finite family of disjoint neighborhoods.*

Proof. Using elementary techniques as in [7] one can construct a locally finite family $\{U_\alpha\}$ of disjoint foliation charts such that each leaf of $\mathscr{F}_0$ passes through some open subset V_α of ∂U_α homeomorphic to $\mathbb{R}^{k-1}$.

Let $h_\alpha : I^k \to U_\alpha$ be a diffeomorphic embedding such that $h_\alpha(W_k)$ contains V_α. If $k > 3$, use h_α to pull over the foliation $\mathscr{G}_1$ of Corollary 1 onto U_α, for each α, to obtain a new foliation $\mathscr{F}'_0$.

Consider any leaf L of $\mathscr{F}_0$. Since $L \cap \partial \bar{U}_\alpha$ is connected and the only changes are made inside U_α, then L corresponds to a leaf L' of $\mathscr{F}'_0$ such that $L = L'$ outside $\bigcup U_\alpha$. By construction L meets some $V_\alpha \subset h_\alpha(W_k)$. Hence L' is not closed. Thus there is one closed leaf inside each U_α and no others.

If $k = 3$, use h_α to pull over X_1 of Lemma 1. In this dimension $L \cap \partial \bar{U}_\alpha$ is not connected, but part (iii) of Lemma 1 enables us to use the preceding argument.

The next step is to use the C^2 "flow plug" of [3] to open the closed leaves of $\mathscr{F}'_0$.

LEMMA 2. *There exists a C^2 non-singular vector field X defined on a neighborhood of I^3 in $\mathbb{R}^3$ satisfying*

(i) *$X(p) = -\partial/\partial z$ for p in a neighborhood of ∂I^3;*

(ii) *X has no periodic trajectories;*

(iii) *if an orbit of* X *enters at* $(x, y, 1)$ *in the top face of* I^3 *and exits* I^3 *at* $(x', y', -1)$ *in the bottom face of* I^3, *then* $x = x'$ *and* $y = y'$;

(iv) *at least one orbit enters* I^3 *at* $(0, 0, 1)$, *say, and never exits.*

Proof. See [3, §4].

We construct a similar "plug" for codimension two foliations with isolated closed leaves.

COROLLARY 2. *There exists a* C^2 *codimension two foliation* $\mathcal{G}$ *defined on a neighborhood of* I^k *in* $\mathbb{R}^k$ $(k > 3)$ *satisfying*

(i) *near* ∂I^k *the first two coordinates of a leaf of* $\mathcal{G}$ *are constant;*

(ii) $\mathcal{G}$ *has no compact leaves;*

(iii) *at least one leaf of* $\mathcal{G}$ *meeting* ∂I^k *is not closed as a subset of* I^k.

Proof. In the proof of Corollary 1, replace $\mathcal{H}_1$ by $\mathcal{H}$, X_1 by X, $\mathcal{G}_1$ by $\mathcal{G}$ and Lemma 1 by Lemma 2.

Proof of Theorem A. Apply Proposition 1 to obtain $\mathcal{F}'_0$ with only isolated closed leaves. Recall the isolated closed leaves $L_\alpha \subset U_\alpha$ of Proposition 1. Let $W_\alpha \subset U_\alpha$ be a flow box meeting L_α in its interior. If $k > 3$ apply Corollary 2 to give a new foliation structure to W_α which "opens" L_α (see Corollary 2, parts (i) and (iii)) and introduces no new closed leaves (part (ii)). If $k = 3$ use Lemma 2.

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