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CONSTRUCTIVE APPROXIMATION

Embedded Continued Fractals and Their Hausdorff Dimension

J. Harrison

Abstract. A continued fractal $Q_\alpha \subset \mathbf{R}^2$ is a curve which is associated to a real number $\alpha \in [0, 1]$. Properties of the continued fraction expansion of α appear as geometrical properties of Q_α. It is shown how number theoretic properties of α affect topological and geometric properties of Q_α such as existence, continuity, Hausdorff dimension, and embeddedness.

Introduction

A continued fractal is a curve Q_α in Euclidean space $\mathbf{R}^2$ associated to a real number $\alpha \in [0, 1]$. Properties of the continued fraction expansion of α appear as geometrical properties of Q_α. For example, we know that α is a quadratic irrational if and only if α has periodic continued fraction. Its continued fractal is self-similar [see H4]. In this paper we study how number theoretic properties of α affect topological and geometric properties of Q_α such as existence, continuity, Hausdorff dimension, and embeddedness.

Continued fractals appear as strange attractors in smooth dynamical systems. The first known example was the key element in the construction of a smooth $C^{2+\varepsilon}$ vector field on the three sphere S^3 with no zeros and no closed integral curves (see [H1] and [H3]). Continued fractals can be Julia sets for *diffeomorphisms* f of the two-sphere S^2. They may be contrasted with the original fractal Julia sets of Julia, Fatou, Sullivan *et al.*, which arise from *noninvertible* mappings (see [B], for example). The invertibility of f gives it a basic position in the world of two-dimensional dynamics.

1. The Embedding $h: I \to \mathbf{R}^2$

We give two descriptions of h. The first is geometric. It provides a guide for programming computers to draw $h(I)$. The second is analytic and is used for proving the theorems.

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Notation

Let $\mathbf{R}$ denote the real numbers and $Q \subset \mathbf{R}$ the rationals. Let π_1 and π_2 denote the projections onto the first and second factors of $\mathbf{R}^2$, respectively. If $x \in \mathbf{R}$, define $\langle x \rangle = x \pmod 1$, $\operatorname{int}(x) = x - \langle x \rangle$, and $\|x\|$ = distance to the nearest integer. Let $I = [0, 1)$. Any function $h: I \to \mathbf{R}^2$ canonically induces a function $\tilde{h}: S^1 \to S^1 \times \mathbf{R}^1$. (Define $h^*: \mathbf{R}^1 \to \mathbf{R}^2$ by $h^*(x) = h(\langle x \rangle) + (\operatorname{int}(x), 0)$. This h^* induces a function $\tilde{h}: S^1 \to S^1 \times \mathbf{R}^1$ via the identifications $x \sim x + n$ on $\mathbf{R}^1$ and $(x, y) \sim (x + n, y)$ on $\mathbf{R}^2$.) For simplicity of notation, we work entirely with $h: I \to \mathbf{R}^2$. Fix $\alpha \in \mathbf{R}$ and define

$$x_n = \langle n\alpha \rangle, \qquad n \geq 0.$$

Normalizing Constants. Let $g: Z^+ \to \mathbf{R}$ be a monotone decreasing function. For $k \geq 1$, we define two normalizing constants:

$$g(0) = \sum (-1)^n g(n), \qquad n = 1, \ldots, k$$

and

$$m_k = 1 + \sum g(n), \qquad n = 0, \ldots, k.$$

The Computer Algorithm

The algorithm depends only on the sequence x_n and weights $g(n)$. Let $x_{n_0} = 0$, $x_{n_1}, \ldots, x_{n_k}$ denote the first $k+1$ terms of the sequence x_n, ordered from left to right in I. Let

$$I_{n_i} = (x_{n_i}, x_{n_{i+1}}), \qquad i = 0 \text{ to } k-1.$$

In the plane, draw a horizontal line segment with left endpoint the origin $(0, 0)$ and length $m_k|I_{n_0}|$. Attach to its right endpoint a line segment sloping backward with fixed, acute angle $\pi/4$ and length $g(n_1)/(\cos \pi/4) = \sqrt{2}g(n_1)$. We call this segment a *diagonal*, Δ_{k,n_1}. It points down if n_1 is even and up if n_1 is odd. Attach to the free endpoint of Δ_{k,n_1}, forming the angle $\pi/4$, a horizontal line segment of length $m_k|I_{n_1}|$. Attach to the free endpoint of this second horizontal line a diagonal Δ_{k,n_2} of length $\sqrt{2}g(n_2)$ again sloping backward with angle $\pi/4$ and up or down according to whether n_2 is odd or even. Continue until k horizontal and diagonal lines have been drawn, one pair for each of the intervals I_{n_i}. Finally, draw a diagonal $\Delta_{k,0}$ with endpoint the rightmost endpoint of the curve and length $\sqrt{2}g(0)$. Call the resulting curve Q_k (see Figs. 1-3).

We prove that for some α and g, there is a subsequence k_i such that the curves Q_{k_i} converge to a limit curve Q. In this case we denote

$$\Delta_n = \lim_{i \to \infty} \Delta_{k_i,n}.$$

An Analytical Definition of Q_k

Let $\rho: I \to I$ be a monotonic, continuous mapping such that $\rho^{-1}(x_n)$ is an interval $\Delta'_n = [y'_n, z'_n]$, $n \geq 0$, and ρ is $1-1$ on the complement of the $\{\Delta'_n\}$. (If x_n is dense,

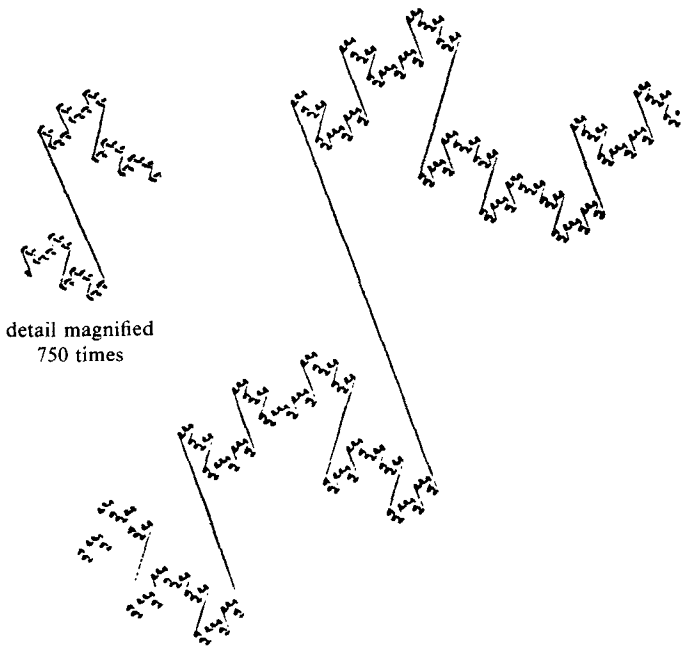
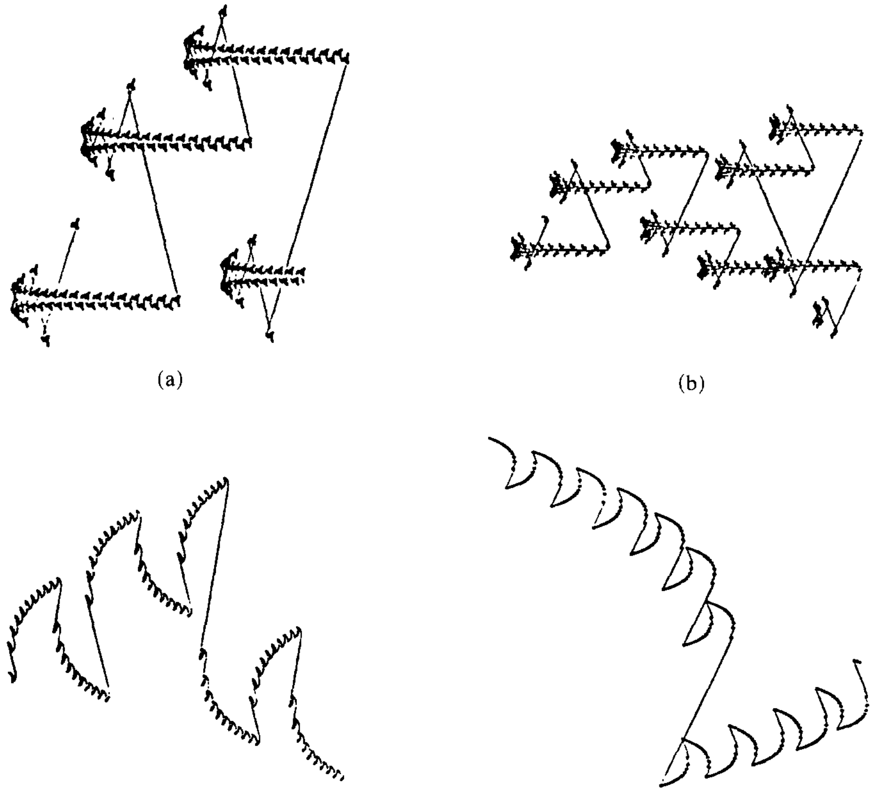


Fig. 1. Continued fractal for $\alpha = (\sqrt{5}-1)/2$ = the Golden Mean, $g(n) = 0.7/n^{0.7}$, $n \leq 5000$.

Fig. 2. Continued fractal for $x_n = n\alpha$, $\alpha = \pi/2$ (detail magnified 75 times), $g(n) = 0.7/n^{0.7}$, $n \leq 5000$.



(a) (b) (c) (d)

Fig. 3. Details of continued fractals: (a) $\alpha = 2^{1/4}$ (detail magnified 10 times), (b) $\alpha = 2^{1/4}$ (detail magnified 2000 times), (c) $\alpha = 7^{1/4}$ (full curve), (d) $\alpha = 7^{1/4}$ (detail magnified 10 times).

then ρ is a Cantor function.) We define immersions $h_k: I \to \mathbf{R}^2$, $k \geq 1$, so that $h_k(I) = Q_k$ and $h_k(\Delta'_n) = \Delta_n$. Let χ_A denote the standard characteristic function. We also need a function $d: I \to \mathbf{R}^2$ which adjusts the location of $h_k(x)$ for x in a segment Δ'_n. It plays a very minor part in the estimates. (Indeed, the reader may assume $d(x) = 0$ without much loss.)

$$d(x) = \begin{cases} (-1)^n g(n) \dfrac{|x - y'_n|}{|z'_n - y'_n|} & \text{if } x \in \Delta'_n, \\ 0 & \text{if } x \notin \bigcup \Delta'_n. \end{cases}$$

Definition. Let $W = [0, \rho(x))$. Define $h_k: I \to \mathbf{R}^2$ by

$$\pi_1 h_k(x) = m_k |W| - \sum_{n=0}^{k} \chi_W \langle i\alpha \rangle g(n) - |d(x)|,$$

$$\pi_2 h_k(x) = \sum_{2n+1 \leq k} \chi_W \langle (2n+1)\alpha \rangle g(2n+1) - \sum_{2n \leq k} \chi_W \langle 2n\alpha \rangle g(2n) - d(x).$$

Define

$$h = \lim h_k \qquad \text{as} \quad k \to \infty.$$

It is not hard to see that if $\sum g(n) < \infty$, then h exists and is continuous. If $\sum g(n) = \infty$, then the construction poses more difficult and interesting problems. The critical exponent of the series $\sum g(n)$ is the Hausdorff dimension in the embedded examples. Hence, the "longer" the curve, the higher the Hausdorff dimension and the more "unfolded" the sequence x_n becomes.

Remark. Q_k may be defined for *any* 1-1 sequence x_n and weight function $g(n)$. In this paper we restrict ourselves to $x_n = \langle n\alpha \rangle$, mod 1.

2. Number Theory Prerequisites

Definition 2.1. Let a_n be a sequence of integers ≥ 1 and

$$\frac{p_n}{q_n} = \left[\cfrac{1}{a_2 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots \cfrac{1}{a_n}}}} \right],$$

where $(p_n, q_n) = 1$. Then

$$\alpha = \lim_{n \to \infty} \frac{p_n}{q_n}$$

exists and is a positive irrational <1. We write $\alpha = (a_1, a_2, a_3, \ldots)$. The fractions p_n/q_n are *rational convergents of* α. The integers p_n and q_n satisfy the recursive relations:

$$p_0 = 0, \quad p_1 = 1, \quad \text{and} \quad p_n = a_n p_{n-1} + p_{n-2};$$

$$q_0 = 1, \quad q_1 = a_1, \quad \text{and} \quad q_n = a_n q_{n-1} + q_{n-2}.$$

Define $r_0 = 1$, $r_1 = \alpha$, and $r_{n-1} = a_n r_n + r_{n+1}$.

The first lemma follows readily from these definitions.

Lemma 2.2.

(i) $q_n r_n + q_{n-1} r_{n+1} = 1$.
(ii) $q_n p_{n-1} - p_n q_{n-1} = (-1)^n$.
(iii) $p_n - q_n \alpha = (-1)^{n+1} r_{n+1}$.
(iv) *If* $\alpha = [c, c, c, \ldots]$ *then* $r_n = \alpha^n$.

Lemma 2.3. *If $\alpha \in \mathbf{R}\backslash Q$ has continued fraction expansion $[a_1, a_2, \ldots]$, then for each $n \geq 1$*

$$\left|\alpha - \frac{p_n}{q_n}\right| = \frac{1}{a_n^2(a_{n+1} + (a_{n+2}, a_{n+3}, \ldots) + (a_n, \ldots, a_1))},$$

where p_n/q_n is the nth *convergent of α.*

Proof. See Lemma 1.5 of [HN]. ■

Let $n \in \mathbf{Z}^+$. Define

$$t_n = q_n + q_{n-1} - 1.$$

We define W_n to be the collection of intervals in I complementary to $\{\langle 0\alpha\rangle, \langle 1\alpha\rangle, \ldots, \langle t_n\alpha\rangle\}$: each of these intervals is a W_n*-interval* and $\langle j\alpha\rangle$, $0 \leq j \leq t_n$, is a W_n*-point*. By Lemma 2.2(iii) $\langle q_{2n+1}\alpha\rangle$ converges monotonically to 0 and $\langle q_{2n}\alpha\rangle$ converges monotonically to 1. Let $I_0(n)$ be the W_n-interval with endpoints $\langle q_{n-1}\alpha\rangle$ and an endpoint of I and $J_0(n)$ the W_n-interval bounded by $\langle q_n\alpha\rangle$ and the other endpoint of I. Define

$$R_\alpha(x) = x + \alpha \pmod 1.$$

Lemma 2.4. *The collection W_n consists of the first q_n iterates of $I_0(n)$ and the first q_{n-1} iterates of $J_0(n)$ under the transformation R_α. In particular, all W_n-intervals have length r_n or r_{n+1}.*

Proof. This follows from Lemma 2.2(i). (See also Lemma 1.5 of [H2].) ■

Definition. An irrational number α satisfies a *Diophantine condition* σ if there exists $c > 0$ such that

$$\left|\alpha - \frac{p}{q}\right| > \frac{c}{q^{1+\sigma}} \qquad \text{for every } \frac{p}{q} \in Q.$$

α has *Diophantine type δ if*

$$\delta = \underline{\lim}\{\sigma\colon \alpha \text{ satisfies a Diophantine condition } \sigma\}.$$

Estimates on the Discrepancy of $\langle n\alpha\rangle$

The remainder of this section is devoted to estimates of "weighted discrepancy" which are fundamental to the study of continued fractals.

Theorem 2.5

(i) *Let $\alpha \in \mathbf{R}\backslash Q$ and p/q be a rational convergent of α. If J is an interval of I with $|J| = \|q\alpha\|$, then for every $0 \leq s < t$*

$$\left|\sum_{i=s}^{t-1} (\chi_J\langle i\alpha\rangle - |J|)\right| < 2.$$

(ii) *Assume $\alpha \in \mathbf{R}\setminus Q$ has Diophantine type δ and $\varepsilon > 0$. There exists $C > 0$ such that if U is an interval of I and $0 \le s < t$, then*

$$\left|\sum_{i=s}^{t-1} (\chi_U\langle i\alpha\rangle - |U|)\right| < C(t-s)^{1-(1/\delta)+\varepsilon}.$$

Proof. For (i), see [K] or Theorem 2.2(i) of [H2]. (ii) follows immediately from Theorem 3.2 of [KN, p. 123]. ∎

The next lemma is used in weighted versions of Theorem 2.5.

Lemma 2.6. *Let $C > 0$ and $f(i)$ be a monotone increasing sequence of positive real numbers. Let $b(i)$ be a sequence of real numbers satisfying*

$$\left|\sum_{i=j}^{j+N-1} b(i)\right| < Cf(N) \qquad \text{for all} \quad j, N \ge 0.$$

Let $g(i) > 0$ be a monotone decreasing sequence. Then:

(i) $$\left|\sum_{i=s}^{t-1} b(i)g(i)\right| \le Cf(t-s)g(s) \qquad \text{for all} \quad 0 \le s \le t.$$

(ii) *Define n and r by $2^n \le t-1 < 2^{n+1}$ and $2^r \le s < 2^{r+1}$. Then*

$$\left|\sum_{i=s}^{t-1} b(i)g(i)\right| \le C\left|\sum_{\lambda=r}^{n} f(2^\lambda)g(2^\lambda)\right|.$$

Proof. This is essentially "summation by parts." (For details, see Proposition 2.3 of [H2].) ∎

Corollary 2.7.

(i) *Let $\alpha \in \mathbf{R}\setminus Q$. If $J \subset I$ is an interval with $|J| = \|q_n\alpha\|$ and $g: \mathbf{Z} \to \mathbf{R}^+$ is monotone decreasing, then*

$$\left|\sum_{i=1}^{t-1} (\chi_J\langle i\alpha\rangle - |J|)g(i)\right| < 2g(s).$$

(ii) *Assume $\alpha \in \mathbf{R}\setminus Q$ has Diophantine type δ. Let $\varepsilon > 0$ and $\eta = 1-(1/\delta)+\varepsilon$. There exists $C > 0$ such that if U is an arbitrary interval of I and $g(i) \le 1/i^\gamma$ where $\gamma > \eta + 1/2\delta$, then*

$$\left|\sum_{i=s}^{t-1} (X_U\langle i\alpha\rangle - |U|)g(i)\right| < Cs^{\eta-\gamma}.$$

Proof. (i) This follows immediately from Kesten's theorem (Theorem 2.5(i)) and summation by parts (Theorem 2.6(i)).

(ii) Let C_1 be the constant depending on α and ε obtained from Theorem 2.5(ii). Apply Lemma 2.6(ii) to $b(i)=\chi_U\langle i\alpha\rangle-|U|$, $g(i)\le 1/i^\gamma$, and $f(i)=i^\eta$ to obtain

$$\left|\sum_{i=s}^{t-1} b(i)g(i)\right|<C_1\left|\sum_{\lambda=r}^{n} 2^{\lambda(\eta-\gamma)}\right|<C_1C_2s^{\eta-\gamma}.$$

Note that C_2 depends on δ since $1/2\delta<\lambda-\eta$. Let $C=C_1C_2$. Then C depends on α and ε. ∎

3. Sufficient Conditions for Q to Exist

Most irrational numbers are Diophantine. If α has Diophantine type δ we show that the curve Q generated by a monotone function $g(n)\le c/n^\gamma$ and α exists and is continuous for $\gamma>1-1/2\delta$. The Hausdorff dimension of Q is bounded above by $1/\gamma$ if $\frac{1}{2}<\gamma<1$ and by 1 if $\lambda\ge 1$. This bound is sharp since there are examples with Hausdorff dimension $1/\gamma$ for $\frac{1}{2}<\gamma\le 1$.

We prove that the subsequence of Q_{t_k} converges to a limit curve where $t_k=q_k+q_{k-1}-1$. Henceforth, for simplicity of notation, set

$$Q_k=Q_{t_k},$$
$$h_k=h_{t_k},$$
$$m_k=m_{t_k}.$$

Theorem 3.1. *Let $\alpha\in\mathbf{R}\backslash Q$ have Diophantine type δ. Assume $\gamma>1-1/2\delta$. Let Q_k be the sequence of curves generated by $g(n)\le c/n^\gamma$ and α. Then $Q=\lim Q_{t_k}$ exists and is continuous.*

Proof. Recall that $Q_k=h_k(I)$. Let U be an interval in I with endpoints $p<q$. Define

$$H_k=H_k(U)=\pi_1h_k(q)-\pi_1h_k(p),$$
$$V_k=V_k(U)=\pi_2h_k(q)-\pi_2h_k(p).$$

See Fig. 4.

In order to prove that h_k converges uniformly to a continuous function h, it suffices to show that H_k and V_k converge uniformly over intervals of I. Let C be the constant of Theorem 2.7(ii). Recall

$$\Delta'_n=(y'_n, z'_n)=\rho^{-1}\langle n\alpha\rangle.$$

In order to prove Theorem 3.1 we verify the uniform convergence of H_k and V_k over two basic types of intervals U with endpoints $p<q$:

$$p\notin\bigcup\,[y'_n, z'_n)\quad\text{and}\quad q\notin\bigcup\,(y'_n, z'_n]; \tag{3.1}$$

$$[p, q]\subset[y'_n, z'_n]\qquad\text{for some}\quad n\in\mathbf{Z}. \tag{3.2}$$

Uniform convergence over all intervals of I follows.

In the hypotheses we have $\gamma>1-1/2\delta$, $\delta\ge 1$. Choose $\varepsilon>0$ with

$$\gamma>1+\varepsilon-1/2\delta.$$

Define $\eta=1+\varepsilon-1/\delta$.

Preliminary Estimate. Let $c=1$. Recall the normalizing constant

$$m_k = 1+\sum g(n)+(-1)^n g(n), \qquad n=1,\ldots,t_k.$$

Then

$$m_k < 1+\frac{t_k^{1-\gamma}}{1-\gamma} < 1+\frac{2q_k^{1-\gamma}}{1-\gamma}.$$

Since $r_k q_k < 1$ (by Lemma 2.2) it follows that $m_k r_k \to 0$ as $k\to\infty$. Therefore, for $\beta>0$, there exists N such that if $t_k \ge N$, then

$$\max\{m_k r_k, 2Ct_k^{\eta-\gamma}, 2t_k^{-\gamma}\} \le \frac{\beta}{3}.$$

Proof of the Uniform Convergence of H_k. Let U be an interval of type (3.1) with endpoints $p<q$. Then $d(q)=0$. Let $W=\mathrm{int}(\rho(U))$ and $N<t_j<t_k$. It follows that

$$\left(H_k(U) = 1+\sum_{n=0}^{t_k} g(n)\right)|W| - \sum_{n=m}^{t_k} \chi_W\langle n\alpha\rangle g(n), \tag{3.3}$$

where $m=\min\{n\colon \langle n\alpha\rangle \in \mathrm{int}(W)\}$. (Whenever $m>t_k$, the last sum is zero.) Assume $m\le t_j<t_k$. Apply Theorem 2.7(ii). Then

$$|H_j - H_k| = \left|\sum_{n=t_j}^{t_k} (|W|-\chi_W\langle n\alpha\rangle) g(n)\right| < Ct_j^{\eta-\gamma} < \beta.$$

Assume $t_j<t_k<m$. Then $|W|\le t_k$. Since the last sum in (3.3) is 0 for both H_j and H_k we have

$$|H_j - H_k| = |m_j - m_k|\,|W| < m_k t_k < \beta.$$

If $t_j<m\le t_k$, let l satisfy $t_j\le t_l<m\le t_{l+1}\le t_k$. Then $|W|\le r_l$. The last sum in (3.3) for H_j is zero so

$$\begin{aligned}
|H_j - H_k| &= \left||W|\sum_{n=t_j+1}^{t_k} g(n) + \sum_{n=t_l+1}^{t_k} \chi_W\langle n\alpha\rangle g(n)\right| \\
&= \left||W|\sum_{n=t_j+1}^{t_l} g(n) + \sum_{n=t_l+1}^{t_k} (|W|-\chi_W\langle n\alpha\rangle) g(n)\right| \\
&< m_l r_l + Ct_l^{\eta-\gamma} \qquad \text{(by Theorem 2.7(ii))} \\
&< \beta.
\end{aligned}$$

Thus H_k converges uniformly over the intervals U of type (3.1).

Now assume $U\subset \Delta'_n$. In general, if $n\le t_l$ then $|H_l|\le g(n)$. If $n>t_l$, then $|H_l|\le m_l r_l$. Therefore, if $t_j<t_k<n$,

$$\begin{aligned}
|H_j - H_k| &< \max\{|H_j|, |H_k|\}, \\
&\le \max\{m_j r_j, m_k r_k\} \\
&< \beta.
\end{aligned}$$

If $n \le t_j < t_k$, then $|H_j - H_k| = 0$. Finally, if $t_j < n \le t_k$,

$$\begin{aligned} |H_j - H_k| &< \max\{|H_j|, |H_k|\} \\ &\le \max\{m_j r_j, g(n)\} \\ &< \max\{m_j r_j, g(t_j)\} \\ &< \beta. \end{aligned}$$

Proof of Uniform Conference of V_n. Let $U \subset I$ be an interval of type (3.1) and $W = \operatorname{int}(\rho(U))$. Then

$$V_k = \sum_{2n+1 \le t_k} \chi_W \langle (2n+1)\alpha \rangle g(2n+1) - \sum_{2n \le t_k} \chi_W \langle 2n\alpha \rangle g(2n). \tag{3.4}$$

Let $N < t_j < t_k$. Then

$$\begin{aligned} |V_j - V_k| &= \left| \sum_{t_j < 2n+1 \le t_k} \chi_W \langle (2n+1)\alpha \rangle g(2n+1) - \sum_{t_j < 2n \le t_k} \chi_W \langle 2n\alpha \rangle g(2n) \right| \\ &= \left| \sum_{t_j < 2n+1 \le t_k} (\chi_W \langle (2n+1)\alpha \rangle - |W|) g(2n+1) \right. \\ &\quad + \sum_{t_j < 2n \le t_k} (|W| - \chi_W \langle 2n\alpha \rangle) g(2n) \\ &\quad \left. + |W| \sum_{t_j < 2n, 2n+1 \le t_k} (g(2n+1) - g(2n)) \right|. \end{aligned}$$

Apply the triangle inequality and Theorem 2.7(ii) to estimate the first two sums. (The third sum is bounded by $2g(j)$ since $|W| < 1$.) Thus

$$|V_j - V_k| < 2Ct_j^{\eta-\gamma} + 2g(t_j) < \beta.$$

Suppose $U \subset \Delta'_n$. Then V_l is zero for $t_l < n$. If $n \le t_l$ then V_l is a constant depending only on U. It is bounded by $g(n)$. Therefore, if $n \le t_j < t_k$ or $t_j < t_k < n$, then $|V_j - V_k| = 0$. If $t_j < n \le t_k$, then $|V_k - V_j| = |V_k| < g(n) < g(j) < \beta$. Hence V_k converges uniformly.

The proof for arbitrary $c > 0$ is similar. ■

4. Embedded Continued Fractals and Number Theory

It is more delicate to show Q is embedded. In [H2] there is a proof for $\alpha = \sqrt{2} - 1$. Here, we give a new proof for a class of numbers containing α. The definition of g given below makes it possible to provide sharp estimates for the embedding.

Definition of g. Let $\alpha \in \mathbf{R} \backslash Q$, $c > 0$, and $\frac{1}{2} < \gamma < 1$. Define

$$g(n) = c\alpha^{k\gamma} \qquad \text{for} \quad q_{k-1} \le n < q_k.$$

Theorem 4.1. *Assume $\alpha = [N, N, N, \dots]$, N even. There exists constants $c_N > 0$ and $\gamma_N < 1$ such that if $0 < c < c_N$ and $\gamma_N < \gamma < 1$, then $h = h_{\gamma,c}: I \to \mathbf{R}^2$ is an embedding onto Q.*

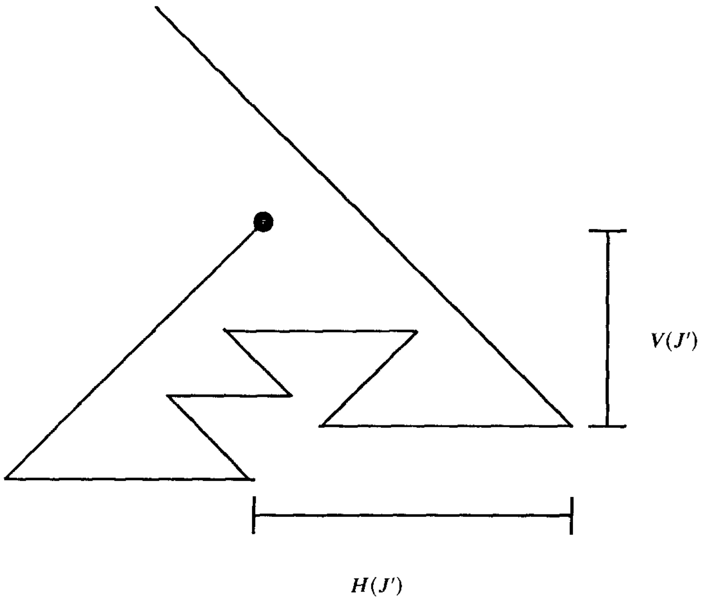


Fig. 4

Proof. The idea of the proof is fairly simple. The line segment I is covered by W_k-intervals. Let J be one of these. We may assume $|J| = \alpha^k$, otherwise it appears as a W_{k+1}-interval. See Lemmas 2.4 and 2.2(iv). Assume that J includes the endpoint with larger index, n, say. Then $q_{k-1} \leq n < q_k$. Let $J' = \rho^{-1}(J)$. Define $H = H(J')$ and $V = V(J')$ as in the previous section. If $H > V$ then the two diagonals attached to the endpoints of $h(J')$ are disjoint. By induction the curve is embedded (see Fig. 4). ■

Let W denote the covering of I by W_k-intervals, $k \geq 1$.

Theorem 4.2.

(i) *There exist* $c_N > 0$ *and* $\frac{1}{2} < \gamma_N < 1$ *so that* $|H/V| > 1$ *for all* $0 < c < c_N$, $\gamma_N < \gamma < 1$ *and* $J \in W$.

(ii) $|H/V|$ *is uniformly bounded: given* c *and* γ, *there exists* $L > 0$ *with* $|H/V| < L$, *for all* $J \in W$.

Remark. We only need Theorem 4.2(i) for this proof; Theorem 4.2(ii) is needed later in the estimates for the Hausdorff dimension of Q.

Proof of Theorem 4.2. It follows from Theorems 3.1 and 3.3 that $H = \alpha^k + A + B$ where

$$A = \alpha^k \sum_{i=1}^{q_{k-1}-1} g(i)$$

and

$$B=\sum_{i=q_{k-1}}^{\infty}(|J|-\chi_J(\langle i\alpha\rangle))g(i).$$

By the choice of g,

$$A=c\alpha^k[(q_1-q_0)\alpha^{\gamma}+(q_2-q_1)\alpha^{2\gamma}+\cdots+(q_{k-1}-q_{k-2})\alpha^{(k-1)\gamma}].$$

To compute A, we need some simple, preliminary facts: by Lemmas 2.2(iii) and 2.3,

$$\alpha^{n+1}=|q_n\alpha-p_n|=\frac{1}{q_n(N+\alpha+p_n/q_n)}=\frac{1}{q_n(N+2\alpha+\varepsilon_n)},$$

where $|\varepsilon_n|<2/q_n^2$. Therefore

$$q_n=\frac{\alpha^{-(n+1)}}{(N+2\alpha+\varepsilon_n)},$$

since $\alpha=[N,N,N,\ldots]$, $\alpha^{-1}=N+\alpha$. Hence

$$q_n-q_{n-1}=\frac{\alpha^{-n}(N+\alpha-1)(1+\beta_n)}{(N+2\alpha)},$$

where $\beta_n\to 0$. Let $\Delta=c\alpha^{k\gamma}$. It is straightforward to verify that as $k\to\infty$

$$\frac{A}{\Delta}\to\frac{N+\alpha-1}{(N+2\alpha)(\alpha^{\gamma-1}-1)}. \tag{4.1}$$

We next estimate B; let $b(i)=|J|-\chi_J(\langle i\alpha\rangle)$. We apply summation by parts:

$$\begin{aligned}B=\lim[b(q_{k-1})+\cdots+b(q_m)]g(q_m)&+b(q_{k-1})[g(q_{k-1})-g(q_{k-1}+1)]\\&+[b(q_{k-1})+b(q_{k-1}+1)][g(q_{k-1}+1)-g(q_{k-1}+2)]\\&+\cdots+[b(q_{k-1})+\cdots+b(q_m-1)][g(q_m-1)-g(q_m)].\end{aligned}$$

Now $\sum b(i)$ is bounded (according to Kesten's Theorem 2.5(i)) and $g(q_m)\to 0$, so the first term vanishes in the limit. Note that $g(i)-g(i-1)=0$ unless $i=q_j$. Also, $g(q_m-1)=g(q_{m-1})$. Hence,

$$\begin{aligned}B=[b(q_{k-1})&+\cdots+b(q_k-1)][g(q_{k-1})-g(q_k)]\\&+[b(q_{k-1})+\cdots+b(q_{k+1}-1)][g(q_k)-g(q_{k+1})]+\cdots.\end{aligned}$$

Next we substitute the identity $g(q_{n-1})-g(q_n)=\alpha^{n\gamma}(1-\alpha^{\gamma})$. Assume that k is even. According to Theorem 4.2 of [HN], we know

$$\sum_{i=q_{k-1}}^{q_n-1} b(i)=\sum_{i=0}^{q_n-1} b(i)-\sum_{i=0}^{q_{k-1}-1} b(i)=q_{k-1}(\pm\alpha^{n+1}-\alpha^k),$$

+ if n is even, − if n is odd. If k is odd, we get

$$q_{k-1}(\pm\alpha^{n+1}-\alpha^k)=q_{k-1}(\pm\alpha^{n+1}-\alpha^k),$$

− if n is even, + if n is odd. We write these in terms of α, using the above identities and get

$$\sum b(i)=\frac{(-1\pm\alpha^{n+1-k})}{(N+2\alpha+\varepsilon_k)},$$

+ if and only if k and n have the same sign. Putting all of this together yields

$$\begin{aligned}B&=\frac{\alpha^\gamma-1}{N+2\alpha+\varepsilon_k}[(-\alpha+1)\alpha^{\gamma k}+(-\alpha^3+1)\alpha^{\gamma(k+2)}+\cdots+(\alpha^2+1)\alpha^{\gamma(k+1)}\\&\qquad+(\alpha^4+1)\alpha^{\gamma(k+3)}+\cdots]\\&=\left[\frac{\alpha^\gamma-1}{N+2\alpha+\varepsilon_k}\right]\left[\frac{-\alpha^{1+\gamma k}+\alpha^{2+\gamma(k+1)}}{1-\alpha^{2+2\gamma}}+\frac{\alpha^{\gamma k}}{1-\alpha^\gamma}\right].\end{aligned}$$

Thus

$$\frac{B}{\Delta}\to\frac{\alpha-\alpha^\gamma(\alpha^2+\alpha)+2\alpha^{2+2\gamma}-1}{(N+2\alpha)(1-\alpha^{2+2\gamma})}. \tag{4.2}$$

Last of all, we estimate V/Δ. Recall (3.4). We have

$$V=\sum_{n=q_{k-1}}^{\infty}[(\chi_J\langle n\alpha\rangle, n\text{ odd})-(\chi_J\langle n\alpha\rangle, n\text{ even})]g(n).$$

Since N is even, the sequence q_n alternates parity, starting with $q_0=1$, $q_1=2$, $q_{n+1}=2q_n+q_{n-1}$. Therefore, if $k-1$ is odd, q_{k-1} is even. Thus the diagonals attached to J are parallel. This will pose no difficulty for the embedding. (See Fig. 5.) We study the case with $k-1$ even:

$$|V/c|=\alpha^{k\gamma}-2[\alpha^{(k+1)\gamma}+\alpha^{(k+3)\gamma}+\cdots]=\alpha^{k\gamma}-2\alpha^{(k+1)\gamma}/(1-\alpha^{2\gamma}).$$

Hence $|V/\Delta|\to 1-2\alpha^\gamma/(1-\alpha^{2\gamma})$.

Let γ_N be the solution for the limit equation $H/V=1$. As $N\to\infty$, $\gamma_N\to 1$. If $\gamma_N<\gamma<1$, there is k_N such that if $k>k_N$, then $|H/V|>1$. The choice of the constant c affects H for k small. Recall

$$H/\Delta=(\alpha^k+A+B)/\Delta=\alpha^{k(1-\gamma)}/c+(A+B)/\Delta>\alpha^{k(1-\gamma)}/c.$$

We know that $|V/\Delta|<C$ for some $C>0$ and all $k>0$. It suffices to find c so that

$$\alpha^{k(1-\gamma)}/c>C \qquad \text{for} \quad k\le k_N.$$

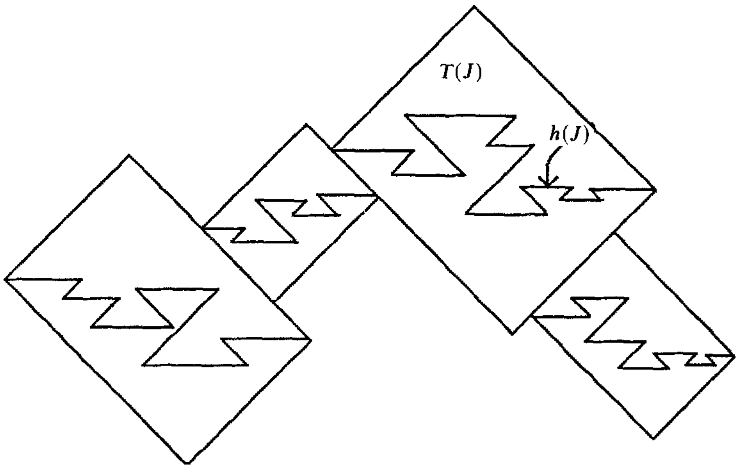


Fig. 5

It will follow that $|H/V|>1$ for all k and that h is an embedding. Fix $c_N = \alpha^k N^{(1-\gamma)}/C$. So $\alpha^{k(1-\gamma)}/c_N > C$ for $k \le k_N$. Then $|H/V|>1$ for $\gamma_N < \gamma < 1$ and $0 < c < c_N$. It is easy to see that $|H/\Delta|$ is uniformly bounded for fixed c and γ. Hence $|H/V|$ is uniformly bounded.

This completes Theorem 4.2. ∎

Remarks. The constant γ_N is sharp. That is, if $\gamma < \gamma_N$, then $H/\Delta > 1$ for k sufficiently large. It follows that Q is not embedded for such γ.

Proof of Embedding. Begin with γ_N and c_N of Theorem 4.2. Let J be a W_k-interval and $H = H(J)$, $V = V(J)$, and $\Delta = \Delta(J)$ be defined as above. Recall that J contains the endpoint with the largest index. We have $|H/V|>1$. Let $T = T(J)$ be the parallelogram determined by the endpoints of $h(\text{int}(J))$, see Fig. 5. By Theorem 4.2, the smaller diagonal attached to the end of $h(\text{int}(J))$ is contained in T. This is the fundamental geometric observation for it implies that $h(J) \subset T$.

Let x and $y \in I$. Let k be the first integer such that $\rho(x)$ and $\rho(y)$ are separated by a W_k-point. Then $\rho(x) \in J_1$ and $\rho(y) \in J_2$ where J_1 and J_2 are W_k-intervals. The chain of W_k-parallelograms T are disjoint by Theorem 4.2. Therefore $h(x) \neq h(y)$.

5. Hausdorff Dimension of Q

Let Q be the curve generated by $\alpha = [N, N, N, \ldots]$, and let c and γ be as in the previous section.

Definition. Given a metric space X, for each nonnegative real number s there is a corresponding s-dimensional Hausdorff measure μ_s defined as follows. Let $B \subset X$ be an arbitrary set. The zero-dimensional measure $\mu_0 B$ is the number of points in B. For $s > 0$, $\alpha > 0$, let

$$\mu_{s,\alpha} B = \inf \sum_i [\operatorname{diam}(B_i)]^s,$$

where the infimum is taken over all covers $\{B_i\}$ of B such that $\operatorname{diam}(B_i) < \alpha$ for each i. Then

$$\mu_s B = \lim_{q \to 0^+} \mu_{s,\alpha} B.$$

A set B has *Hausdorff dimension* s iff $\mu_r B = 0$ for all $r > s$ and $\mu_r B = \infty$ for all $r < s$. If $\dim(B) = s$, then $\mu_s B$ is the *Hausdorff measure of* B *within its dimension.*

Let Γ denote the Cantor set $Q \setminus \bigcup \{\operatorname{int} \Delta_n\}$.

Theorem 5.1. *The Hausdorff dimension of* Γ *is* $1/\gamma$. *The Hausdorff measure of* Γ *within its dimension is a positive, real number.*

Proof. Let $I_0 = I \setminus \{\langle n\alpha \rangle\}$. Define $g \colon I_0 \to \mathbf{R}^2$ by $g = h \circ \rho^{-1}$.

Claim. There exist constants A_1, A_2, A_3, and K_1 such that:

(1) If J is a W_k-interval, $k \geq 0$, then

$$A_2 |J|^\gamma < |g(J)| < A_1 |J|^\gamma.$$

(2) If J is an arbitrary interval of S^1, then

$$|g(J)| > A_3 |J|^\gamma.$$

(3) If p, $q \in Q$, the arc connecting p and q is contained in a disk of radius $K_1 d(p, q)$. Hence if $B \subset \mathbf{R}^2$ is a disk with $B \cap g(I_0) \neq \emptyset$, then

$$\operatorname{cl}(g^{-1}(K_1 B)) \supset \text{interval} \supset g^{-1}(B).$$

Proof of Claim. (1) By (4.1) and (4.4), $H \to C\Delta$ for some $C > 0$. The definitions imply $\Delta = c\alpha^{k\gamma} = c|J|^\gamma$. Finally, Theorem 4.2 implies $|H/V|$ is bounded away from 0 and ∞. This establishes (1).

(2) Let I be a W_k-interval contained in J, where k is minimal. Then

$$(2N+1)|I| > |J| > |I|$$

since any $2N+1$ adjacent W_k-interval contains a W_{k-1}-interval. Then

$$|g(J)| \geq |g(I)| > A_2 |I|^\gamma > A_3 |J|^\gamma$$

for some A_3.

(3) The proof that Q is embedded proves this claim (3) that Q is a quasi-circle: let $p=h(x)$ and $q=h(y)\in Q$. As before $h(x)$ lies in $T(J_1)$ or its left attached diagonal and $h(y)$ lies in $T(J_2)$ or its right attached diagonal. The arc connecting $h(x)$ and $h(y)$ is contained in the union D of these sets. Then there is a constant K_1 such that $d(x,y)>K_1|D|$.

By (1) the cover $\bigcup g(J)$, $J\in W_k$, satisfies

$$\sum |g(J)|^{1/\gamma}\le A_1^{1/\gamma}\sum |J|\le A_1^{1/\gamma}.$$

Hence $\mu_1/\gamma(g(I_0))<\infty$ implying

$$\dim(g(I_0))\le\frac{1}{\gamma}.$$

Suppose that $\dim(g(I_0))\le 1/\gamma$. Then $\mu_1/\gamma(g(I_0))=0$. Then there is a cover of $g(I_0)$ by disks B_i such that $\sum |B_i|^{1/\gamma}<\delta$ for any δ. Notice that $\sum |K_iB_i|^{1/\gamma}<K_1^{1/\gamma}\delta$. Consider the cover $\{K_1B_i\}$. Using (3) for each i we pick J_i such that

$$g^{-1}(B_i)\subset J_i\subset \mathrm{cl}(g^{-1}(K_1B_i)).$$

Therefore by (2) we have

$$A_3|J_i|^{\gamma}<|g(J_i)|\le K_1|B_i|.$$

Hence $|J_i|<(K_1|B_i|/A_3)^{1/\gamma}$. Since $\bigcup J_i$ covers I_0 we have

$$1\le\sum |J_i|\le (K_1/A_3)^{1/\gamma}\sum |B_i|^{1/\gamma}.$$

This contradicts the assumption that $\mu_{1/\gamma}=0$. From the preceding paragraph we have $0<\mu_{1/\gamma}(g(I_0))<\infty$. The theorem follows since $\mathrm{cl}(g(I_0))=\Gamma$. ■

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J. Harrison
Department of Mathematics
University of California
Berkeley
California 94720
U.S.A.