Proc. Indian Acad. Sci. (Math. Sci.), Vol. 99, No. 2, August 1989, pp. 113–122.


# Dynamics on Ahlfors quasi-circles

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MS received 20 September 1987; revised 24 October 1988

**Abstract.** The celebrated theory of Denjoy introduced a topological invariant distinguishing $C^1$ and $C^2$ diffeomorphisms of the circle. A $C^2$ diffeomorphism of the circle cannot have an infinite minimal set other than the circle itself. However, this is possible for $C^1$ diffeomorphisms. In dimension two there is a related invariant distinguishing $C^2$ and $C^3$ diffeomorphisms.

**Theorem.** Let $Q$ be a quasi-circle contained in a surface. If $\Gamma$ is an infinite minimal isometry set in $Q$ for a $C^3$ diffeomorphism, then $\Gamma$ equals $Q$. There exists a $C^2$ diffeomorphism of the annulus with a minimal Cantor set contained in a quasi-circle.

**Keywords.** Quasi-circle; minimal set; rotation number; Cantor sets; Denjoy counterexample.

## 1. Introduction

Poincaré and Birkhoff proved that a measure preserving homeomorphism of the two-dimensional annulus which twists the two boundary components in opposite directions must have fixed points in the interior of $A$. What is known as KAM theory emerged from this and is currently being developed and refined. (See [7], [8] and [9], for example.) The theory produces global topological and dynamical conclusions from local assumptions—$C^3$ differentiability and infinitesimal 'twisting'. The $C^3$ hypothesis is sharp; there exist counterexamples in the $C^2 + \delta$ category. (See [6].) In this paper we consider a problem with some rudimentary resemblance to twist theory.

Let $f$ be a $C^r$ diffeomorphism of the two-dimensional annulus $A = S^1 \times [-1, 1]$ to itself which is repelling at one boundary component $A^+ = S^1 \times \{1\}$ and attracting at the other $A^- = S^1 \times \{-1\}$. Suppose $f$ has no periodic points in $\mathrm{int}(A_0)$. If one orbit 'gets across', must they all? That is, if the $\alpha$-limit set of $x_0$ is in $A^+$ and its $\omega$-limit set is in $A^-$ for some $X_0 \in A$, then is this true for every $X \in A$? There is growing evidence that the answer is in the affirmative for $r = 3$. We pose an equivalent version of this question.

## 2. The north pole, south pole conjecture

Suppose $f: S^2 \to S^2$ is a $C^3$ diffeomorphism with the north pole N a repeller, the south pole S an attractor and no other periodic points. If one orbit is asymptotic to both

*Partially supported by NSF grant No. MCS-83202062.

S and N then $f$ is dynamically equivalent to the standard north pole, south pole diffeomorphism.

Formally, the conclusion means that $f$ is the time one map of the gradient flow on $S^2$.

A $C^r$ diffeomorphism $f:M\to M$ is of class $C^{r+\delta}$ if the $r$th derivative satisfies a $\delta$-Hölder condition. That is, there exists $C>0$ such that $\|D^r f_x - D^r f_y\| < C\|x-y\|^\delta$ for $x, y \in M$.

There exist counter-examples to the NP-SP conjecture if $f$ is $C^{2+\delta}$. (See [5], [4] and [3].) We show in this paper that $C^3$ is a natural bound to these examples. Hence the NP-SP conjecture concerns the topological-dynamical invariants that might distinguish $C^{2+\delta}$ and $C^3$.

We find the coordinates of the annulus more convenient to work with and put this spherical formulation aside.

A non-empty, closed, invariant set $\Gamma$ of a homeomorphism $f$ is said to be *minimal* if it is closed and contains no smaller non-empty, closed, invariant sets.

If there is a counter-example to the NP-SP conjecture then there exists a $C^r$ diffeomorphism $f$ of the annulus without periodic points which has one orbit asymptotic to both boundary components and one orbit whose closure $\Gamma$ stays bounded away from $\partial A$. Furthermore, $\Gamma$ may be taken to be a minimal set. In [4] it is shown that the existence of such a diffeomorphism implies the existence of a $C^r$ Seifert counter-example. That is, there exists a $C^r$ vector field on the three-sphere $S^3$ with neither zeroes nor closed integral curves. Hence the NP-SP conjecture is 'contained' in the Seifert conjecture.

A recent theorem of John Franks is useful in analyzing the dynamics of $f$.

**Theorem** (Franks). *Let $f:A\to A$ be a homeomorphism of the open annulus $A$ and $x\in A$. Let $g$ be a lift of $f$ to the universal cover $\mathbb{R}\times[-1,1]$ of $A$ and $y$ a lift of $x$. Let $y_n$ denote the first component of $g^n(y)$. If there exists a rational number $p/q$ with*

$$\underline{\lim}\,\frac{y_n}{n} \leqslant \frac{p}{q} \leqslant \overline{\lim}\,\frac{y_n}{n}$$

*then there exists a point $z\in A$ with $f^q(z)=z$.*

By the theorem of Franks, $\Gamma$ must be an infinite, perfect minimal set which has irrational rotation number—the cyclic order is preserved by $f$. Certainly $\Gamma$ could not be a circle, otherwise no orbit would get across it; however, it is not known if $\Gamma$ must be a Cantor set.

## 3. The Denjoy Cantor sets

The reader might be reminded of the Denjoy's theory where the critical degree of differentiability is 2 and the dimension is 1. Denjoy [1] found that the degree of differentiability of a circle diffeomorphism $f$ influences its topological type. If $f$ is $C^2$ and has no periodic points then $f$ has simple dynamics—it is topologically conjugate to a rotation through an irrational angle. This is not the case in the $C^{1+\delta}$ category.

Suppose $\Gamma\subset S^1$ is a Cantor set. If there exists a homeomorphism $f:S^1\to S^1$ for which $\Gamma$ is minimal, then the pair $(f,\Gamma)$ is a *Denjoy Cantor set*. Denjoy Cantor sets provide the key ingredient to classifying homeomorphisms of the circle. Poincaré defined the *rotation numbers* and showed that all homeomorphisms of the circle have them. Furthermore, any homeomorphism of the circle with irrational rotation number $\alpha$ is either topologically conjugate to a rigid rotation through $\alpha$ or has a minimal Cantor set. Denjoy proved that these examples can all exist as $C^1$ diffeomorphisms but not $C^2$ (actually $C^{1+bv}$ is impossible). We call these examples Denjoy Cantor sets. More generally, a homeomorphism $g:\Gamma' \to \Gamma'$, $\Gamma'$ contained in an $n$-manifold $M$ is also called a Denjoy Cantor set if the pair $(g, \Gamma')$ is topologically conjugate to a Denjoy Cantor set $(f, \Gamma)$ in $S^1$. That is, there exists an embedding $h:\Gamma \to M$ such that $h(\Gamma) = \Gamma'$ and $h \cdot f = g \cdot h$.

It is not completely understood under what circumstances Denjoy Cantor sets can exist. Hall [2] showed that it is possible to have a Denjoy Cantor set in a $C^\infty$ annular diffeomorphism. However, it is attracting, and so no orbit is asymptotic to both boundary components. Do there exist $C^3$ diffeomorphisms $f$ of $A$ with no periodic points, a Denjoy Cantor set $(f, \Gamma)$ and one orbit asymptotic to both boundary components? We consider some possibilities.

There are two features of $\Gamma$ for us to study—its structure as a subset of $A$ and the properties of the first derivative of $f$ at $\Gamma$, the *distortion* of $f$ at $\Gamma$.

Using the methods of Denjoy, one can rule out any Denjoy Cantor set $\Gamma$ contained in a smooth Jordan curve as long as $f$ is $C^2$. It is not possible for $\Gamma$ to have totally arbitrary topological structure since minimality implies homogeneity. A natural question arises—how wild can $\Gamma$ be?

In Denjoy's theory, the simplest $C^1$ examples have first derivative, the identity at the minimal Cantor set. It is quite easy to show there are no $C^2$ diffeomorphisms of the circle with this condition at the Cantor set: Let $L_n$ denote the intervals complementary to $\Gamma$, indexed so that $f(L_n) = L_{n+1}$. Let $a_n = |L_n|$. Since $f$ is $C^1$, we can apply the mean value theorem and continuity of the first derivative to conclude that $a_n/a_{n+1} \to 1$ as $n \to \infty$. If $f$ were $C^2$, we could apply the mean value theorem to $f'$ and use continuity of the second derivative to conclude that

$$\frac{1 - \dfrac{a_n}{a_{n+1}}}{a_n} \to 0 \quad \text{as } n \to \infty.$$

It follows that $\sum a_n = \infty$, contradicting the finite arc length of $S^1$.

The same proof extends to $\mathbb{R}^2$ and shows there are no $C^3$ annular diffeomorphisms with the first derivative the identity and the second derivative the 0 bilinear transformation at a square rectifiable Denjoy Cantor set.

## 4. Isometry sets

In this paper we consider a large class of 'simplest' examples. We assume that the first derivative of $f$ at each point of $\Gamma$ is an isometry for some Riemannian metric on $A$. The isometry may vary from point to point. We call $\Gamma$ an *isometry* set. Note that this isometry condition on $\Gamma$, even for the usual metric on $A$, is weaker than the identity hypothesis. The annulus may be replaced by any Riemannian manifold $M$.

We need a little more background before we can state the main results.

Smooth curves $Q$ are *n-rectifiable* for all $n \geqslant 1$. That is, there exists a constant $L > 0$ such that $\sum |x_i - x_{i+1}|^n < L$ for all partitions $x_1 < x_2 < \cdots < x_i < x_{i+1} < \cdots$ of $Q$. A curve is *square rectifiable* if it is 2-rectifiable. One can similarly define the notion of

$n$-rectifiable for subsets of curves or for any set on which there is a well-defined order. In particular, we may consider whether Denjoy Cantor sets are $n$-rectifiable. Topological curves and Cantor sets may be so wild that they fail to be $n$-rectifiable, for any $n$. However, additional restrictions guarantee that curves be square-rectifiable.

A curve $Q$ is called a *quasi-curve* if there exists $K > 0$ such that if $x, y \in Q$, the arc connecting $x$ and $y$ is contained in a disc of radius $Kd(x, y)$. A *quasi-circle* is a Jordan curve which is the union of quasi-arcs. We prove

PROPOSITION

*Quasi-arcs are square rectifiable.*

**Theorem.** *Let $f$ be a $C^3$ diffeomorphism of a compact Riemannian n-manifold M and $\Gamma \subset M$ a minimal isometry set. If $\Gamma$ is a Denjoy Cantor set then $\Gamma$ is not square rectifiable.*

The Proposition and Theorem imply the following

COROLLARY

*Let $f$ be a $C^3$ diffeomorphism of the annulus $A$ and $Q \subset A$ a quasi-circle. If $\Gamma \subset Q$ is an infinite, minimal isometry set then $\Gamma = Q$.*

*Proof.* Since $\Gamma$ is an infinite minimal set, the rotation number of $f|_\Gamma$ is irrational. Then $\Gamma$ can only be a Cantor set or all of $Q$.

These results depend on a general estimate for the asymptotic behaviour of pairs of orbits of isometry minimal sets. This is an example of an estimate of 'non-linear' distortion.

**Theorem.** *Let $E$ be an isometry minimal set of a $C^3$ mapping $f$ of a compact Riemannian 2-manifold $M$. If $y \in M$ and $z \in E$ then*

$$\sum_{n=1}^{\infty} \mathrm{d}(y_n, x_n)^2 = \infty.$$

The proof of Denjoy's result depends on the divergence of the Poincaré series for $C^2$ maps $f$

$$\sum_{n=1}^{\infty} |Df_x^n|^1.$$

(See Sullivan [10], for example). In practice $Df_x^n$ is sometimes replaced by $\mathrm{d}(x_n, y_n)$ for $s$ in the minimal set and $y$ arbitrary in the manifold where $d$ is the Riemannian metric on $M$. The exponent is related to the degree of differentiability of $f$. If $f$ is $C^r$, it is natural to estimate the general dynamic sum

$$\sum_{n=1}^{\infty} \mathrm{d}(y_n, X_n)^{r-1}$$

even on higher dimensional manifolds. In this paper we restrict ourselves to Riemannian two-manifolds and $r = 3$, although generalizations to higher dimensions and degrees of differentiability are possible.

## 5. Isometry and the second derivative

A linear transformation from one normed space to another, say $T:E_1 - E_2$, is an *isometry* if $T$ is a bijection and

$$\| Tx \|_{E_2} = \| x \|_{E_1} \text{ for all } x \in E_1.$$

The quantity

$$\rho(T) = \max\left[ \sup_{\|x\|_{E_1}=1} \| Tx \|_{E_2},\ \sup_{\|x\|_{E_1}=1} 1/\| Tx \|_{E_2} \right]$$

measures how non-isometric is $T$. Then $\rho(T) \geqslant 1$; $T$ is an isometry if and only if $\rho(T) = 1$ and $T$ is a bijection.

*Lemma 5.1. Let $f$ be a $C^2$ diffeomorphism of a Riemannian m-manifold $M$ and $p \in M$, fixed. Suppose there exists $p_n \to p$ such that $T_{p_n} f$ is an isometry respecting the given Riemann structure $g$ of $M$. Let $\bar{f}$ be the lift of $f$ from $M$ to $T_p M \approx R^m$ at $p$.*

$$\begin{array}{ccc} T_pM & \xrightarrow{\ \bar{f}\ } & T_{fp}M \\ \Big| \exp_p & & \Big| \exp f_p \\ M & \xrightarrow{\ f\ } & M \end{array} \qquad \bar{p}_n = \exp_p^{-1}(p_n)$$

*Let $\bar{p}_n = \exp_p^{-1}(p_n)$, then the sequence of linear maps*

$$(D\bar{f})\bar{p}_n\colon \quad T_pM - T_{fp}M$$

*is non-isometric only to the extent:*

$$| 1 - \rho(D\bar{f}\bar{p}_n) | = O(\| \overline{p_n} \|^2)$$

*Remarks.* The norms on $T_pM$ and $T_{fp}M$ are $g(p)$ and $g(fp)$ respectively. The length $\| \overline{p_n} \|$ is also calculated with respect to $g(p)$, although we could replace $\| \overline{p_n} \|$ with $d(p_n, p)$ since

$$\frac{\| \overline{p_n} \|}{d(p_n, p)} \to 1 \quad \text{as } p_n \to p.$$

*Proof.* To calculate $\rho((D\bar{f})\overline{p_n})$ one considers the pulled-up Riemann structure on $T_pM$ and $T_{fp}M$, namely

$$\bar{g}_p\colon \bar{g}_p(w_p; u, v) = \langle T_{w_p} \exp_p(u), T_{w_p} \exp_p(v) \rangle_{\exp_p(w_p)}$$

for all $w_p \in T_pM$ near $O_p$ and

for all $u, v \in T_{w_p}(T_pM) \approx T_pM$

$$\bar{g}_{fp}\colon \bar{g}_{fp}(w_{fp}; u, v) = \langle T_{w_{fp}} \exp_{fp}(u), T_{w_{fp}} \exp_{fp}(v) \rangle_{\exp_{fp}(w_{fp})}.$$

The map $D\bar{f}$ at the point $\overline{p_n} = \exp_p^{-1}(p_n)$ is an isometry from the tangent space to

$T_pM$ at $\overline{p_n}$, equipped with the metric $\bar{g}_p(\overline{p_n};*)$ to the tangent space to $T_{fp}M$ at $\overline{f(p_n)}$ equipped with the metric $\bar{g}_{fp}(\overline{fp_n};*)$.

Let $e^1,\ldots,e^m$ be an orthonormal basis for $T_pM$.

The map $T_pf:T_pM - T_{fp}M$ is an isometry (being the limit of isometries) so $T_pf(e^1),\ldots,T_pf(e^m)$ is an orthonormal basis at $T_{fp}M$. Both these bases give rise to $\bar{g}_{ij}$-expressions for the metrics $\bar{g}_p$ and $\bar{g}_{fp}$ on $T_pM$ and $T_{fp}M$. Besides,

$$\bar{g}_{ij}p(w_p)=\bar{g}_p(w_p;e^i,e^j)=\delta_{ij}+O(\|w_p\|^2)$$

$$\bar{g}_{ij}f_p(w_{fp})=\bar{g}_{fp}(w_{fp};\ T_pf(e^i),\ T_pf(e^j))=\delta_{ij}+O(\|w_{fp}\|^2).$$

Thus

$$\langle(D\bar{f})\bar{p}_n(u),\quad (D\bar{f})\bar{p}_n(u)\rangle_{g(fp)}$$

$$=\sum(i\text{-th component of }v)^2 \text{ where } v=(D\bar{f})\overline{p_n}(u)$$

$$=\sum\delta_{ij}(i\text{-th component of }v)(j\text{-th component of }v)$$

$$=\sum\bar{g}_{ijfp}(\overline{fp_n})(i\text{-th component of }v)(j\text{-th component of }v)$$

$$+\sum(\delta_{ij}-\bar{g}_{ij}f_p)(\overline{fp_n})(i\text{-th component of }v)(j\text{-th component of }v)$$

$$=\langle(D\bar{f})_{\bar{p}_n}(u),\quad (D\bar{f})_{\bar{p}_n}(u)\rangle_{\bar{g}_{fp}}(\overline{fp_n})$$

$$+\sum(\delta_{ij}-\bar{g}_{ijfp})(\overline{fp_n})(i\text{-th component of }v)(j\text{-th component of }v)$$

$$=\langle u,u\rangle_{\bar{g}_p(p_n)}+O(\|\overline{fp_n}\|^2)\cdot\|D\overline{f_{pn}}(u)\|^2$$

$$=\|u\|^2+O(\|\overline{p_n}\|^2)\cdot\text{constant }\|u\|^2.$$

Since $f$ is a diffeomorphism and $T_pf$ is an isometry we have

$$\|\overline{p_n}\|/\|\overline{fp_n}\|\to 1\quad\text{as } n\to\infty.$$

Hence $\|D\overline{f_{p_n}}\|=(1+O(\|\overline{p_n}\|^2)^{1/2}=1+O(\|\overline{p_n}\|^2)$. q.e.d.

Let $x_1$ be a sequence of points in $\mathbb{R}^n$. Suppose there exists a finite set of unit vectors $v^1, v^2,\ldots,v^m$ in $\mathbb{R}^n$ which is the limit set of $\{x_i/\|x_i\|\}$. If $\|x_i\|\to 0$, we say that the sequence $x_i$ converges to 0 from $m$ directions $v^1, v^2,\ldots,v^m$.

*Lemma 5.2. Let $g$ be a $C^2$ diffeomorphism of $R^n$. Suppose there exist points $x\in R^n$ converging to 0 from $m$ directions $v^1,\ldots,v^m$ such that $|1-\rho(Dg_x)|=O(\|x\|^2)$. Then $D^2g_0(v,w)\cdot Dg_0(u)=0$ for $u,v,w\in sp(v^1,\ldots,v^m)$. If $\mathrm{sp}(v^1,\ldots,v^m)\cong R^n$ then $D^2g_0(v,w)=0$.*

The author is grateful to M Shub and C Robinson for the following proof.

*Proof.* Observe that $Dg_0$ is an isometry since $\rho(Dg_0)=1$.

The set of linear transformations $\{Dg_0^{-1}Dg_x\}$ is tangent to the orthogonal metrices at the identity (where $x=0$). The antisymmetric metrices form the tangent space to the orthogonal matrices based at the identity. Hence, if $A=Dg_0^{-1}D^2g_0$, then $A(v)=A_v$

is antisymmetric for $v \in \mathrm{sp}(v^1,\ldots,v^m\}$. Thus $A_v(w_1)\cdot w_2 = -w_1\cdot A_v(w_2)$ for all vectors $w_1$ and $w_2$.

Let $v, w \in \mathrm{sp}\{v^1,\ldots,v^m\}$. Then $A_v(v)\cdot w = -v\cdot A_v(w) = -v\cdot A_w(v) = A_w(v)\cdot v = A_v(w)\cdot v = -w\cdot A_v(v) = 0$. Write $A_v(v) = x + y \in \mathrm{sp}\{v^1,\ldots,v^m\} \times R^l$ where $R^l$ is the orthogonal complement to $\mathrm{sp}\{v^1,\ldots,v^m\}$. Then $0 = A_v(v)\cdot x = x\cdot x + y\cdot x$. Since $y\cdot x = 0$ we have $x\cdot x = 0$. Thus $x = 0$ and $A_v(v) \in R^l$. Since $A_{v+w}(v+w) = A_v(v) + 2A_v(w) + A_w(w) \in R^l$, then $A_v(w) \in R^l$. Hence $u\cdot A_v(w) = 0$ for all $u, v, w \in \mathrm{sp}\{v^1,\ldots,v^m\}$. Note that if $\mathrm{sp}\{v^1,\ldots,v^m\} \cong R^n$ then $A_v(w) = 0$.

Hence $0 = u\cdot Dg_0^{-1}D^2g_0(v,w) = (Dg_0u)\circ D^2g_0(v,w)$. If $\mathrm{sp}\{v^1,\ldots,v^m\} \cong R^n$ then $0 = Dg_0^{-1}D^2g_0(v,w)$. Since $Dg_0$ is an isometry, $0 = D^2g_0(v,w)$. q.e.d.

*Lemma 5.3. Let $g$ be a $C^2$ diffeomorphism of $R^n$. Let $E \subset R^n$. Suppose there exist points $x \in E$ converging to 0 where $x$ is a limit point of $E$ in the direction $v_x$. Suppose $\{v^1,\ldots,v^m\}$ are limit vectors of $v_x$ as $x \to 0$. If $|1 - \rho(Dg_x)| = O(\|x\|^2)$ for all $x \in E$ then $D^2g_0(v,w)\cdot Dg_0(u) = 0$ for $u, v, w \in \mathrm{sp}\{v^1,\ldots,v^m\}$. If $\mathrm{sp}\{v^1,\ldots,v^m\} \cong R^n$ then $D^2g_0(v,w) = 0$.*

The proof is identical to that of Lemma 5.2.

## 6. Minimal isometry sets

DEFINITION

A set $E \subset M$ is *minimal* under $f$ if it is invariant and contains no invariant subsets. A set $E$ is an *isometry set* if $Df$ is an isometry at each $x \in E$.

**Theorem 6.1.** *Let $f: M \to M$ be a $C^3$ mapping of a compact Riemannian 2-manifold $M$. Let $E \subset M$ be an isometry minimal set. Let $y, z \in E$ where $z$ is in a local geodesic coordinate chart about $y$. Then $\|Df_y(z-y) + \frac{1}{2}D^2f_y(z-y)^2\| \leq \|z-y\| + O(\|z-y\|^3)$.*

Here, "$(z-y)$" refers to the vector $u \in R^2$ such that $\exp_y(u) = z$; "$Df_y$" and "$D^2f_y$" are the first and second derivatives at 0 of the lift $\bar{f}: T_yM \to T_{fy}M$.

Notice that Taylor's theorem alone merely implies that $\|Df_y(z-y) + \frac{1}{2}D^2f_y(z-y)^2\| \leq \|z-y\| + O(\|z-y\|^2)$ if $Df_y$ is an isometry. The proof of Theorem 2.1 uses the dynamics of $f$ as well as the isometry condition on $Df$ to sharpen the estimate.

*Proof.* We may assume that $E$ is an infinite set, otherwise the result is trivial. It follows that $E$ is perfect since it is an infinite minimal set.

First, suppose there exists a point $x \in E$ with sequences $p$ and $q$ in $E$ approaching $x$ from two directions.

Since $f$ is $C^1$ (in fact $C^3$) and $E$ is invariant, each point $f^n(x) = x_n$ has limit points in $E$ from two directions. Since $Df$ is an isometry at these points it follows from Lemma 5.1 and Lemma 5.2 that $D^2f = 0$ at $x_n$. Since $E$ is minimal, $\{x_n\}$ is dense in $E$. Since $y \in E$, $D^2f_y = 0$. The estimate follows.

Now suppose there is no point in $E$ with limit points from two directions. Then for each $x \in E$ there is a unique unit vector $v_x$ along which points in $E$ are converging towards $x$. Either $v_x$ varies continuously or it does not. Suppose $v_x$ is not continuous at $p$. Then there exist at least two distinct limits $v'$ and $v''$ of $v_x$ as $x \to p$. By

Lemma 5.3 we have $D^2f_p = 0$. By the smoothness of $f$ this discontinuity is preserved, so $D^2f_x = 0$ for each orbit point $x = x_n$ and hence for each $x \in E$.

Suppose that $v_x$ is a continuous function of $x$. We know by Lemma 5.2 only that $D^2f_x(v_x, v_x)\cdot Df_x(v_x) = 0$. We need to estimate the dot product $|D^2f_y(w, w)\cdot Df_y(w)|$ where $w = z - y$.

(6.2) *Let* $g: R^2 \to R^2$ *be* $C^2$. *If* $v\cdot D^2g_0(v, v) = 0$ *then* $|w\cdot D^2g_0(w, w)| \leqslant O\|w\|^3\cdot\sin(w, v)$.

*Proof of 6.2.* Assume $v = (1, 0)$ and $w = (1, y)$. Then $v\cdot D^2g_0(v, v) = (1, 0)\cdot(\partial^2 g_1/\partial x\partial x, \partial^2 g_2/\partial x\partial x) = 0$. Thus $\partial^2 g_1/\partial x\partial x = 0$. Using the Hessian matrix to express $D^2g_0(w, w)$ in coordinates we obtain

$$|w\cdot D^2g_0(w, w)| = \left|(1, y)\cdot\left(y\frac{\partial^2 g_1}{\partial y\partial x} + y\left(\frac{\partial^2 g_1}{\partial x\partial y} + y\frac{\partial^2 g_1}{\partial y^2}\right),\ \frac{\partial^2 g_2}{\partial x^2} + y\frac{\partial^2 g_2}{\partial y\partial x} + y\left(\frac{\partial^2 g_2}{\partial x\partial y} + \frac{y\partial^2 g_1}{\partial y^2}\right)\right)\right|.$$

Therefore there exist constants $C > 0$ and $C' > 0$, depending on the second derivative of $g$, such that $|w\cdot D^2g_0(w, w)| \leqslant C\|y\| \leqslant C'|\sin(w, v)|$. The estimate 6.2 follows from the linearity of the dot product and the second derivative. Hence

(6.3) $$|D^2f_y(z-y)^2\cdot Df_y(z-y)| \leqslant O\|z-y\|^3\cdot|\sin(v_y, z-y)|.$$

We next estimate $|\sin(v_y, z-y)|$ for $y, z \in E$. The function $v_y$ is continuous. It is uniformly continuous by compactness of $M$. Since $v_y$ approximates $(z-y)/\|z-y\|$ it follows that $|\sin(v_y, z-y)| \leqslant 0\|z-y\|$. Hence, $|D^2f_y(z-y)^2\cdot Df_y(z-y)| \leqslant 0\|z-y\|^4$.

(6.4) *If* $a, b \in R^2$, $\|a\| \leqslant 1$, $|a\cdot b| \leqslant O\|a\|^4$ *and* $\|b\| \leqslant O\|a\|^2$ *then* $\|a+b\| \leqslant \|a\| + O\|a\|^3$.

*Proof of 6.4.* Let $c$ be the unique vector such that $a\cdot(b-c) = c\cdot(b-c) = 0$. Then $|\cos\theta| = \|c\|/\|b\|$ where $\theta$ is the angle between $a$ and $b$. We have $|a\cdot b| = \|a\|\,\|b\|\,|\cos\theta| \leqslant 0\|a\|^4$. Hence $\|a\|\,\|c\| \leqslant 0\|a\|^4$ and thus $\|c\| \leqslant 0\|a\|^3$. Note $\|b-c\| \leqslant \|b\| \leqslant 0\|a\|^2$. Since $a\cdot(b-c) = 0$ it follows that $\|a+(b-c)\| = \sqrt{(\|a\|^2 + \|b-c\|^2)} \leqslant \sqrt{(\|a\|^2 + C\|a\|^4)} \leqslant \|a\| + 0\|a\|^3$. Hence $\|a+b\| \leqslant \|a+(b-c)\| + \|c\| \leqslant \|a\| + 0\|a\|^3$.

By (6.3) and (6.4) we conclude

$$\|Df_y(z-y) + \tfrac{1}{2}D^2f_y(z-y)^2\| \leqslant \|z-y\| + 0(\|z-y\|^3). \qquad \text{q.e.d.}$$

COROLLARY 6.5

*Let* $E$ *be an isometry minimal set of a* $C^3$ *mapping* $f$ *of a compact Riemannian 2-manifold* $M$. *If* $y \in M$ *and* $z \in E$ *then*

$$\sum_{n=1}^{\infty} (\mathrm{d}(f^n(y), f^n(z)))^2 = \infty$$

*Proof.* Let $y \in M$ and $z \in E$. Since $E$ is compact, there exists a constant $\mu > 0$ and geodesic coordinate charts $U_{zn} = U_n$ based at $f_n(z) = z_n$ with radius $> \mu$. We can assume that there exists a positive integer $N$ such that $y_n$ lies in $U_n$ for $n \geqslant N$. Otherwise the result is immediate. Assume $n \geqslant N$.

By Taylor's theorem, since $f$ is $C^3$ and $E$ is compact, there exists a constant $A > 0$ such that in the coordinates of the chart $U_n$,

$$\frac{\| f(y_n) - (f(z_n) + Df_{z_n}(y_n - z_n) + \tfrac{1}{2} D^2 f_{z_n}(y_n - z_n)^2) \|}{\| y_n - z_n \|^3} < A.$$

But

$$\frac{| \| Df_{z_n}(y_n - z_n) + \tfrac{1}{2} D^2 f_{z_n}(y_n - z_n)^2 \| - \| y_n - z_n \| |}{\| y_n - z_n \|^3} < A'$$

Hence

$$\frac{\| f(y_n) - (f(z_n) + (y_n - z_n)) \|}{\| y_n - z_n \|^3} < A''.$$

Since $f(y_n) = y_{n+1}$ lies in $U_{n+1}$ we may replace $f(y_n)$ by $y_{n+1}$. By the triangle inequality

$$\left| \frac{\| y_{n+1} - z_{n+1} \| - \| y_n - z_n \|}{\| y_n - z_n \|^3} \right| \leqslant \frac{\| y_{n+1} - (z_{n+1} - (y_n - z_n)) \|}{\| y_n - z_n \|^3} \leqslant A''.$$

6.6 *If a sequence of positive numbers $a_n \to 0$ satisfies*

$$\frac{|a_{a+1} - a_n|}{|a_n|^r} < A \textit{ and } \frac{a_{a+1}}{a_n} > B > 0$$

then

$$\sum_{a=1}^{\infty} a_n^{r-1} = \infty.$$

*Proof of* (6.6). Since $a_n \to 0$ it follows that

$$\prod_{a=1}^{N} \frac{a_n}{a_{n+1}} = \frac{a_1}{a_N} \to \infty \quad \text{as } N \to \infty.$$

Hence

$$\sum_{n=1}^{\infty} \left| 1 - \frac{a_n}{a_{n+1}} \right| = \infty.$$

By hypothesis $(a_{n+1}/a_n) > B$ and $a_n^r > (|a_{n+1} - a_n|/A)$, so

$$a_n^{r-1} > \frac{1}{A} \left| \frac{a_{n+1}}{a_n} - 1 \right| = \frac{1}{A} \frac{a_{n+1}}{a_n} \left| 1 - \frac{a_n}{a_{n+1}} \right| > \frac{B}{A} \left| 1 - \frac{a_n}{a_{n+1}} \right|.$$

It follows from the comparison test that $\sum_{n=1}^{\infty} a_n^{r-1} = \infty$.

With (6.6) we may complete the proof of Corollary 6.5. Since $Df$ is an isometry at $z_n$, there exists a constant $B > 0$ such that $\| y_{n+1} - z_{n+1} \| / \| y_n - z_n \| > B$. Let $a_n = \| y_n - z_n \|$ and $r = 3$. q.e.d.

COROLLARY 6.7

*Let $f$ be a $C^3$ diffeomorphism of a compact Riemannian 2-manifold $M$ and $Q \subset M$ a minimal isometry set. If $Q$ is a Denjoy Cantor set then $Q$ is not square-rectifiable.*

*Proof.* By definition, if $Q$ is a Denjoy Cantor set in $M$ there exists $h:Q \to S^1$ mapping $Q$ onto a Denjoy Cantor set $\Gamma$ in $S^1$. Let $x_0$ and $y_0$ be the inverse image of endpoints of an interval complementary to $\Gamma$ in $S^1$. Since $f$ preserves the order of $Q$ the intervals $(x_n, y_n)$ are each disjoint from $Q$ and each other. By Corollary 6.5, $\sum d(x_n, y_n)^2 = \infty$. Thus $Q$ is not square-rectifiable. q.e.d.

DEFINITION

A Jordan curve $Q$ in a Riemannian two-manifold is a *quasi-circle* if there exists a positive constant $K$ such that if $x, y \in Q$ then one of the arcs of $Q$ connecting $x$ and $y$ is contained in a disk of radius $Kd(x, y)$.

PROPOSITION 6.8

*A quasi-circle Q is square-rectifiable.*

*Proof.* Let $x_1 < x_2 < \cdots$ be a partition of $Q$.

Let $a_n = d(x_n, x_{n+1})$ and $B_n$ the disk of radius $a_n/8K$ centered at $x_n$ where $K$ is the quasi-constant for $Q$. The result follows from compactness of $Q$ if the $B_n$ are disjoint. Suppose $B_n \cap B_m \neq \varnothing$. Then $d(x_n, x_m) < (a_n + a_m)/8K \leqslant a_n/4K$, assuming $a_m \leqslant a_n$. Since $Q$ is a quasi-circle, one of the arcs connecting $x_n$ and $x_m$ is contained in a disk of radius $a_n/4$. Let $z$ be a point in this arc equidistant from $x_n$ and $x_m$. Then $d(x_n, z) = d(x_m, z) < a_n/2$. Thus $a_n = d(x_n, x_m) \leqslant d(x_n, z) + d(z_n, x_m) < a_n$. q.e.d.

COROLLARY 6.9

*Let f be a $C^3$ diffeomorphism of a compact Riemannian 2-manifold M and $Q \subset M$ a minimal isometry set. If Q is a Denjoy Cantor set then Q is not square-rectifiable.*

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