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The Loxodromic Mapping Problem

JENNY HARRISON

Department of Mathematics, Yale University,
New Haven, Connecticut 06511

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The Seifert conjecture asserts

Every vector field on the three sphere S^3 has either a zero or a closed integral curve.

Paul Schweitzer [6] showed that this conjecture is false for C^1 vector fields. The author [3] has constructed C^2 counterexamples.

In this paper we describe a reduction of the problem by one dimension. That is, a vector field on S^3 with no zeroes and no closed integral curves necessarily exists if there is a diffeomorphism of S^2 with a few dynamical properties.

Begin with a Denjoy diffeomorphism [1]

$$g: S^1 \to S^1.$$

Dynamically, g has no periodic points and has an invariant Cantor set. (The Appendix gives a detailed discussion of g.) We embed g in a diffeomorphism of S^2

$$f: S^2 \to S^2.$$

The north pole N is repelling, the south pole S is attracting. One orbit is asymptotic to both N and S. Apart from that, the dynamics of f are like that of g: there are no periodic points in $S^2 \setminus (N \cup S)$ and there is an invariant Cantor set.

The last step is to suspend f to obtain a tangent vector field

$$X: S^3 \to \mathbb{R}^4$$

which has no zeroes and no closed integral curves. It is as smooth as the diffeomorphism f.

THEOREM A. *Suppose there exists a C^r orientation preserving diffeomorphism $f: S^2 \to S^2$ with no other periodic points than the north pole N*

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which repels and the south pole S which attracts. Suppose there exist one orbit asymptotic to both N and S and one orbit that is not. Then there exists a C^r vector field on S^3 which has no zeroes and no closed integral curves.

Theorem A describes a dynamic component $f: S^2 \to S^2$ more basic than a Seifert counterexample. Since it is a diffeomorphism in dimension two rather than a vector field in dimension three, it is more amenable for study.

Identify S^1 with $\mathbb{R}^1 \backslash \mathbb{Z}^1 \cong [0, 1)$. Denote the annulus $S^1 \times [-1, 1]$ by A and its boundary components $S^1 \times \{1\}$ by $\partial^+ A$ and $S^1 \times \{-1\}$ by $\partial^- A$. Let $B^+ = \{(x, t): \frac{1}{2} < t < 1\}$ and $B^- = \{(x, t): -1 < t < -\frac{1}{2}\}$. Let Id be the identity transformation, $\text{Id}(x) = x$.

The existence of $f: S^2 \to S^2$ in the hypothesis of Theorem A implies the existence of a C^r orientation preserving diffeomorphism $f_1: A \to A$ such that

(i) $f_1 \mid \text{int}(A)$ has no periodic points;

(ii) $f_1 \mid \partial A$ is the identity transformation.

(iii) $f_1(B^+) \cap f_1^{-1}(B^-) = \varnothing$; if $(x, t) \in B^+ \cup f_1^{-1}(B^-)$ then $f_1(x, t) = (x, t')$, where $t' < t$.

(iv) There exists a point $p \in f_1(B^+)$ such that $f_1(p) \in f_1^{-1}(B^-)$.

(v) There exists $q \in A$ with its orbit bounded away from ∂A.

It is relatively straightforward to obtain (i)–(iii) from the fact that N is an attracting fixed point and S is repelling. Wlog we may assume that there are disjoint neighborhoods C_0^+ of N and C_0^- of S, homeomorphic to discs, such that f is the identity precisely on smaller disc neighborhoods D_0^+ of N and D_0^- of S and simply repelling or attracting in the complementary regions $B_0^+ = C_0^+ \backslash D_0^+$ and $B_0^- = C_0^- \backslash D_0^-$. We may also assume that B_0^+ and B_0^- are sufficiently small that $f(B_0^+) \cap f^{-1}(B_0^-) = \varnothing$. This may be achieved without introducing new periodic points outside of $B_0^+ \cup B_0^-$ and maintaining the existence of $p' \in S^2$ which is asymptotic to both N and S and q' which is not. We may assume that $p' \in f(B_0^+)$. A power f^n will satisfy $f^n(p') \in f^{-1}(B_0^-)$. Now remove D_0^+ and D_0^- from S^2 and replace f with f^n. Reparametrize to obtain $f_1: A \to A$ satisfying (i)–(iv).

Since there is a point $q' \in S^2$ which is not asymptotic to both N and S, we may assume that its entire f-orbit is bounded away from S, say. Since N is repelling, the Ω-limit set κ of q' is bounded away from N. Thus κ is bounded away from both N and S. Since κ is closed, invariant, and non-empty, it contains the entire orbit of some point q''. Hence the orbit of q'' is bounded away from N and S. Use q'' to find $q \in A$ with its orbit bounded away from ∂A.

Since f_1 is orientation preserving, it is isotopic to $f_0 = \text{Id}$ by a C^r isotopy f_s (see [7]). By (iii) we may assume the isotopy decreases t-levels in

$B^+ \cup f_1^{-1}(B^-)$, i.e., $f_s(x, t) = (x, t')$, where $t' \leqslant t$ and $f_s \mid \partial A = \mathrm{Id}$. For s near 0, let $f_s = \mathrm{Id}$ and for s near 1, let $f_s = f_1$.

Let $A \times S^1$ have coordinates $(x, t\, s)$, where $(x, t) \in A$ and $s \in [0, 1)$. The isotopy defines a suspension flow $F_u: A \times S^1 \to A \times S^1$ by

$$F_u(x, t, x) = (f_{s+u} \circ f_s^{-1}(x, t), s + u).$$

The flow conditions $F_{u+v} = F_u \circ F_v$ and $F_0 = \mathrm{Id}$ are easily verified. By the chain rule, F_u is clearly C^r away from the slice $A \times \{0\}$. Since the isotopy is constant near $s = 0$ or $s = 1$, the flow is trivial in a neighborhood of $A \times \{0\}$: $F_u(x, t, s) = (x, t, s + u)$. Thus F_u is C^r on $A \times S^1$.

If $K \subset A$, denote the suspension of K to be $\{F_u(K \times \{0\}) : 0 \leqslant u \leqslant 1\}$.

Let B denote the suspension of $B^+ \cup f_1^{-1}(B^-)$. Then the suspension η of the entire orbit of q is disjoint from B. Otherwise the closure of the orbit of q meets ∂A, contradicting (v). The suspension ξ of the orbit of p is a C^r graph except at its endpoints. Its interior meets each s-slice only once. Its interior is disjoint from B since p is not in $B^+ \cup f_1^{-1}(B^-)$, but its endpoints lie in B.

Let F' denote the C^{r-1} tangent vector field of the flow F_u. It follows from (i) that F' has no closed integral curves on $\mathrm{int}(A \times S^1)$. It has no zeroes since it is a suspension.

Let T be the thickened torus $S^1 \times [-2, 2] \times S^1$ with coordinates (x, t, s). It contains $A \times S^1$ in its interior. Let N be the vector field $-\partial/\partial t$ defined on T.

Choose a smooth, real-valued function ψ which is 1 on $(A \times S^1) \backslash B$, 0 on $T \backslash (A \times S^1)$ and $0 < \psi < 1$ on the interior of B. Let

$$Y = \psi F' + (1 - \psi) N.$$

Y has no zeroes and has no closed integral curves: On $T \backslash (A \times S^1)$, where $\psi = 0$, we have $Y = N = -\partial/\partial t$. On B, we have $0 < \psi < 1$ and both F' and N are t-level reducing. Since N strictly reduces t-levels, Y is strictly t-level reducing on $(T \backslash A \times S^1) \cup B$. The dynamics on $(A \times S^1) \backslash B$ are identical to that of F'. Therefore there are no zeroes and no closed integral curves.

Since $\psi \mid (A \times S^1) \backslash B = 1$ and η and ξ are disjoint from the interior of B, they are contained in maximal integral curves η' and ξ' of Y. But $\eta = \eta'$ since η is already maximal. The curve ξ' enters on the outer boundary of T at a point p' and exists on the inner boundary of T at q'. Let us verify that ξ' is unknotted: The "ends" of ξ', that is, the two components of $\xi' \backslash \xi$ may be continuously isotoped to become vertical without disturbing ξ. Since ξ is a graph, the new curve may be isotoped to become a graph disjoint from the $s = 0$ slice of T. These isotopes may be realized by an ambient isotopy of T. Therefore ξ' is unknotted.

There exists a small disk $D \subset \partial T$ containing p' such that $U = \{$integral curves of Y meeting $D\}$ is a C^r tubular neighborhood of ξ'. Then Y is tangent to ∂U. Let $T_0 = T \setminus U$. Since ξ' is unknotted there exists a C^r diffeomorphism h of $\mathbb{R}^3$ such that $h(T_0)$ is the Schweitzer "clerical collar." (See Fig. 1.)

The non-zero vector field $Z = hYh^{-1}$ on $h(T_0)$ satisfies the necessary properties to make a flow plug. (See [2].) That is, in $\mathbb{R}^3$ coordinates (x, y, z), $Z = -\partial/\partial z$ in a neighborhood of $\partial h(T_0)$ there are no closed integral curves in $h(T_0)$, and there is one integral curve $h(\eta)$ contained entirely in $h(T_0)$. Extend with Z' which has the "mirror image property" with respect to Z. (See [8, 6] or Fig. 1.) Assume that the domain of $Z \cup Z'$ is contained in the unit cube C in $\mathbb{R}^3$. Use $-\partial/\partial z$ to extend $Z \cup Z'$ to a non-zero vector field P defined on all of C. Then at least one integral curve of P enters the top face of C and never exits. Otherwise, the entering integral curve would completely foliate C, contradicting the existence of $h(\eta)$. By the mirror image property, if any integral curve entering C also exists, it does so directly below where it entered. Therefore P has no closed integral curves.

Choose a C^∞ non-zero vector field on S^3 with only finitely many closed integral curves. For each of these curves choose a flow box meeting it. They may be chosen to be disjoint. Replace the vector field in each flow box by a copy of P so that the previously closed integral curve enters it and never exits. No new closed integral curves are introduced. The resulting vector field V on S^3 has no zeroes and no closed integral curves. Its flow G_t is of class C^r.

According to Hart [5] there exists a C^r diffeomorphism of S^3

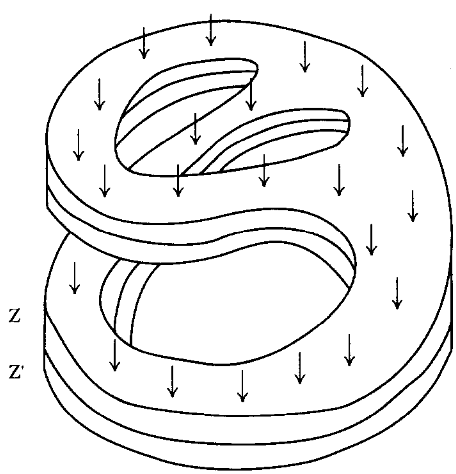

FIGURE 1

conjugating G_t to a flow which is generated by a C^r vector field X. Since X is conjugate to V it also has no zeroes and no closed integral curves.

THEOREM. *There exists a* $C^{2+\delta}$ *diffeomorphism* $f: A \to A$ *satisfying* (i)–(v).

This is one of the main results of [4].

COROLLARY. *There exists a* $C^{2+\delta}$ *counterexample to the Seifert conjecture.*

A C^1 EXAMPLE

With the help of Theorem A, Schweitzer's example has a simple description: Let $f_1: S^2 \to S^2$ be a C^1 diffeomorphism which has a Denjoy diffeomorphism g of the circle on its equator and each latitude circle. (See the appendix for a discussion of g.) Then make the equator semi-stable by gently pushing points above the equator closer to it, and points below farther away, towards the south pole. Modify the map slightly near S and N so that it will be C^1 there. Finally, perturb the map so that some points pass through one of the Denjoy "gaps." See Fig. 2.

This gives a C^1 diffeomorphism of S^2 satisfying the conditions of Theorem A. In order to make a C^2 example, the equator circle is made into a fractal. The higher its Hausdorff dimension, the higher the fractional differentiability of f. (See Fig. 3.)

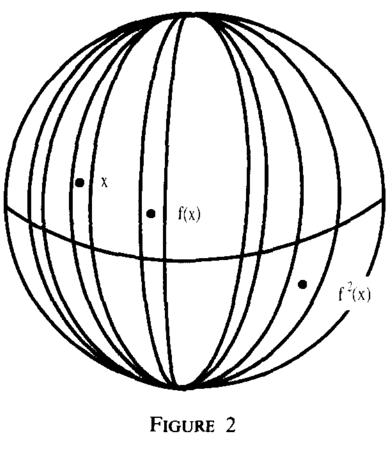

FIGURE 2

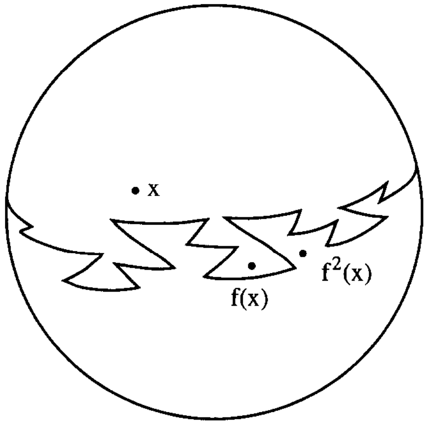

FIGURE 3

THE LOXODROMIC MAPPING PROBLEM

The loxodromic (or "chocolate fudge") diffeomorphism of the two-sphere is the standard "north-pole, south-pole" diffeomorphism $L: S^2 \to S^2$. In spherical coordinates, $L(\theta, \varphi) = (\theta, g(\varphi) + \varphi)$, where $g: \mathbb{R} \to \mathbb{R}$ is C^∞, $g(0) = g(\pi) = 0$, $g(\varphi) > 0$ and $g'(\varphi) > -1$ for $0 < \varphi < \pi$, and $\partial^r g/\partial\varphi^r = 0$, $r \geqslant 1$, at $\varphi = 0$ and $\varphi = \pi$.

A loxodromic diffeomorphism is any diffeomorphism of S^2 which is topologically conjugate to L.

Several interesting problems arise from this work. The simplest can be stated as a

Conjecture. Suppose $f: S^2 \to S^2$ is a C^3 diffeomorphism which repels N, attracts S, and has no other periodic points. If one orbit is asymptotic to both N and S then f is a loxodromic diffeomorphism.

To put it simply, the conjecture states that if one orbit gets across then they all do.

If the conjecture is true, it is an interesting fact about 2-dimensional dynamics. If it is false, then there is a C^3 counterexample to the Seifert conjecture.

This loxodromic conjecture is related to the Birkhoff conjecture:

Suppose $f: S^2 \to S^2$ is an area-preserving diffeomorphism and has no other periodic points than the fixed poles. Then f is C^0 conjugate to a rigid rotation.

NUMBER THEORY AND THE LOXODROMIC MAPPING PROBLEM

The examples of [4] use rotations of the circle with restricted rotation number α. It must be a quadratic irrational. The methods of [4] will not produce examples with Liouville rotation number on the invariant Cantor set. This leaves open the general question of how number theory affects the existence of C^2 Seifert counterexamples. Schweitzer's C^1 examples exist for all rotation numbers α.

APPENDIX: DENJOY DIFFEOMORPHISMS

Let $S=\sum 1/n^2$, where $n \in \mathbb{Z}$. Let $a_n = 1/Sn^2$ so that $\sum a_n = 1$ and $a_{n+1}/a_n \to 1$. Let $0<\alpha<1$ be an irrational number and x_n the fractional part of $n\alpha$. For $J \subset \mathbb{R}$, let χ_J denote the characteristic function over J.

Define ρ: $[0, 1) \to [0, 1)$ by $\rho^{-1}(t) = \sum a_n \chi_{[0,t)}(x_n)$ if $t \neq x_i$ for any i. Let $\rho^{-1}(x_i)$ be the closed interval $[\sum a_n \chi_{[0,t)}(x_n), a_i + \sum a_n \chi_{[0,t)}(x_n)]$.

The intervals $\rho^{-1}(x_i)$ are called Denjoy intervals. The complement of the Denjoy intervals is the Denjoy Cantor set C.

Let R_α: $[0, 1) \to [0, 1)$ be the rotation $R_\alpha(t) = t + \alpha \pmod 1$. Define f: $C \to C$ by $\rho f(t) = R_\alpha \rho(t)$. One may extend f to the entire interval $[0, 1)$ with a cubic polynomial on each Denjoy interval or apply the Whitney extension theorem. Because the Denjoy intervals have full measure in $[0, 1)$, all intervals may be expressed as unions of them up to sets of measure zero. This together with the fact that $a_{n+1}/a_n \to 1$ is enough to verify the conditions of the Whitney extension theorem. It provides a C^1 extension g: $S^1 \to S^1$ with $g \mid C = f$ and $Dg = 1$ on C. Denjoy proved that g is not topologically conjugate to any C^2 diffeomorphism.

Remark. By choosing $a_n = 1/n(\log n)^2$, the diffeomorphism if $C^{2-\varepsilon}$ for all $\varepsilon > 0$.

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