NUMERICAL INTEGRATION OF VECTOR FIELDS
OVER CURVES WITH ZERO AREA

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ABSTRACT. The boundary of a Jordan domain A may be a nonsmooth curve γ. If F is a smooth vector field defined near γ, then F is integrable over γ provided γ has two-dimensional Lebesgue measure zero and F is sufficiently smooth. When actually computing the integral $\int_\gamma F \cdot ds$, one might hope that $\lim_{k\to\infty} \int_{\gamma_k} F \cdot ds = \int_\gamma F \cdot ds$ for PL or smooth approximators γ_k of γ. Several examples show that this is not the case. However, there are algorithms for choosing γ_k so that $\int_{\gamma_k} F \cdot ds$ converges to $\int_\gamma F \cdot ds$ exponentially quickly.

1. INTRODUCTION AND PRELIMINARIES

Let S^1 denote the unit circle, oriented positively. A *Jordan curve* γ is the image of a 1-1, continuous mapping $\varphi: S^1 \to \mathbb{R}^2$. The mapping φ is called a *parametrization* of γ and induces an orientation on γ. The bounded region D which γ bounds is called its *Jordan domain*. Let m denote the two-dimensional Lebesgue measure.

Let $F: U \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a vector field defined on an open subset U of $\mathbb{R}^2$ and γ a Jordan curve contained in U. In case there is a smooth parametrization φ of γ, the line integral $\int_\gamma F \cdot ds$ is classically defined by $\int_0^1 F(\varphi(t)) \cdot \varphi'(t)\, dt$.

The answer is independent of φ. This definition does not make sense if γ is without tangent vectors. The line integral has recently been extended to arbitrary continuous Jordan curves with two-dimensional Lebesgue measure zero, provided F is sufficiently smooth. (see [H-N].)

Definitions. Let K be a bounded set in $\mathbb{R}^2$. Let $N(\varepsilon)$ denote the minimum number of squares of side length epsilon required to cover K. The *box dimension* of K, denoted $\dim(K)$, is defined by

$$\dim(K) = \lim_{\varepsilon \to 0} -\frac{\log N(\varepsilon)}{\log \varepsilon},$$

if the limit exists.

A *k-square* from the binary grid is a square of the form

$$Q = [j2^{-k}, (j+1)2^{-k}] \times [l2^{-k}, (l+1)2^{-k}]$$

where j, $l \in \mathbb{Z}$. One need only consider k-squares to calculate $\dim(K)$.

$$\dim(K) = \lim_{k \to \infty} -\frac{\log N(k)}{k \log 2},$$

where $N(k)$ is the minimum number of k-squares from the binary grid required to cover K.

Let U be an open subset of $\mathbb{R}^2$. A continuous function $f: U \to \mathbb{R}$ is said to be of class C^0. A function $f: U \to \mathbb{R}$ is said to be of class C^α, $0 < \alpha < 1$, if there exists a constant $M > 0$ such that $|f(x) - f(y)| < M|x - y|^\alpha$ for all $x, y \in U$. Define $\mathbb{Z}^+$ to be the set of integers ≥ 1. We say f is of class C^s, $s = r + \alpha$, $r \in \mathbb{Z}^+$, $0 \leq \alpha < 1$, if the rth partial derivatives of f exist and are of class C^α. If f is defined on a closed subset K of $\mathbb{R}^2$, it is said to be of class C^s if it extends to a function of class C^S defined in an open neighborhood U of K. A function is *smooth* if it is of class C^1. A vector field $F: U \subset \mathbb{R}^2 \to \mathbb{R}^2$ is of class C^S if each of its component functions F_1 and F_2 is of class C^S. For $p \in U$, define $|F(p)| = \sqrt{F_1(p)^2 + F_2(p)^2}$. Define $|F|_0 = \sup\{|F(p)| : p \in U\}$. If $F \in C^\alpha$, $0 < \alpha < 1$, the α-norm of F is defined by

$$|F|_\alpha = |F|_0 + \sup\left\{\frac{|F(p) - F(q)|}{|p - q|^\alpha}, \ p, q \in U, \ p \neq q\right\}.$$

For curves with two-dimensional Lebesgue measure zero, the existence of $\int_\gamma F \cdot ds$ is a consequence of the assumption that F has Hölder exponent larger than the box dimension of γ minus 1. (See [H-N].)

An *arc* σ is the image of a 1-1 continuous mapping $\varphi: [0, 1] \to \mathbb{R}$, oriented so that $\sigma(0) < \sigma(1)$. We will show that the theory of integration over Jordan curves does not extend to all arcs in the plane, but it is valid for arcs with finitely accessible endpoints: σ has *finitely accessible endpoints* if there exists a rectifiable arc τ so that $\sigma \cup \tau$ is a Jordan curve and $\sigma \cap \tau = \{\sigma(0), \sigma(1)\}$.

For $1 < d \leq 2$, let $\Lambda_d = \{(\gamma, F) : \gamma$ is a Jordan curve or an arc with finitely accessible endpoints, $\dim(\gamma) < d$, $m(\gamma) = 0$ and F is a vector field defined in a neighborhood of γ of class $C^{d-1}\}$. It is proved in [H-N] that the line integral extends to elements of Λ_d. That is, there is a real valued function $\int$ defined on Λ_d such that $\int(\gamma, F) = \int_\gamma F \cdot ds$ as long as there is a smooth parametrization of γ. In general, we denote $\int(\gamma, F) = \int_\gamma F \cdot ds$ for $(\gamma, F) \in \Lambda_d$. Moreover, the following is proved in [H-N]:

Theorem 1. *The mapping* $\int: \Lambda_d \to \mathbb{R}$ *has the following properties*:
(i) *Bilinearity*:

$$\int_\gamma (aF + bG) \cdot ds = a \int_\gamma F \cdot ds + b \int_\gamma G \cdot ds \quad \text{for } (\gamma, F), (\gamma, G) \in \Lambda_d;$$

$$\int_{\gamma_1} F \cdot ds + \int_{\gamma_2} F \cdot ds = \int_{\gamma_1 \cup \gamma_2} F \cdot ds \quad \text{for } (\gamma_1, F), (\gamma_2, F) \in \Lambda_d,$$

and $\gamma_1 \cap \gamma_2$ *with one-dimensional Lebesgue measure zero.*

(ii) *Independence of the extension*: If (γ, F), $(\gamma, G) \in \Lambda_d$ and $F|_\gamma = G|_\gamma$ then $\int_\gamma F \cdot ds = \int_\gamma G \cdot ds$.

(iii) *Fundamental theorem of calculus*: $\int_\sigma \nabla f \cdot ds = f(\sigma(1)) - f(\sigma(0))$ for an arc σ, $(\sigma, \nabla f) \in \Lambda_d$.

(iv) *Green's theorem*: Suppose F is smooth away from γ, D is its Jordan domain and $(\gamma, F) \in \Lambda_d$. Then $\int_\gamma F \cdot ds = \int_D (\nabla \times F) \cdot dA$.

Remarks.

• One may not replace "box dimension" with "Hausdorff dimension". That is, there is no generalized integral defined for all pairs (γ, F) with the Hölder exponent of F exceeding the Hausdorff dimension of γ minus one that satisfies both (ii) and (iv). (see [H1].)

• Theorem 1 does not extend to all arcs. Let σ be the arc depicted in Figure 1 and F the analytic vector field $F(x, y) = (y, 0)$. Let σ_k denote the portion of σ outside the $1/k$-neighborhood of the origin so that $\sigma_k \to \sigma$, pointwise. Clearly, if there are enough loops in the spiral the integral of F along σ_k will diverge as $k \to \infty$.

Note that σ does not have finitely accessible endpoints and therefore does not contradict Theorem 1.

• If $m(\gamma) > 0$ then (iv) cannot be satisfied. Let D be the Jordan domain of γ. Assume that γ is a subset of the unit disk A. Let $A' = A \setminus D$ and $F(x, y) = (0, x)$. If the integral $\int_\gamma x\, dy$ exists and satisfies (iv) then it equals $\int_D dx\, dy$. Applying (iv) again gives

$$\int_\gamma x\, dy = \int_{S^1} x\, dy - \int_{A'} dx\, dy = \int_A dx\, dy - \int_{A'} dx\, dy.$$

Hence $m(A) = m(D) + m(A')$, contradicting the assumption that $m(\gamma) > 0$.

• If $\dim(\gamma) = 1 + \alpha$ the result fails. Hassler Whitney constructed a function "nonconstant on a connected set γ of critical points". Properties (ii) and (iii) of Theorem 1 cannot both be satisfied. Otherwise, by (iii), $\int_\sigma \nabla f \cdot ds = f(\sigma(b)) - f(\sigma(a))$. The left-hand side of the equation is zero by (ii) applied to $G \equiv 0$. The right-hand side is nonzero since f is nonconstant on σ. It is not hard to show that $\nabla f \in C^\alpha$ and $\dim(\gamma) = 1 + \alpha$ so Theorem 1 is "sharp". (See [W, N, H-N].)
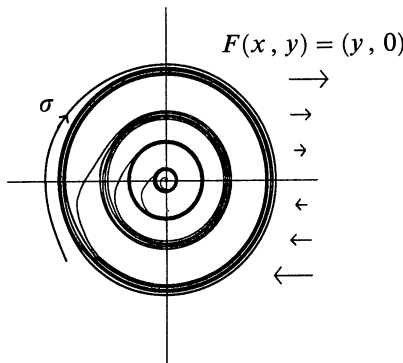

FIGURE 1. An arc σ for which the line integral is not defined.

2. Two counterexamples

Since tangent vectors do not generally exist for continuous Jordan curves, conventional methods of integration are not valid. However, the theory *guarantees* the existence of $\int_\gamma F \cdot ds$, provided $(\gamma, F) \in \Lambda_d$. If F is of class C^1 and $m(\gamma) = 0$, one may always calculate $\int_A (\nabla \times F) \cdot dA$ and this will equal $\int_\gamma F \cdot ds$.

If F is only of class C^α, the integral is defined using the Whitney extension theorem: F may be defined on all of $\mathbb{R}^2$. Restrict F to γ, then extend $F|_\gamma$ to a new vector field G on $\mathbb{R}^2$ that is C^∞ away from γ and globally of class C^α. The old values of F away from γ are completely discarded. Now one may use the curl formulation and define

$$\int_\gamma F \cdot ds = \int_A (\nabla \times G) \cdot dA.$$

This integral exists and is independent of the extension G. (See [H-N].) However, it is not practical to use the Whitney extension theorem in applications. A natural approach to actually finding the integral would be to integrate over approximating curves that are smooth or PL. There is no hope that the limit will generally converge to anything meaningful if the hypotheses in Theorem 1 are not met, so we henceforth assume that $(\gamma, F) \in \Lambda_d$. That is, $m(\gamma) = 0$, $\dim(\gamma) < d$ and $F \in C^{d-1}$.

Even if the hypotheses are met, there can be problems as the next examples show.

Let γ be a Jordan curve. We say $\{\gamma_k\}$ *approximates* γ if γ_k converges to γ in the Hausdorff metric and γ_k is oriented in the same way as γ. We write $\gamma_k \to \gamma$. There are two natural classes of approximators of γ:

(a) S is a *secant approximator* of γ if S is PL and its vertices are contained in γ and have the same cyclical order in S as in γ. That is, suppose $x_i \in \gamma$, $0 \le i \le n$, with $x_0 < x_1 < \cdots < x_n = x_0$ in γ. Then $X = \{x_i, 0 \le i \le n\}$ forms an *ordered partition* of γ. An ordered partition X determines an oriented PL curve S_X called a *secant approximator* of γ which is the union of the oriented straight line segments in $\mathbb{R}^2$ with endpoints $x_i < x_{i+1}$.

(b) S is an internal approximator of γ if S is contained in the Jordan domain bounded by γ. (Of course, one may equally consider external approximators.) We require that S be either smooth or PL.

We next show even if the integral $\int_\gamma F \cdot ds$ exists, not all secant or internal approximators of γ may be used to approximate the integral. That is, there exist $\gamma_k \to \gamma$ with $\int_{\gamma_k} F \cdot ds \nrightarrow \int_\gamma F \cdot ds$.

Example 1. The integral of an analytic vector field F over a secant approximator of a Jordan curve γ does not necessarily approximate the integral of F over γ.

Consider the vector field $F(x, y) = (y, 0)$ and the Jordan curve γ illustrated in Figure 2. The curve γ is the countable union of arcs σ_i. Each arc σ_i contains many "thin" half-moon turns along which the integral of F is virtually zero so that $\int_\gamma F \cdot ds > 0$. The secant approximator γ_k closely approximates $\sigma_1, \ldots, \sigma_k$ so that the integrals of F over $\bigcup \sigma_i$, $i = 1, \ldots, k$, and γ_k are
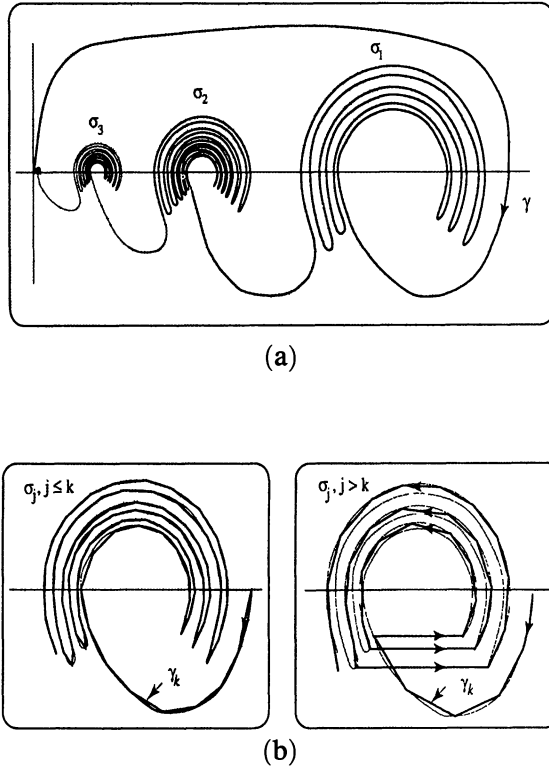
(a)

(b)

FIGURE 2. (a) The curve γ is a countable union of arcs σ_l. (b) The secant approximator γ_k on each subarc of γ.

essentially the same. The figure indicates how γ_k is defined over the arcs σ_i, $k < i \leq k^2$. It is not hard to see that with a little care, $\int_{\sigma_k} F \cdot ds < 0$.

This is a counterexample to a conjecture of P. Jones that $\int_{\gamma_k} F \cdot ds \to \int_\gamma F \cdot ds$ if γ_k is a secant approximator of γ with $\gamma_k \to \gamma$ and the pair $(\gamma, F) \in \Lambda_d$. In the next section we provide an algorithm for finding secant approximators over which the integral do converge to the correct limit $\int_\gamma F \cdot ds$.

Example 2. The integral of a vector field F over an internal approximator of a Jordan curve γ does not approximate the integral of F over γ.

Let γ denote the boundary of the unit square, $[0, 1] \times [0, 1]$ and γ_k the internal approximator depicted in Figure 3 on the next page. Let

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \text{ or } x > 1, \\ (-1)^{n+1}/n & \text{for } x = 1/n \text{ and } n = 2, 3, \ldots. \end{cases}$$

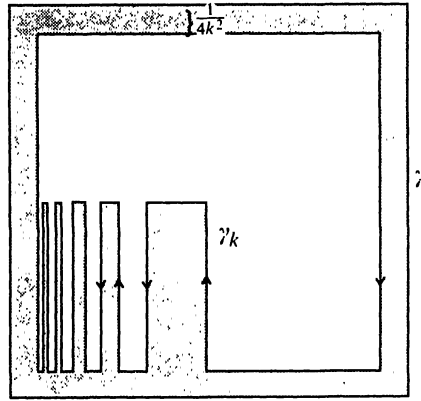Extend f to the rest of of $\mathbb{R}$ so that f has Hölder exponent $1/2$.

FIGURE 3

Let F be the vector field $F(x, y) = (0, f(x))$. By Theorem 1, the integral of F over γ exists. It is 0 by (ii). For $n \geq 1$, let

$$I_n = \left\{ \frac{1}{n} \right\} \times \left[\frac{1}{n}, 1 - \frac{1}{n} \right]$$

and

$$J_n = \begin{cases} \left[\dfrac{1}{n+1}, \dfrac{1}{n} \right] \times \left\{ \dfrac{1}{n} \right\} & \text{if } n \text{ is odd}, \\[3mm] \left[\dfrac{1}{n+1}, \dfrac{1}{n} \right] \times \left\{ 1 - \dfrac{1}{n} \right\} & \text{if } n \text{ is even}. \end{cases}$$

Define $\gamma_k = \bigcup I_n \cup J_n$, $2k \leq n \leq (2k)^2 - 1$, together with the five segments of the $(1 - 1/4k^2)$-square which make γ_k into a Jordan curve. (See Figure 3.) Note $\int_{I_n \cup J_n} F \cdot ds = (1/n)(1 - 2/n)$. It follows easily that $\lim \int_{\gamma_k} F \cdot ds$ grows as $\log k$.

A version of this example may be found in [H-N].

3. TWO ALGORITHMS

We describe algorithms for choosing secant approximators and internal approximators that can be used to estimate the integral of F over γ.

Definitions. If $U \subset \mathbb{R}^2$ is nonempty, the *diameter* of U is defined by $|U| = \sup\{|x - y| : x, y \in U\}$. Let $\lambda = \{\lambda_i\}$ be a collection of subsets of $\mathbb{R}^2$ and $d > 0$. The number $\sum_i |\lambda_i|^d$ is called the *d-sum* of λ. If $A = \bigcup T_i$ and the T_i are closed domains with disjoint interiors, we call $T = \{T_i\}$ a *decomposition* of A.

Let $K \subset \mathbb{R}^2$ be closed. We decompose $\mathbb{R}^2 \setminus K$ into k-squares Q of the form $Q = [j2^{-k}, (j + 1)2^{-k}] \times [l2^{-k}, (l + 1)2^{-k}]$ where $j, l \in \mathbb{Z}$: Fix $k \in \mathbb{Z}$ and consider the collection of k-squares in $\mathbb{R}^2$ that do not touch K or a k-square that touches K. Take the union over k, discarding any squares that are contained in larger ones. The final collection W is a decomposition of $\mathbb{R}^2 \setminus K$ and is called the *Whitney decomposition* of $\mathbb{R}^2 \setminus K$ (see Figure 4). It satisfies $|Q| \leq d(Q, K) \leq 4|Q|$, $Q \in \mathbf{W}$. Define A_k to be the union of all j-squares of $\mathbf{W}$, $j \leq k$.
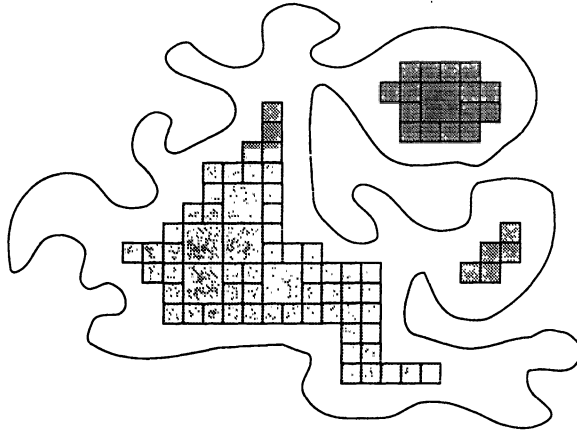
FIGURE 4. The Whitney decomposition of the Jordan domain of γ.

Theorem 2 (Continuity). *Let $d > 1$. There exists C such that if (γ_1, F), $(\gamma_2, F) \in \Lambda_d$ and γ_1 and γ_2 bound a region A decomposed by λ then*

$$\left| \int_{\gamma_1} F \cdot ds - \int_{\gamma_2} F \cdot ds \right| \le C|F|_{d-1} \times d\text{-sum}(\lambda).$$

For the proof, see [H-N, Theorem 3.7].

This theorem gives us an idea of which curves are good approximators. The region bounded by γ_k and γ should have a decomposition with small d-sum. We make an explicit choice.

Definition. Let γ be a Jordan curve and $C \ge 1$. A sequence $\{\gamma_k\}$ of secant approximators of γ is said to be *C-regular* if there exists N such that if $k \ge N$ then (i) γ_k is a Jordan curve oriented in the same direction as γ; (ii) any pair of successive vertices of γ_k is contained in the boundary of a k-square meeting γ; (iii) there are $\le C$ such pairs in any single k-square meeting γ.

A sequence $\{\sigma_k\}$ of secant approximators of an arc σ is said to be *C-regular* if σ_k contains $\sigma(0)$ and $\sigma(1)$ and (i) σ_k is embedded; (ii) any pair of successive vertices of σ_k are contained in the boundary of a k-square meeting σ, possibly excepting the endpoints; (iii) there are $\le C$ such pairs in any k-square meeting σ.

Algorithm for choosing secant approximators. *Let σ be an arc. There exists a 1-regular sequence $\{\sigma_k\}$ of secant approximators of σ. If γ is a Jordan curve, there exists a 3-regular sequence of secant approximators of γ.*

Construction. Let p and q be the endpoints of σ. Fix $k \ge 1$. Let T_1 be the union of k-squares that contain $x_0 = p$. If σ exists at all from T_1, let x_1 be the last exit point. (If σ does not exit T_1, let $x_1 = q$ and stop the process.) Let T_2 be the union of those k-squares not in T_1 that contain x_1. Let x_2 be the last exit point of σ from T_2, if it exists T_2 at all. (If σ does not exist T_2, let $x_2 = q$.) Inductively define x_i, forming a partition $X_k = \{x_i\}$, stopping when q is reached. Then $\sigma_k = S_{X_k}$ contains the endpoints p and q. The T_i

have disjoint interiors. Since σ_k contains at most one segment in each T_i, it is embedded. (The only possible ambiguity is at the vertices x_i, but since σ is embedded and $x_i \in \sigma$, there is no intersection at the vertices.) A pair of successive vertices of σ_k lies on the boundary of some k-square. Since there is only one such pair, at most, $\{\sigma_k\}$ is 1-regular.

While constructing secant approximators to Jordan curves we prove a slightly stronger statement that will be useful later:

($*$) *If $\{\sigma_k\}$ is a C-regular sequence of secant approximators of σ and τ is any connecting arc for σ, there exist arcs $\tau_k \to \tau$ so that $\{\tau_k\}$ is a $(C + 1)$-regular sequence of secant approximators to τ and $\{\sigma_k \cup \tau_k\}$ is a $(2C + 1)$-regular sequence of secant approximators of γ.*

Let $\{\sigma_k\}$ be a regular sequence of secant approximators converging to σ and τ a connecting arc for σ. We construct a partition Y_k of τ so that $\tau_k = S_{Y_k}$ meets σ_k only at p and q, and $\{\tau_k\}$ is a regular sequence of secant approximators of τ. The method is similar to that preceding. Replace the k-squares with the components of $T \setminus \sigma_k$ for T a k-square. By assumption, there are at most $C + 1$ of these components U. Let U_1 denote the union of those components with q in their closure and set $y_0 = q$. Let y_1 be the last exit point of τ from U_1. Observe that the line segment with endpoints y_0 and y_1 misses σ_k except at q. Begin the process again with y_1 and obtain a partition $Y = \{q = y_0, y_1, \ldots, y_n = p\}$. The secant approximators $\tau_k = S_Y$ form a $(C + 1)$-regular sequence. Hence $\{\sigma_k \cup \tau_k\}$ is $(2C + 1)$-regular.

Theorem 3. *Suppose $(\lambda, F) \in \Lambda_d$. If $\{\lambda_k\}$ is a C-regular sequence of secant approximators of λ then*

$$\int_\lambda F \cdot ds = \lim_{k \to \infty} \int_{\lambda_k} F \cdot ds.$$

Proof. Suppose that $\lambda = \gamma$ is a Jordan curve. By the Continuity Theorem we merely have to show that the region Y_k bounded by γ and γ_k has a decomposition with small d-sum.

Recall the sets A_n determined by the Whitney decomposition $\mathbf{W}$ of $\mathbb{R}^2 \setminus \gamma$:

$$A_n = \bigcup Q \colon Q \text{ is a } j\text{-square of } \mathbf{W}, \qquad j \le n.$$

The region Y_k is contained in $\mathbb{R}^2 \setminus A_{n_k}$ with n_k sufficiently large. Each Y_k is the union of those $Q \in \mathbf{W}$ that are completely contained in Y_k and the closure of the components of $Q \setminus \gamma_k$ that are contained in Y_k. This induces a decomposition $\mathscr{T}_k$ of Y_k. The curve γ_k only intersects k-squares meeting γ and does so at most C times. Thus it meets each $Q \in \mathbf{W}$ at most C times. It follows that the d-sum of $\mathscr{T}_k$ is bounded $(C + 1) \times$ the d-sum of $\mathbf{W}$ restricted to $\mathbb{R}^2 \setminus A_{n_k}$. The latter sum $\to 0$ as $k \to \infty$; therefore the former sum $\to 0$.

For arcs $\lambda = \sigma$, first connect the endpoints of σ with a rectifiable arc τ. Assume that τ is smooth away from its endpoints. Apply ($*$) to find τ_k so that $\{\sigma_k \cup \tau_k\}$ is $(2C + 1)$-regular sequence of secant approximators of $\sigma \cup \tau$. Since $\sigma \cup \tau$ is a Jordan curve we may apply the first part to conclude that $\int_{\sigma_k \cup \tau_k} F \cdot ds \to \int_{\sigma \cup \tau} F \cdot ds$. By Theorem 1(i) we have $\int_{\sigma_k} F \cdot ds + \int_{\tau_k} F \cdot ds \to \int_\sigma F \cdot ds + \int_\tau F \cdot ds$. Since τ is smooth away from its endpoints standard methods show that $\int_{\tau_k} F \cdot ds \to \int_\tau F \cdot ds$. Therefore $\int_{\sigma_k} F \cdot ds \to \int_\sigma F \cdot ds$. $\square$

Speed of convergence. Let $d > 1$ and $(\gamma, F) \in \Lambda_d$. For any $0 < \varepsilon < d - \dim(\gamma)$ there exists $C > 0$ such that

$$\left| \int_{\gamma_k} F \cdot ds - \int_{\gamma} F \cdot ds \right| \le C|F|_{d-1} 2^{-k(d - \dim(\gamma) - \varepsilon)}.$$

To see this, let Y_k be the region bounded by γ and γ_k. Recall the decomposition T_k of Y_k induced by the Whitney decomposition of $\mathbb{R}^2 \setminus \gamma$ (see the proof of Theorem 3). By Theorem 2

$$\left| \int_{\gamma_k} F \cdot ds - \int_{\gamma} F \cdot ds \right| \le C_1 |F|_{d-1} \times d\text{-sum}(T_k).$$

It follows easily from the definition of box dimension that

$(**)$ *The number of k-squares in the Whitney decomposition of $\mathbb{R}^2 \setminus \gamma$ is bounded by $2^{k(\dim \gamma + \varepsilon k)}$, where $\varepsilon_k \to 0$.*

Thus

$$d\text{-sum}(T_k) \le \sum_{n=k}^{\infty} 2^{(\dim \gamma + \varepsilon_n - d)n} \le C_2 \sum_{n=k}^{\infty} 2^{(\dim \gamma + \varepsilon - d)n}.$$

Since $\varepsilon < d - \dim(\gamma)$, there exists C_3 such that $2^{\dim \gamma + \varepsilon - d} < 1 - 1/C_3$. Therefore, $d\text{-sum}(T_k) \le C_2 C_3 2^{-k(d - \dim(\gamma) - \varepsilon)}$.

Algorithm for choosing internal approximators. Recall A_k determined by the Whitney decomposition $\mathbf{W}$ of $\mathbb{R}^2 \setminus \gamma$. Define $\gamma_k = \partial A_k$. For k sufficiently large, γ_k is nonempty. It is a union of segments of length 2^{-k}, each from the boundary of a k-square of $\mathbf{W}$. One may use γ_k to approximate the integral (see [H-N]) or, more simply, consider the sum of the integrals around each k-square that is contained in $\mathbb{R}^2 \setminus \gamma$ and does not touch any k-square which touches γ. It is easy to show, using $(**)$, that the integral along the approximating curves converge exponentially quickly to the limit integral along γ.

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