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DENJOY FRACTALS

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INTRODUCTION

Poincaré defined the rotation number $\rho(f)$ for a homeomorphism f of the circle S^1:

$$\rho(f) = \lim_{n\to\infty} \frac{\tilde{f}^n(x) - x}{n}$$

where $\tilde{f}$ is any lift of f to the real line R^1 and $x \in R^1$. The limit $\rho(f)$ is a topological invariant of f and is independent of the lift $\tilde{f}$ and the starting point x. Every homeomorphism of a compact manifold has a minimal set. If $\rho(f) = \alpha$ is rational then every minimal set for f is finite and conversely. Henceforth we assume α is irrational. Since every infinite minimal set is perfect and homogeneous, an infinite minimal set of S^1 is either a Cantor set or S^1 itself.

Denjoy Theorem (1932) [3]. *If f is C^2 then f is topologically conjugate to a rotation through α.*

As a complement to this theorem, Denjoy produced uncountably many topologically distinct examples of C^1 diffeomorphisms D having Cantor minimal sets. Each D permutes the countable set of intervals $\{I_n\}$ complementary to the minimal set.

In this paper we give examples of $C^{2+\delta}$ diffeomorphisms of the annulus A permuting a countable set of disjoint disks $\{R_n\} \subset A$.

Theorem A. *For $\delta > 0$ sufficiently small there exists a Jordan curve $Q \subset A$, a family of disjoint disks $\{R_n\} \subset A$ with $R_n \cap Q \neq \varnothing$ and a $C^{2+\delta}$ diffeomorphism f: $A \to A$ such that $Q \cup \{R_n\}$ is f-invariant and has no periodic points.*

The curve Q has Hausdorff dimension $1+\delta$.

The derivative of f at the minimal set in Q is an isometry, a feature shared by the canonical Denjoy counterexample D. This property is useful in [6] where $C^{2+\delta}$ counterexamples to the Seifert conjecture are found.

There may be periodic points of f in a neighborhood of $Q \cup \{R_n\}$. In [6] f is made to be semi-stable so there is no longer any periodicity.

An overview of this paper and its sequel [6] may be found in [11].

Problem. If f: $M \to M$ is a C^3 diffeomorphism of a compact surface M with no periodic points and $B \subset M$ is a disk with $\{f^n(B)\}$ disjoint, must the shape of $f^n(B)$ become distorted? That is, must (outer diameter $(f^n(B))$)/(inner diameter $(f^n(B))$) be unbounded?

Denjoy proved that if D were C^2 then $\Sigma\,|I_n| = \infty$, contradicting the finite arc length of S^1. The fact that there is bounded distortion in D is important – that is, $\|Df^q\|$ is bounded for a subsequence $q \to \infty$. There are several examples of embeddings of one-dimensional Denjoy examples in the plane where the distortion of f is unbounded and f is at least C^2. In each of these $\Sigma\ |I_n| < \infty$, but there is no contradiction.

The first was due to R. Knill [16]. He embedded the Denjoy minimal set in the plane as part of a C^∞ diffeomorphism. Mather [18] proved there exist embedded Denjoy minimal sets in some area-preserving twist maps of the annulus. In 1980 G. R. Hall [5] embedded an entire Denjoy counterexample in the plane as part of a C^∞ diffeomorphism. M. Herman [13] produced an area-preserving $C^{3-\varepsilon}$ example.

These examples have unbounded distortion in the derivative and putting limits on the distortion makes such examples impossible. The author proved that if Df is an isometry at the Cantor set and Q is a quasi-circle then f cannot be C^3 [7]. Ghys [4] showed that if α satisfies a Diophantine condition then f cannot be complex analytic and restrict to a Denjoy counterexample on Q.

The construction of R. Knill preceded and influenced this paper. He embedded the forward Denjoy intervals horizontally and the backward intervals vertically. His diffeomorphism was hyperbolic and had infinitely many periodic orbits in a neighborhood of the Denjoy minimal set. At the time the author was looking for a two-dimensional "Denjoy" example which had intervals of length $1/n^\gamma$, $\gamma < 1$ and disjoint, invariant disks. The longer intervals were needed to satisfy analytic requirements of [8, 9] for a $C^{2+\delta}$ extension and the disks were needed to help rid the example of periodic points. Although Knill's example was two-dimensional, it did not satisfy these two additional properties. The author learned of Knill's work from C. Rourke. In January, 1978 Rourke and she made an unsuccessful attempt to incorporate these properties into a modification of Knill's example.

There are several mathematicians whose help I have appreciated while doing this research. My Seifert education began with M. Handel. G. Levitt and H. Helson suggested the possibility of a relationship between the estimates of my initial work on weighted uniform distribution and the "unweighted" estimates of Ostrowski and Kesten. As a result, I was able to simplify the proof significantly. Yoccoz made the suggestion of using two orbits instead of one in the construction of $h: S^1 \to A$. This makes the embedding easier to describe. Unfortunately, the number theoretic estimates become more difficult.

I thank A. Norton and J. Robbin for their careful reading of the manuscript and helpful suggestions for improving the exposition. I thank M. Hirsch, C. Pugh, M. Shub, L. Siebenmann and J. Yorke for educational conversations and encouragement.

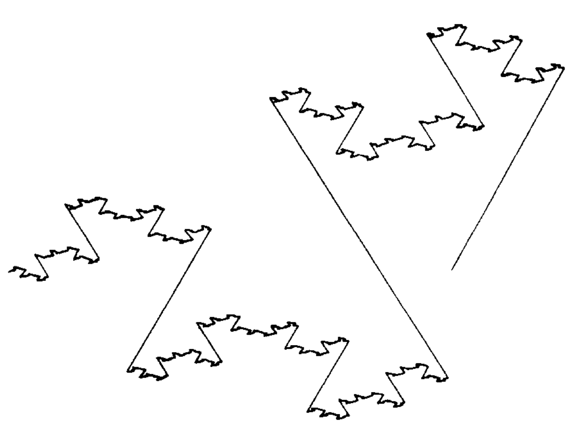


Fig. 1. A Denjoy fractal Q.

§1. GEOMETRY OF CONTINUED FRACTIONS

Let a_n be a sequence of positive integers and,

$$\alpha = \lim_{n\to\infty} \left[\cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots \cfrac{1}{a_n}}}} \right]$$

The limit α exists and is a positive irrational <1. If there exists a positive integer N such that $a_n \le N$ then α is of *constant type*.

Definition 1.1. Define sequences of integers p_n and q_n:

$$p_0 = 0,\ p_1 = 1 \quad \text{and} \quad p_n = a_n p_{n-1} + p_{n-2};$$
$$q_0 = 1,\ q_1 = a_1 \quad \text{and} \quad q_n = a_n q_{n-1} + q_{n-2}.$$

Let

$$r_0 = 1,\ r_1 = \alpha, \quad \text{and} \quad r_{n+1} = r_{n-1} - a_n r_n.$$

The fractions p_n/q_n are *rational convergents* of α and $\lim_{n\to\infty} p_n/q_n = \alpha$. Any rational p/q satisfying $(p, q) = 1$ and $|\alpha - p/q| < 1/q^2$ is called a *rational approximate* of α. Every rational convergent of α is a rational approximate of α. (see Khintchine [15], for example)

LEMMA 1.2. *For* $n \ge 1$,

(i) $q_n r_n + q_{n-1} r_{n+1} = 1$ *and*

(ii) $p_n r_n + p_{n-1} r_{n+1} = \alpha$.

Proof. For $n=1$ both statements follow from (1.1). Assume (i) is true for fixed $n \ge 1$. Then

$$\begin{aligned} q_{n+1} r_{n+1} + q_n r_{n+2} &= (a_{n+1} q_n + q_{n-1}) r_{n+1} + q_n r_{n+2} \\ &= (a_{n+1} r_{n+1} + r_{n+2}) q_n + q_{n-1} r_{n+1} \\ &= r_n q_n + q_{n-1} r_{n+1} \quad \text{by} \quad (1.1) \end{aligned}$$

Now assume (ii) is true for $n \ge 1$. Then

$$\begin{aligned} p_{n+1} r_{n+1} + p_n r_{n+2} &= (a_{n+1} p_n + p_{n-1}) r_{n+1} + p_n (r_n - a_{n+1} r_{n+1}) \\ &= p_n r_n + p_{n-1} r_{n+1} = \alpha. \end{aligned}$$

Q.E.D.

Let $S^1 = \mathbf{R}^1/\mathbf{Z}$. If $x \in \mathbf{R}^1$ let $\langle x \rangle = x \pmod 1) \in S^1$. For $0 < \alpha < 1$ and $x \in S^1$, let $R_\alpha(x) = \langle x + \alpha \rangle$ be the rotation of the circle through α. If $A \subset S^1$ let $A + \alpha = R_\alpha(A)$. If $n \in Z$, we call $\langle n\alpha \rangle$ the *orbit point* of R_α with *index* n.

Give S^1 the orientation inherited from $[0, 1)$. Any two points $x \ne y \in S^1$ bound a unique oriented arc (x, y). If its length is $\le 1/2$ define $x < y$.

LEMMA 1.3. *For* $n \ge 0$, $p_n - q_n \alpha = (-1)^{n+1} r_{n+1}$. *Hence* $\langle q_n \alpha \rangle$ *alternates on either side of* $\langle 0\alpha \rangle$.

Proof. If $n=0$ this follows from (1.1). Assume $p_{n-1} - q_{n-1}\alpha = (-1)^n r_n$. Multiplying by $-r_{n+1}/r_n$ gives

$$(\alpha q_{n-1} r_{n+1}/r_n) - (p_{n-1} r_{n+1}/r_n) = (-1)^{n+1} r_{n+1}.$$

On the other hand, the L.H.S. is precisely $p_n - q_n\alpha$ by (1.2). Q.E.D.

LEMMA 1.4. *If* $a_n(\alpha)=N$ *for* $n\geq 1$ *then* $r_k=\alpha^k$ *for* $k\geq 0$.

Proof. Definition (1.1) implies the result for $k=0$ or 1. Assume $r_k=\alpha^k$ and $r_{k-1}=\alpha^{k-1}$. Then by (1.1) $r_{k+1}=\alpha^{k-1}-N\alpha^k=\alpha^k(\alpha^{-1}-N)$. But $\alpha^{-1}=N+\alpha$. Thus $r_{k+1}=\alpha^{k+1}$.

Q.E.D.

Let $n\in z^+$. We define $\mathbf{W}'_n$ to be the collection of intervals I in

$$S^1\backslash\{\langle 0\alpha\rangle, \langle 1\alpha\rangle, \ldots, \langle (q_n+q_{n-1}-1)\alpha\rangle\}.$$

I is a *W'_n-interval* and $\langle j\alpha\rangle$, $0\leq j<q_n+q_{n-1}$, is a *W'_n-point*. Let $\mathbf{I}'_0(\mathbf{n})$ be the W'_n-interval with endpoints $\langle 0\alpha\rangle$ and $\langle q_{n-1}\alpha\rangle$ and $\mathbf{J}'_0(\mathbf{n})$ the interval bounded by $\langle 0\alpha\rangle$ and $\langle q_n\alpha\rangle$. By (1.3), $I'_0(n)$ and $J'_0(n)$ are on opposite sides of $\langle 0\alpha\rangle$ in S^1.

LEMMA 1.5. *The intervals of W'_n consist of the first q_n iterates of $I'_0(n)$ and the first q_{n-1} iterates of $J'_0(n)$ under rotation by R_α. In particular, all W'_n-intervals have length r_n or r_{n+1}.*

Proof. Without loss of generality, assume $I'_0=(\langle 0\alpha\rangle, \langle q_{n-1}\alpha\rangle)$ and $J'_0=(\langle q_n\alpha\rangle, \langle 0\alpha\rangle)$. Denote by V the collection of q_n-iterates of I'_0:

$$(\langle 0\alpha\rangle, \langle q_{n-1}\alpha\rangle), (\langle \alpha\rangle, \langle (q_{n-1}+1)\alpha\rangle), \ldots, (\langle (q_n-1)\alpha\rangle, \langle (q_n+q_{n-1}-1)\alpha\rangle)$$

together with the q_{n-1}-iterates of J'_0;

$$(\langle q_n\alpha\rangle, \langle 0\alpha\rangle), (\langle (q_n+1)\alpha\rangle, \langle \alpha\rangle), \ldots, (\langle (q_n+q_{n-1}-1)\alpha\rangle, \langle (q_{n-1}-1)\alpha\rangle).$$

Then V consists precisely of the W'_n-intervals: The collection of endpoints of V-intervals and W'_n-intervals is the same. By (1.2) the total length of V-intervals is one. If two V-intervals had intersection then there would exist some interval of the circle not covered by V with its left endpoint, say, a W'_n-point. But each W'_n-point appears as a left endpoint of one of the intervals of V.

Q.E.D.

We now remove the orbit points $\langle n\alpha\rangle$ for integers $n<0$ as well.

1.6. Define $t_n=[(q_n+q_{n-1})/2]$. Then

$$(q_n+q_{n-1}-1)/2\leq t_n\leq (q_n+q_{n-1})/2.$$

LEMMA 1.7.

(i) $1/2<q_nr_n<1$ *for all* $n\geq 1$;
(ii) *If* $a_n(\alpha)\leq N$ *for all* $n\geq 1$ *then* $t_n<q_n<2\,(N^2+1)t_{n-2}$.

Proof. (i) By (1.2), $q_nr_n<1$. By (1.1) $r_{n-1}>2r_{n+1}$. Then

$$r_nq_n=1-q_{n-1}r_{n+1}>1-r_{n+1}/r_{n-1}>\tfrac{1}{2}.$$

(ii) By (1.1) and (1.6)

$$t_n<q_n\leq (N^2+1)q_{n-2}+Nq_{n-3}<(N^2+1)(q_{n-2}+q_{n-3}-1)\leq 2(N^2+1)t_{n-2}.$$

Q.E.D.

An interval I of S^1 is in the collection $\mathbf{W}_n$ if it is the image of a W'_n-interval under rotation by $-t_n\alpha$; I is called a *W_n-interval* and its endpoints are *W_n-points*.

Let $I_0(n)=R_{-t_n\alpha}(I_0'(n))$ and $J_0(n)=R_{-t_n\alpha}(J_0'(n))$.

LEMMA 1.8. *W_n is isometric to W_n' and consists of the first q_n iterates of $I_0(n)$ and the first q_{n-1} iterates of $J_0(n)$ under rotation by R_α. The intervals in the former set have length r_n and those in the latter have length r_{n+1}. Each $J_0(n)$-iterate in W_n is also an $I_0(n+1)$-iterate in W_{n+1}.*

Proof. Let I be a $J_0(n)$-iterate in W_n. Assume that

$$J_0(n)=(\langle(q_n-t_n)\alpha\rangle, \langle -t_n\alpha\rangle) \text{ and } I_0(n+1)=(\langle(q_n-t_{n+1})\alpha\rangle, \langle -t_{n+1}\alpha\rangle).$$

Then the index p of the right endpoint of I satisfies $-t_n\leq p<-t_n+q_{n-1}$. By (1.6) $-t_{n+1}<p<-t_{n+1}+q_{n+1}$; these bounds are also the bounds for the indices of the right endpoints of the $I_0(n+1)$-iterates.

Q.E.D.

Remarks 1.9. The W_n-intervals I are open. Sometimes we need I to be half closed or closed. Even so, we refer to I as a W_n-interval and specify its type.

The largest positive index t_n' of a W_n-point will equal t_n only if q_n+q_{n-1} is odd. This will be the case when $a_n(\alpha)$ is even. We assume in our proofs that $t_n'=t_n=(q_n+q_{n-1}-1)/2$ for our notation to be tractable. Otherwise t_n' differs from t_n by 1. (But all theorems are valid as stated.)

§2. NUMBER THEORETIC ESTIMATES

In this section we discuss the nature of a weighted distribution of the orbit points $\langle n\alpha\rangle$ in S^1.

$$\left|\sum_{i=s}^{t-1}\left(\chi_U\langle i\alpha\rangle-|U|\right)\Big/ i^\gamma\right|$$

The estimates will depend on the degree of "irrationality" of α, the given "weight" i^γ and on the type of interval U over which we are testing the distribution.

The classical Denjoy–Koksma theorem (2.1) gives an upper bound of two for this series with $\gamma=0$, $s=0$ and $t=q$, a denominator of a rational approximate of α. If $U=[x, x+\langle q\alpha\rangle)$ and $\gamma=0$ then, according to Kesten, two is again an upper bound for arbitrary integers s and t.

THEOREM 2.1. *Let $\alpha\in\mathbf{R}\backslash\mathbf{Q}$. Let p/q be a rational approximate of α. Let f be a homeomorphism of the circle with rotation number $\rho(f)=\alpha$. Let θ: $S^1\to\mathbf{R}^1$ be a function with bounded variation and μ an invariant probability measure of f. Then, for every $x\in S^1$,*

$$\left|\sum_{i=0}^{q-1}\left(\theta\circ f^i(x)-\int_{S^1}\theta\,\mathrm{d}\mu\right)\right|\leq \mathrm{Var}(\theta).$$

Proof. See [17].

THEOREM 2.2.

(i) (Kesten [14]) *Let $\alpha\in\mathbf{R}\backslash\mathbf{Q}$ and p/q a rational approximate of α. If $I=[x, \langle x+q\alpha\rangle)\subset S^1$, for $x\in S^1$, then for every $s<t\in Z$*

$$\left|\sum_{i=s}^{t-1}\left(\chi_I\langle i\alpha\rangle-|I|\right)\right|<2.$$

(ii) (Hecke [12] and Ostrowski [19]) *Let α be of constant type. There exists a constant $C>0$ such that if U is an interval of S^1 and $s<t\in\mathbf{Z}$ then*

$$\left|\sum_{i=s}^{t-1}\left(\chi_U\langle i\alpha\rangle-|U|\right)\right|<C\log(t-s).$$

Proof of (i). Let $I_0=[0, \langle q\alpha\rangle)$. Note that for any real numbers a and b,

$$* \qquad \langle b-a\rangle=\chi_{[0,\langle a\rangle)}\langle b\rangle+\langle b\rangle-\langle a\rangle$$

Then

$$\begin{aligned}\left|\sum_{i=s}^{t-1}\chi_I\langle i\alpha\rangle-|I|\right| &= \left|\sum_{i=0}^{t-s-1}\chi_{I_0}\langle (i+s)\alpha-x\rangle-|I_0|\right|\\ &= \left|\sum_{i=0}^{t-s-1}\langle (i+s-q)\alpha-x\rangle-\langle (i+s)\alpha-x\rangle\right|\\ &= \left|\sum_{i=0}^{q-1}\langle (i+s-q)\alpha-x\rangle-\langle (i-q+t)\alpha-x\rangle\right|\\ &= \left|\sum_{i=0}^{q-1}\chi_{[0,(s-t)\alpha)}\langle (i+s-q)\alpha-x\rangle-\langle (s-t)\alpha\rangle\right|<2 \qquad \text{(by (2.1))}\end{aligned}$$

Proof of (ii). See [10, 15, 17].

Q.E.D.

The next proposition involves "summation by parts".

PROPOSITION 2.3. *Let $C>0$ and f_n a monotone increasing sequence of positive real numbers. Let b_i be a sequence of real numbers satisfying*

$$\left|\sum_{i=j}^{j+N-1} b_i\right|<Cf_N \qquad \textit{for all } j,\ N\geq 0.$$

Let $d_j>0$ be a monotone decreasing sequence and $0\leq k\leq m$. Then

$$\text{(i)} \qquad \left|\sum_{i=k}^{m-1} b_i d_i\right|\leq Cf_{m-k}d_k.$$

(ii) *Define n and r by $m-1=2^n+l$, $0\leq l<2^n$ and $k=2^r+p$, $0\leq p<2^r$. Then*

$$\left|\sum_{i=k}^{m-1} b_i d_i\right|\leq C\left[\sum_{\lambda=r}^{n} f_{2^\lambda}d_{2^\lambda}\right].$$

Proof. By Abel's partial summation formula,

$$\begin{aligned}\left|\sum_{i=s}^{t-1} b_i d_i\right| &= |(b_s+\cdots+b_{t-1})d_{t-1}+b_s(d_s-d_{s+1})\\ &\quad +(b_s+b_{s+1})(d_{s+1}-d_{s+2})+\cdots+(b_s+\cdots+b_{t-2})(d_{t-2}-d_{t-1})|\\ &\leq |b_s+\cdots+b_{t-1}|d_{t-1}+|b_s|(d_s-d_{s+1})\\ &\quad +|b_s+b_{s+1}|(d_{s+1}-d_{s+2})+\cdots+|b_s+\cdots+b_{t-2}|(d_{t-2}-d_{t-1})\\ &\leq Cf_{t-s}d_{t-1}+Cf_{t-s}(d_s-d_{s+1})+\cdots+Cf_{t-s}(d_{t-2}-d_{t-1})\\ &= Cf_{t-s}d_s.\end{aligned}$$

Thus

$$\left|\sum_{i=k}^{m-1} b_i d_i\right| = \sum_{\lambda=r+1}^{n-1}\left|\sum_{i=2^\lambda}^{2^{\lambda+1}-1} b_i d_i\right| + \left|\sum_{i=2^n}^{m-1} b_i d_i\right| + \left|\sum_{i=k}^{2^{r+1}-1} b_i d_i\right|$$

$$\leq C\left[\sum_{\lambda=r}^{n} f_{2^\lambda} d_{2^\lambda}\right]. \qquad \text{Q.E.D.}$$

We conclude with the estimates underlying Lemmas 3.2 and 3.7.

THEOREM 2.4.

(i) Let $\alpha \in \mathbf{R}\backslash\mathbf{Q}$. *If* $I = [x, \langle x + q_n\alpha\rangle)$, $g: \mathbf{Z} \to \mathbf{Z}^+$ *is monotone decreasing and* $0 < s < t$ *then*

$$\left|\sum_{i=s}^{t-1}\left(\chi_I\langle i\alpha\rangle - |I|\right)g(i)\right| < 2g(s).$$

(ii) *Let* $\alpha \in \mathbf{R}\backslash\mathbf{Q}$ *be of constant type. There exists a constant* $C > 0$ *such that if* $\frac{1}{2} < \gamma < 1$, U *is an interval of* S^1 *and* $s < t$, *then*

$$\left|\sum_{i=s}^{t-1}\left(\chi_U\langle i\alpha\rangle - |U|\right)\Big/ i^\gamma\right| < C \log s/s^\gamma.$$

Proof of (i). The result follows immediately from (2.2(i)) and (2.3(i)).

Proof of (ii). Let C_1 be the constant depending on α obtained from (2.2(ii)). Apply (2.3(ii)) to $b_i = \chi_U\langle i\alpha\rangle - |U|, d_i = 1/i^\gamma$ and $f_i = \log(i)$. We have

$$\left|\sum_{i=s}^{t-1} b_i d_i\right| < C_1\left[\sum_{\lambda=r}^{n} \log(2^\lambda)/2^{\lambda\gamma}\right]$$

where $s = 2^r + p, 0 \leq p < 2^r$. The series is bounded by a geometric series since the ratio of successive terms is $(\lambda+1)/\lambda 2^{\lambda\gamma}$ which is less than 1 for $\lambda > 1$. Thus the series is bounded by $(\log(2^r)/2^{r\gamma}) \times (\Sigma(\lambda+1)/\lambda 2^{\lambda\gamma})$. The latter series is bounded independently of $\frac{1}{2} < \gamma < 1$. The result follows.

Q.E.D.

§3. THE EMBEDDING

Let $\alpha \in \mathbf{R}\backslash\mathbf{Q}$ and $D_\alpha: S^1 \to S^1$ a Denjoy counterexample with rotation number α. Let $\rho: S^1 \to S^1$ be a monotonic, continuous mapping semi-conjugating D_α to the rotation R_α. Assume that $\rho^{-1}\langle n\alpha\rangle$ is an interval with interior denoted by $\Delta'_n = (y'_n, z'_n)$ and that the left endpoint of Δ'_0 is 0. Define $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$ for $(x, y) \in S^1 \times R^1$.

We construct a mapping $h: S^1 \to S^1 \times \mathbf{R}^1$ as a limit of mappings $h_n: S^1 \to S^1 \times \mathbf{R}^1$. The image of h resembles the Denjoy circle except that each Denjoy interval Δ'_n is replaced by a diagonal Δ_n with slope $(-1)^n$. The total length of the Δ_n is unbounded.

More precisely, let $\frac{1}{2} < \gamma < 1$ and $a > 0$ be real numbers; define $g(i) = a/|i|^\gamma$ for integers $i \neq 0$. (The constants γ and a will be specified later.) Recall the sequence t_n, (see 1.6), and define

$$g(0) = \sum_{|i|=1}^{t_n} (-1)^{i+1} g(i).$$

The lengths of the diagonals will be $\sqrt{2}\cdot g(i)$. The choice of $g(0)$ will insure $\pi_2 h_n\langle 0\alpha\rangle$ is well defined. Note that $g(0)$ is positive and depends on n.

Define
$$m_n = 1 + \sum_{|i|=0}^{t_n} g(i).$$

This constant will insure that $\pi_1 h_n\langle 0\alpha\rangle$ is well-defined.

Use the isomorphism $S^1 \cong \mathbf{R}^1 \backslash \mathbf{Z}$ to carry the Euclidean metric $d(\cdot,\cdot)$ and norm $\|\cdot\|$, locally, to $S^1 \times \mathbf{R}^1$. Then the meaning of line and slope in $\mathbf{R}^2$ naturally carry over to $S^1 \times \mathbf{R}^1$. Call a line segment in $S^1 \times \mathbf{R}^1$ *horizontal* if its slope is 0.

Now fix $n \in \mathbf{Z}^+$. Order the endpoints of the intervals Δ_i', $0 \le |i| \le t_n$, beginning with 0, from left to right:
$$0 = p_0 < q_0 < p_1 < q_1 < \cdots < p_m = 1 = 0.$$
Let
$$\langle i_k \alpha\rangle = \rho(p_k) = \rho(q_k)$$
for
$$0 \le i < m \text{ and } \langle i_m \alpha\rangle = 1 = \rho(p_m).$$

Define a piecewise linear function $h_n : S^1 \to S^1 \times \mathbf{R}^1$ inductively: Define $h_n(p_0) = (0,0)$ and $h_n[p_0, q_0]$ to be the line segment with slope $+1$, length $\sqrt{2}\cdot g(0)$ and decreasing first coordinate. Suppose that $h_n[p_0, q_{k-1}]$ has been defined for $1 \le k \le m$. Suppose that its endpoints are $(0,0)$ and (x, y). Map $h_n[q_{k-1}, p_k]$ linearly to the horizontal line segment with one endpoint (x, y), increasing first component and length $m_n|\langle i_k\alpha\rangle - \langle i_{k-1}\alpha\rangle|$. Suppose that the image of $[p_0, p_k]$ under h_n has been defined for $1 \le k \le m$. If $k = m$ then h_n has been defined on all of S^1. Otherwise, suppose that its endpoints are $(0,0)$ and (x, y). Map $h_n[p_k, q_k]$ linearly to the diagonal line segment Δ with length $\sqrt{2}\cdot g(i_k)$, slope $(-1)^{i_k}$ so that $h_n(p_k) = (x, y)$ and $h_n[p_k, q_k]$ has decreasing first coordinate. This uniquely specifies a backward sloping diagonal Δ.

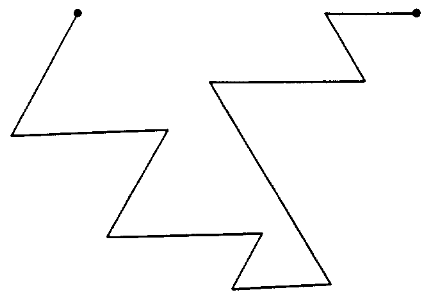
From this description we derive an explicit formula for h_n: Fix $x \in S^1$ and let $W = [0, \rho(x))$. The diagonal associated to $\langle j\alpha\rangle \in W$, $|j| \le t_n$, contributes $-g(j)$ to the first component $\pi_1 h_n(x)$, no matter what its slope. It contributes $(-1)^{j+1} g(j)$ to the second coordinate $\pi_2 h_n(x)$. Each W_n-interval I, $I \cap W \ne \varnothing$, contributes its normalized length $m_n|I \cap W|$ to $\pi_1 h_n(x)$. The total of the normalized lengths is $m_n|W|$. Finally, if x lies in $cl(\Delta_p')$ for $|p| \leqslant t_n$, its relative position is preserved on the diagonal $h_n(cl(\Delta_p'))$ and a correction term $\pm d_x$ is added to each of the coordinates. For such x define
$$d_x = (-1)^p g(p)|x - y_p'|/|z_p' - y_p'|.$$

Otherwise, let $d_x = 0$. Then
$$\pi_1 h_n(x) = m_n|W| - \sum_{|i|=0}^{t_n} \chi_W \langle i\alpha\rangle g(i) - |d_x|. \tag{3.1}$$
$$\pi_2 h_n(x) = \sum_{|2i+1| \le t_n} \chi_W \langle (2i+1)\alpha\rangle g(2i+1) - \sum_{|2i| \le t_n} \chi_W \langle 2i\alpha\rangle g(2i) - d_x.$$

Let $x = 1$, the second copy of 0. Then $W = S^1$ and $d_x = 0$. Hence $h_n(1) = (1,0) = (0,0) = h_n(0)$ and h_n is well-defined. See Fig. 2.

Define $h = \lim h_n$ as $n \to \infty$.

It is not at all clear *a priori* that h exists or is continuous, much less that it is an embedding for any choice of α, γ and a. We first prove that if $\alpha \in \mathbf{R}\backslash\mathbf{Q}$ is of constant type, $a > 0$ and $\frac{1}{2} < \gamma < 1$, then h exists and is continuous. (It can be shown that, indeed, h exists and is continuous for any α satisfying a Diophantine condition [10].) Choosing $a_n(\alpha) = 2$ and placing further restrictions on γ and a enables us to prove that h is an embedding.



h_n (S^1)

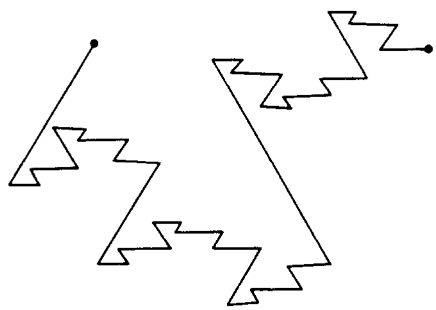


Fig. 2.

Estimates on horizontal and vertical components of the mapping h

Let U be an interval in S^1 with endpoints $p<q$.

Define $\mathbf{H}_n(\mathrm{U})=\pi_1 h_n(q)-\pi_1 h_n(p)$ and $\mathbf{V}_n(\mathrm{U})=\pi_2 h_n(q)-\pi_2 h_n(p)$. If H_n and V_n converge uniformly over intervals of S^1 then h_n converges uniformly to a continuous function h.

LEMMA 3.2. *If* $\alpha\in\mathbf{R}\backslash\mathbf{Q}$ *is of constant type,* $\frac{1}{2}<\gamma<1$ *and* $a>0$ *then* $H_n=H_n(\alpha,\gamma,a)$ *and* $V_n=V_n(\alpha,\gamma,a)$ *converge uniformly over intervals of* S^1.

Proof. Let C be the constant depending on α from (2.4) (ii). Recall $\Delta_i'=(y_i',z_i')=\rho^{-1}\langle i\alpha\rangle$.

We verify the uniform convergence of H_n and V_n over two types of intervals U with endpoints $p<q$.

(1) $p\notin\cup[y_i',z_i')$ and $q\notin\cup(y_i',z_i']$;
(2) $U\subset[y_i',z_i']$ for some $i\in Z$.

Uniform convergence over all intervals of S^1 follows.

Given $\delta>0$ choose N such that if $n\geq N$ then

$$0<m_n r_n+2C\log(t_n)g(t_n)+2g(t_n)<\delta.$$

(Recall r_n (1.1); estimate m_n by an integral and apply (1.7) to see that

$$m_n r_n<(1+4at_n^{1-\gamma}/(1-\gamma))r_n<r_n+4ar_n^{\gamma}/(1-\gamma)\to 0 \text{ as } n\to\infty.$$

By the definition of g, $\log(t_n)g(t_n)\to 0$, so it is possible to find such an N.)

Proof of the uniform convergence of H_n. Let U be an interval of type (1) with endpoints $p<q$. Then in (3.1) $d_q=0$. Letting $W=\text{int}(\rho(U))$ and $N<m<n$ it follows that

$$H_n(U)=\left(1+\sum_{|i|=0}^{t_n} g(i)\right)|W|-\sum_{|i|=j}^{t_n} \chi_W\langle i\alpha\rangle)g(i). \tag{3.3}$$

where $j=\min\{|i|: \langle i\alpha\rangle\in\text{int}(W)\}$. (Whenever $j>t_n$, the last sum is zero.)

Suppose $j\le t_m<t_n$. Apply (2.4) (ii). Then

$$|H_n(U)-H_m(U)|=\left|\sum_{|i|=t_m+1}^{t_n}\left(|W|-\chi_{W'}\langle i\alpha\rangle\right)g(i)\right|$$
$$<C\log(t_m)g(t_m)<\delta$$

Suppose $t_m<t_n<j$. Then W is disjoint from the W_n-points. Therefore W is contained in a W_n-interval and $|W|\le r_n$ by (1.8). Since the last sum in (3.3) is 0 for both $H_n(U)$ and $H_m(U)$ we have

$$|H_n(U)-H_m(U)|=|m_n-m_m||W|<m_nr_n<\delta.$$

If $t_m<j\le t_n$ let l satisfy $t_m\le t_l<j\le t_{l+1}\le t_n$. Then W is disjoint from the W_l-points and $|W|\le r_l$ as in the preceding paragraph. The last sum in (3.3) for $H_m(U)$ is zero so

$$|H_n(U)-H_m(U)|=\left||W|\sum_{|i|=t_m+1}^{t_n} g(i)-\sum_{|i|=t_l+1}^{t_n}\chi_W\langle i\alpha\rangle g(i)\right|$$
$$=\left||W|\left|\sum_{|i|=t_m+1}^{t_l} g(i)+\sum_{|i|=t_l+1}^{t_n}\left(|W|-\chi_W\langle i\alpha\rangle\right)g(i)\right|\right.$$
$$<m_lr_l+C\log(t_l)g(t_l)\quad\text{(by (2.4)(ii))}$$
$$<\delta.$$

Thus H_n converges uniformly over the intervals of type (1).

Now suppose $U\subset\Delta_i'$. If $|i|\le t_n$ then $|H_n(U)|\le g(i)$. If $|i|>t_n$, then $\rho(U)=\langle i\alpha\rangle\subset I_n$, an open W_n-interval. Hence

$$|H_n(U)|=m_n|I_n||U|/|\rho^{-1}(I_n)|\le m_n|I_n|\le m_nr_n.$$

Therefore, if $t_m<t_n<|i|$,

$$|H_n(U)-H_m(U)|<\max\{|H_n(U)|,|H_m(U)|\}$$
$$\le\max\{m_nr_n, m_mr_m\}$$
$$<\delta.$$

If $|i|\le t_m<t_n$ then $|H_n(U)-H_m(U)|=0$. Finally, if $t_m<|i|\le t_n$

$$|H_n(U)-H_m(U)|<\max\{|H_n(U)|, |H_m(U)|\}$$
$$\le\max\{g(i), m_mr_m\}$$
$$\le\max\{g(t_m), m_mr_m\}$$
$$<\delta.$$

Proof of uniform convergence of V_n. Let $U\subset S^1$ be an interval of type (1) and $W=\text{int}(\rho(U))$, as before. Then

$$V_n(U)=\sum_{|2i+1|\le t_n}\chi_W\langle(2i+1)\alpha\rangle g(2i+1)-\sum_{|2i|\le t_n}\chi_W\langle 2i\alpha\rangle g(2i) \tag{3.4}$$

Let $n > m > N$. Then

$$|V_n(U) - V_m(U)| = \left| \sum_{t_m < |2i+1| \le t_n} \chi_W \langle (2i+1)\alpha \rangle g(2i+1) - \sum_{t_m < |2i| \le t_n} \chi_W \langle 2i\alpha \rangle g(2i) \right|$$

$$= \left| \sum_{t_m < |2i+1| \le t_n} \Big(\chi_W \langle (2i+1)\alpha \rangle - |W| \Big) g(2i+1) + \sum_{t_m < |2i| \le t_n} \Big(|W| - \chi_W \langle 2i\alpha \rangle \Big) g(2i) \right.$$

$$\left. + |W| \sum_{t_m < |2i|, |2i+1| \le t_n} \Big(g(2i+1) - g(2i) \Big) \right|$$

Apply the triangle inequality and (2.4 (ii)) to estimate the first two sums. (The third sum is bounded by $2g(t_m)$ since $|W| < 1$.) Thus

$$|V_n(U) - V_m(U)| < 2C \log(t_m) g(t_m) + 2g(t_m) < \delta.$$

Suppose $U \subset \Delta_i'$. Then $V_j(U)$ is zero for $t_j < |i|$. If $|i| \le t_j$ then $V_j(U)$ is a constant depending only on U. It is bounded by $g(i)$.

Therefore, if $|i| \le t_m < t_n$ then $|V_n(U) - V_m(U)| = 0$. If $t_m < |i| \le t_n$, then $|V_n(U) - V_m(U)| = |V_n(U)| < g(i) < g(t_m) < \delta$.

Hence V_n converges uniformly.

Q.E.D.

COROLLARY 3.5. *If* $\alpha \in \mathbf{R} \setminus \mathbf{Q}$ *is of constant type,* $\gamma > 0$, *and* $a > 0$ *then* $h = h(\alpha, \gamma, a)$ *exists and is continuous.*

Q.E.D.

It remains to find constants α, γ and a such that $h = h(\alpha, \gamma, a)$ is an embedding. The next lemma gives us some restrictions. It provides the main estimate for the proof of the embedding.

If U is an interval of S^1 define

$$H(U) = \lim H_n(U) \text{ and } V(U) = \lim V_n(U) \text{ as } n \to \infty.$$

Definition 3.6. Define $\tau: \mathbf{Z} \to \mathbf{Z}^+$ by $\tau(n) = k$ where $t_{k-1} < |n| \le t_k$. Then $\langle j\alpha \rangle$ is an endpoint of a W_k-interval I iff $\tau(j) \le k$. If $|I| = r_k$ then at least one endpoint $\langle j\alpha \rangle$ has $k - 1 \le \tau(j) \le k$. Otherwise both are endpoints of W_{k-2}-intervals implying $|I| \ge r_{k-1}$.

LEMMA 3.7. *Suppose* $\alpha = \sqrt{2} - 1$. *There exist constants* C_1, C_2, C_3 *such that if* $0 < (1 - \gamma) < 1/1000$, $a = (1 - \gamma)/4$ *and* I *is a half closed* W_k*-interval with* $\tau(n) \ge k - 1$ *for any* $\langle n\alpha \rangle \in I$ *then*

(i) $C_1 |I|^\gamma < H(U) < C_2 |I|^\gamma$,

(ii) $|V(U)| < C_3 (1 - \gamma) |I|^\gamma$

where $U = \rho^{-1}(I)$.

For the embedding, we need $|V(U)| < H(U)$. This follows if $1 - \gamma$ is sufficiently small. However, the smaller $1 - \gamma$, the closer the differentiability of the eventual Denjoy example is to two. There is a sharp, but more difficult, version of this lemma which leads to an embedded curve for any $0 < 1 - \gamma < \frac{1}{2}$. One then obtains $C^{3-\varepsilon}$ Denjoy examples for any $\varepsilon > 0$. Here, we sacrifice sharpness for simplicity, time and again.

In what follows, keep in mind that the actual value of the constants C_1, C_2, C_3 is not important. That they exist is a consequence, in part, of (1.1) and (1.7): q_k, t_k, q_{k-1}, $1/r_k$ are all proportional. (We use the fact that $a_n(\alpha)=2$ since $\alpha=\sqrt{2}-1$.) The constant a is only used to keep large diagonals disjoint, γ controls the rest of the diagonals.

Proof. Let $\langle j\alpha\rangle$ be the included endpoint. Since $\tau(n)\geq k-1$ for $\langle n\alpha\rangle\in I$, $k-1\leq\tau(j)\leq k$. Assume $j>0$. Then $t_{k-2}<j\leq t_k$. By (1.7)

$$q_k/10<j<q_k.$$

Proof of (i). Since j is the minimum absolute index of orbit points in I, by (3.1), (cf. (3.3)), we have

$$\left|H_n(U)-\left(|I|+|I|\sum_{|i|=0}^{j-1}g(i)\right)\right|=\left|\sum_{|i|=j}^{t_n}\left(|I|-\chi_I\langle i\alpha\rangle\right)g(i)\right|$$

It follows from the uniform convergence of H_n (3.2) and (2.4(i)) that

$$\left|H(U)-\left(|I|+|I|\sum_{|i|=0}^{j-1}g(i)\right)\right|<4g(j).$$

Therefore

$$\begin{aligned} H(U)&>2a|I|\int_1^{j-1}x^{-\gamma}dx+|I|-4g(j)\\ &=2a|I|(j-1)^{1-\gamma}/(1-\gamma)-4aj^{-\gamma}+(|I|-2a|I|/(1-\gamma))\\ &>\tfrac{1}{2}j^{-\gamma}(|I|j-2(1-\gamma)) &&\text{since } a=(1-\gamma)/4\\ &>\tfrac{1}{2}|I|^{\gamma}(1/20-2(1-\gamma)) &&\text{by the bounds on } j \text{ and (1.7)}\\ &>|I|^{\gamma}/44 &&\text{by the bounds on } \gamma. \end{aligned}$$

Let $C_1=1/44$. On the other hand,

$$\begin{aligned} H(U)&<2a|I|\int_0^{j}x^{-\gamma}dx+|I|g(0)+|I|+4g(j)\\ &=2a|I|j^{1-\gamma}/(1-\gamma)+|I|g(0)+|I|+4g(j)\\ &<2a/q_k^{\gamma}(1-\gamma)+2a/q_k+1/q_k+40a/q_k^{\gamma} \text{ by the bounds on } j,\\ &\qquad\qquad\text{since } q_k|I|<1, \text{ and } g(0)<2g(1)=2a\\ &<2/q_k^{\gamma} &&\text{since } a=(1-\gamma)/4\\ &<4|I|^{\gamma} &&\text{since } q_k|I|>1/2. \end{aligned}$$

Let $C_2=4$.

Proof of (ii). By (3.1) (cf. (3.4))

$$\begin{aligned} |V(U)|&=\left|\sum_{j\leq|2i+1|}\chi_I\langle(2i+1)\alpha\rangle g(2i+1)-\sum_{j\leq|2i|}\chi_I\langle 2i\alpha\rangle g(2i)\right|\\ &<\left|\sum_{j\leq|2i+1|}\left(\chi_I\langle(2i+1)\alpha\rangle-|I|\right)g(2i+1)\right|+\left|\sum_{j\leq|2i|}\left(|I|-\chi_I\langle 2i\alpha\rangle\right)g(2i)\right|+2|I|g(j). \end{aligned}$$

We would like to apply (2.4(i)) at this point. But the rotation number is 2α, not α, and (2.4 (i)) limits application to intervals with length $r_k(2\alpha)$, not $r_k(\alpha)$! (Indeed, Kesten proved that the unweighted sum of (2.2(i)) is unbounded for intervals with length $\neq r_k(2\alpha)$.) However, since $a_n(\alpha)=2$, it follows easily from (1.1), that $r_k(2\alpha)=r_k(\alpha)$ if k is even and $r_k(2\alpha)=2r_k(\alpha)$ if k is odd. Thus for even k, we may apply (2.4(i)) and obtain

$$|V(U)| \leq (8+2|I|)a/j^{\gamma} < 10a/j^{\gamma} \tag{3.8}$$

where j is the index of the included endpoint of I. Hence, for even k,

$$|V(U)| \leq 100a/q_k^{\gamma} < 200a|I|^{\gamma}.$$

Suppose $I = [\langle j\alpha\rangle, \langle (j-q_{k-1})\alpha\rangle)$ where k is odd. (The proof is similar for $I = (\langle j\alpha\rangle, \langle (j-q_{k-1})\alpha\rangle]$ and $\tau(j-q_{k-1}) \geq k-1$.) Then $|I| = r_k$. We decompose I into segments of length r_m with m even:

Let p be the midpoint of I and $I_0 = [\langle j\alpha\rangle, p)$. Let $J_0 = [\langle j\alpha\rangle, \langle (j+q_k)\alpha\rangle)$ and for $n \geq 1$, $J_n = [\langle (j+q_k+q_{k+2}+\dots+q_{k+2n-2})\alpha\rangle, \langle (j+q_k+q_{k+2}+\dots+q_{k+2n})\alpha\rangle)$. Then $|J_n| = r_{k+2n+1} = \alpha^{k+2n+1}$ by (1.4). Since $a_n(\alpha)=2$, $1/\alpha = 2+\alpha$. Therefore $\alpha^2+2\alpha-1=0$ and hence $\alpha^k/2 = \Sigma\, \alpha^{k+2n+1}$, $n \geq 0$. Thus $|I_0| = \Sigma |J_n|$. Since the J_n are consecutive, $I_0 = \cup J_n$.
Let $U_0 = \rho^{-1}(I_0)$. By (3.8)

$$\begin{aligned}
|V(U_0)| &\leq \left|\sum_{n=0}^{\infty} V(\rho^{-1}(J_n))\right| \\
&\leq 10a(1/j^{\gamma} + 1/(j+q_k)^{\gamma} + 1/(j+q_k+q_{k+2})^{\gamma} + \dots) \\
&< 120a/q_k^{\gamma} \text{ since } j > q_k/10 \text{ and } q_{n+2} > 5q_n \text{ for all } n \\
&< 240a|I|^{\gamma} \text{ since } |I| > 1/2q_k. \\
&= 60(1-\gamma)|I|^{\gamma}.
\end{aligned}$$

The estimate may be doubled for $|V(U)|$.
Let $C_3 = 120$. (This estimate is particularly coarse!)

Q.E.D.

Geometry of the embedding $h: S^1 \to S^1 \times \mathbf{R}^1$. Define $h: S^1 \to S^1 \times \mathbf{R}^1$ as in (3.1), so that the hypothesis of (3.7) are satisfied. It depends only on γ.

Let $Q = h(S^1)$. Since Q is compact, there exist real numbers $\varepsilon_1 < \varepsilon_2$ such that $Q \subset A = \{(x,t) \in S^1 \times \mathbf{R}^1 : \varepsilon_1 < t < \varepsilon_2\}$. Denote the diagonal $h(\Delta_n') \subset A$ by Δ_n. Let y_n and z_n be the images of the endpoints $y_n' < z_n'$ of Δ_n'.

For $\beta > 0$ and $x \in \mathbf{R}^2$ define $C_\beta(x) = \{w \in \mathbf{R}^2 : |\text{slope}(w-x)| \leq \beta\}$, the cone of slope $\pm\beta$ based at x. For $x, y \in \mathbf{R}^2$, define $T_\beta(x,y)$ to be the compact component of $C_\beta(x) \cap C_\beta(y)$. If $\|x-y\| < 1$ then $T_\beta(x,y)$ projects to a "parallelogram" in the annulus A, also denoted by T_β.

If $U = (\langle m\alpha\rangle, \langle n\alpha\rangle) \subset S^1$ define $T(U) = T_\beta(U) = T_\beta(z_m, y_n)$. See Fig. 3.

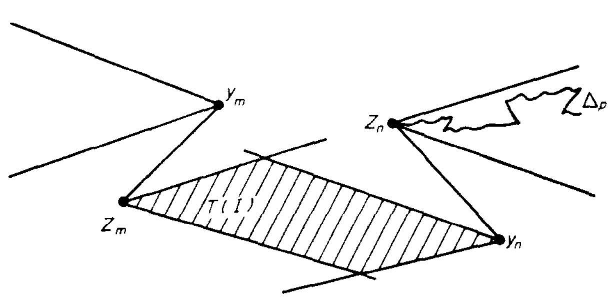


Fig. 3.

Form a rectangle $R_n \subset A$ with one edge containing y_n and slope $(-1)^n/2$. Let the endpoints p and q of this edge satisfy $\pi_1(p) - \pi_1(y_n) = \pi_1(z_n) - \pi_1(q) = 2g(n)$. The opposing parallel edge passes through z_n. This determines R_n. We prove the R_n are disjoint. They are permuted under a C^2 diffeomorphism f of the annulus and have a Cantor limit set. *The disks R_n are analogous to the complementary Denjoy intervals in the circle.*

Now identify S^1 with $S^1 \times \{0\}$ in $S^1 \times \mathbf{R}^1$. Let R'_n be the square centered at the midpoint of Δ'_n; let its sides have length that of Δ'_n. Then the R'_n are disjoint. Extend h continuously to $S^1 \cup \{R'_n\}$ so that $h|R'_n$ is a homeomorphism onto R_n. See Fig. 4. Let π be the normal projection of $S^1 \cup \{R'_n\}$ to S^1.

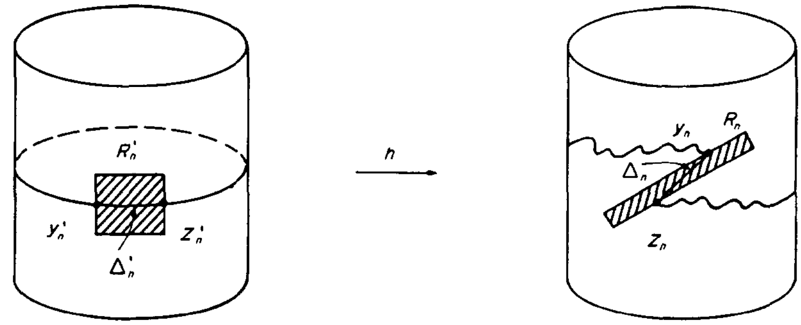


Fig. 4.

The next lemma describes the underlying geometric pattern used to show h is an embedding and Q is a quasi-circle.

Let $U \subset S^1$ and $n \in Z$. Then $\tau(n)$ is minimal over U if $\langle p\alpha \rangle \in U$ implies $\tau(n) \leq \tau(p)$. (See (3.6)). For example, if $U = I$ is a half-closed W_k-interval with endpoint $\langle m\alpha \rangle$ then $\tau(m)$ is minimal over $\mathrm{int}(I)$.

LEMMA 3.9. *Let $0 < \beta < 1$ and $(1-\gamma) < \beta C_1/2C_3$. If $U = (\langle m\alpha \rangle, \langle n\alpha \rangle) \subset S_1$, and $\tau(m)$ and $\tau(n)$ are minimal over U, then $R_p \subset T_\beta(U)$ for every $\langle p\alpha \rangle \in U$.*

Proof. Suppose $\tau(m)$ is minimal over U. Let $\langle p\alpha \rangle \in U$. The idea is to write $(\langle m\alpha \rangle, \langle p\alpha \rangle]$ as a union of right closed W_k-intervals and show the vertical displacement V of each W_k-interval is small compared to its horizontal displacement H (by (3.7)). By "zig-zagging" from z_m to z_p and taking into account the diameter of R_p, we conclude that R_p is to the "right" of z_m and $\sigma = |\mathrm{slope}(x - z_m)| < \beta$ for all $x \in R_p$. The analysis is similar for the interval $[\langle p\alpha \rangle, \langle n\alpha \rangle)$. (See Fig. 3).

More precisely, given $\beta > 0$ let $(1-\gamma) < \beta C_1/2C_3$.

Let $U' = (\langle m\alpha \rangle, \langle p\alpha \rangle] = (\langle m_0\alpha \rangle, \langle p\alpha \rangle]$. Choose $\langle m_1\alpha \rangle \in U'$ with $\tau(m_1)$ minimal over U'. Let $j = \tau(m_1)$. Then $t_{j-1} < m_1 \leq t_j$. By the minimality of $\tau(m_0)$ we have $|m_0|, |m_1| \leq t_j$. Then $U_1 = (\langle m_0\alpha \rangle, \langle m_1\alpha \rangle]$ is a union of right closed W_j-intervals. Let J be one of these and $\langle q\alpha \rangle$ its right endpoint. Then $|J| = r_j$ or r_{j+1} and $\tau(q) = j$. Therefore the estimates (i) and (ii) of (3.7) are valid for this decomposition of U_1.

Since $\tau(m_1)$ is minimal over $U' \backslash U_1$ and $\langle m_1\alpha \rangle$ is an endpoint of $U' \backslash U_1$, we may repeat the process: Choose a point $\langle m_2\alpha \rangle$ in $U' \backslash U_1$ with $\tau(m_2)$ minimal and define $U_2 = (\langle m_1\alpha \rangle, \langle m_2\alpha \rangle]$. Break U_2 into standard intervals over which we may apply (3.7) and compare V with H. Inductively choose $\langle m_k\alpha \rangle$ in $U' \backslash U_1 \cup U_2 \cup \ldots \cup U_{k-1}$ until eventually $\langle p\alpha \rangle$ is chosen. Then U' is a union of W_k-intervals J satisfying

$$|V(\rho^{-1}(J))| < C_3(1-\gamma)|J|^\gamma < \beta C_1|J|^\gamma/2 < \beta H(\rho^{-1}(J))/2.$$

Let J_p be the W_n-interval in this decomposition containing $\langle p\alpha \rangle$ and with $|J_p| = r_n$. Then $\tau(p) \geq n-1$ and $p > q_n/10$. It follows that $g(p) < \beta H(\rho^{-1}(J_p))/2$. The rectangle R_p contributes an extra term of $g(p)$. Hence $\sigma < \beta$. Q.E.D.

Recall the constants C_1, C_3 of (3.7).

COROLLARY 3.10. *If* $(1-\gamma)<C_1/6C_3$ *then* $h:S^1\cup\{R'_n\}\to S^1\times\mathbf{R}^1$ *is an embedding onto* $Q\cup\{R_n\}$.

Proof. Let $\beta=1/3$. Then $1-\gamma<\beta C_1/2C_3$ as required in (3.9).

Let $w\neq z\in S^1\cup\{R'_n\}$ and $x=\rho\pi(w)$, $y=\rho\pi(z)$.

If $x=y$ then w, $z\in R'_j$ for some $j\in\mathbf{Z}$. Since h embeds R'_j, $h(w)\neq h(z)$.

Suppose $x\neq y$. Let U be an interval of S^1 with endpoints x and y and $|U|\leq 1/2$. Choose $\langle p\alpha\rangle\in U$ with $\tau(p)=k$ minimal over U. Let $\langle m\alpha\rangle\leq x$ be the W_k-point closest to x and $y\leq\langle n\alpha\rangle$ be the W_k-point closest to y. Let $J_1=(\langle m\alpha\rangle,\langle p\alpha\rangle)$ and $J_2=(\langle p\alpha\rangle,\langle n\alpha\rangle)$ and $J=J_1\cup\langle p\alpha\rangle\cup J_2$. Then $\tau(p)$ is minimal over J. This implies $\tau(m)$ is minimal over J: otherwise there exists $\langle q\alpha\rangle\in J$ such that $\tau(m)>\tau(q)\geq\tau(p)=k$. But $\tau(m)\leq k$ since $\langle m\alpha\rangle$ is a W_k-point. Similarly $\tau(n)$ is minimal over J. By restriction, the endpoints of J_1, J_2 and J are minimal over their respective intervals.

It follows from (3.9) and the continuity of h that $h(w)\in T_{1/3}(J_1)\cup R_m$ and $h(z)\in T_{1/3}(J_2)\cup R_n$. It is only necessary to show these two sets are disjoint.

By (3.9) $T_{1/3}(J_1)\cup R_p\cup T_{1/3}(J_2)\subset T_{1/3}(J)$. The diagonal Δ_p connects $T_{1/3}(J_1)$ and $T_{1/3}(J_2)$ while Δ_m and Δ_n are attached to the vertices z_m of $T_{1/3}(J_1)$ and y_n of $T_{1/3}(J_2)$, respectively. See Fig. 5.

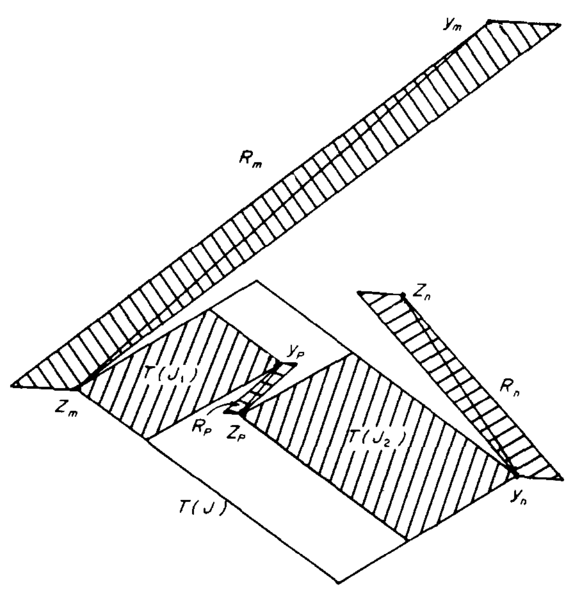


Fig. 5.

3.11. *Either* $\tau(m)$ *or* $\tau(n)\geq k-2$. If $\tau(m)$, $\tau(n)\leq k-3$ then J is a union of W_{k-3} intervals. It must contain at least two since $\langle p\alpha\rangle\in J$ is a W_k-point. Hence J contains an interval of length r_{k-3}. Any such interval is the union of 7 W_k-intervals.

J is also a union of W_k-intervals since its endpoints are W_k-points. If J contained a half closed W_{k-1} interval with its endpoint $\langle j\alpha\rangle$ then $\tau(j)\leq k-1$, contradicting the minimality of $\tau(p)$. Since $a_n(\alpha)=2$, any 5 consecutive W_k-intervals must contain a W_{k-1}-interval. Therefore J can contain no more than 4 half closed consecutive W_k-intervals. Therefore J itself is the union of no more than 6 W_k-intervals.

3.12. *R_m, R_n, R_p, $T(J_1)$ and $T(J_2)$ are pairwise disjoint.* Since $\partial T_{1/3}(J)$ has slope 1/3 and ∂R_p has slope 1/2, it follows that R_p, $T_{1/3}(J_1)$ and $T_{1/3}(J_2)$ are disjoint sets contained in $T_{1/3}(J)$; R_m and R_n each are disjoint from $T_{1/3}(J)$. It remains to see that R_m and R_n are disjoint. By (3.11) we may assume $\tau(n)\geq k-2$. Since $\langle n\alpha\rangle$ is a W_k-point, $t_{k-3}<|n|\leq t_k$. Let $K\subset J$ be the open W_k-interval with right endpoint $\langle n\alpha\rangle$. By (1.7) $|K|\geq r_{k+1}>1/2q_{k+1}$ and $q_{k+1}<10t_{k-1}<60t_{k-3}$. Then $|K|>1/120|n|$. Hence

$$|T_{1/3}(J)|>|T_{1/3}(K)|\geq H(\rho^{-1}(K))>C_1|K|^{\gamma}>C_1/(120|n|)^{\gamma}>3|R_n|.$$

It follows from simple geometry that R_m and R_n are disjoint no matter what the length of Δ_m.

Hence $h(w)\neq h(z)$ if $w\neq z$. Thus h is an embedding.

Q.E.D.

Definition. A Jordan curve J contained in a metric space with metric d is called a *quasi-circle* (in the sense of Ahlfors [2]) if there exists a constant $K>0$ such that if $x\neq y\in J$ then one of the arcs of J connecting x and y is contained in a disk of diameter $Kd(x,y)$.

We prove next that Q is a quasi-circle and slightly more, that $Q\cup\{R_n\}$ has a quasi-structure. Since h is an embedding we may work with h^{-1}.

THEOREM 3.13. *There exists a constant $K>0$ such that if $x,y\in Q\cup\{R_n\}$ then one of the arcs connecting $h\pi h^{-1}(x)$ and $h\pi h^{-1}(y)$ together with the line segments $(x,h\pi h^{-1}(x))$ and $(y,h\pi h^{-1}(y))$ are contained in a disk of radius $Kd(x,y)$. In particular, Q is a quasi-circle*

Proof. Let $x\neq y\in Q\cup\{R_n\}$. As in (3.10) we have $x\in R_m\cup T(J_1)$ and $y\in R_n\cup T(J_2)$. An arc connecting $h\pi h^{-1}(x)$ and $h\pi h^{-1}(y)$ consists of a portion in $T(J_1)$, a portion in $T(J_2)$, the entire diagonal $\Delta_p\subset R_p$ and possible segments of $\Delta_m\subset R_m$ and $\Delta_n\subset R_n$ (if $x\in R_m$ or $y\in R_n$ respectively). By (3.12) these sets are disjoint. The theorem follows by simple plane geometry.

Q.E.D.

PROPOSITION 3.14. *Q has winding number* 1.

Proof. This follows from (3.9) since $Q\subset T(y_0,z_0)\cup cl(\Delta_0)$.

Q.E.D.

Definition. Given a metric space X, for each non-negative real number s there is a corresponding s-dimensional Hausdorff measure μ_s defined as follows. Let $B\subset X$ be an arbitrary set. The zero-dimensional measure $\mu_0 B$ is the number of points in B. For $s>0$, $\alpha>0$, let

$$\mu_{s,\alpha}B=\inf\Sigma_i[\operatorname{diam}(B_i)]^s,$$

where the infimum is taken over all covers $\{B_i\}$ of B such that diam $(B_i)<\alpha$ for each i. Then

$$\mu_s B=\lim_{\alpha\to 0^+}\mu_{s,\alpha}B,$$

A set B has Hausdorff dimension s, denoted $HD(B)=s$, iff $\mu_r B=0$ for all $r>s$ and $\mu_r B=\infty$ for all $r<s$. If $HD(B)=s$, then $\mu_s B$ is the Hausdorff measure of B within its dimension.

THEOREM 3.15. *The Hausdorff dimension of Γ is $1/\gamma$. The Hausdorff measure of Γ within its dimension is a positive, real number.*

Proof. Let $S=S^1\backslash\{\langle n\alpha\rangle\}n=\frac{\infty}{-\infty}$. Define $g:S\to\mathbf{R}^2$ by $g=h\circ\rho^{-1}$.

Claim. There exist constants A_1, A_2, A_3 and K_1 such that, (1) If I is a W_k-interval, $k \geq 0$, then

$$A_2|I|^\gamma < |g(I)| < A_1|I|^\gamma.$$

(2) *If* J is an arbitrary interval of S^1 then,

$$|g(J)| > A_3|J|^\gamma.$$

(3) If $B \subset \mathbf{R}^2$ is a disk with $B \cap g(S) \neq \varnothing$ then

$$cl(g^{-1}(K_1 B)) \supset \text{interval} \supset g^{-1}(B).$$

Proof of claim. (1) This follows from (3.7). (2) The ratios of lengths of W_{k+1}-intervals and W_k-intervals is bounded from 0 and ∞. For each $k \geq 0$ the collection of W_k-intervals forms a cover of S. Hence there exists a constant C (uniform over intervals J) and a W_k-interval $I \subset J$ and $|I| \geq C|J|$. Then $|g(J)| \geq |g(I)| > A_2|I|^\gamma > A_3|J|^\gamma$. (3) This follows since $g(S) \subset \Gamma \subset Q$ and Q is a quasi-circle (3.13).

By (1) the cover $\cup g(I)$, $I \in W_k$ satisfies $\Sigma|g(I)|^{1/\gamma} \leq A_1^{1/\gamma}\Sigma|I| \leq A_1^{1/\gamma}$. Hence $\mu_{1/\gamma}(g(S)) < \infty$ implying

$$HD(g(S)) \leq 1/\gamma.$$

Suppose that $HD(g(S)) < 1/\gamma$. Then $\mu_{1/\gamma}(g(S)) = 0$. Then there is a cover of $g(S)$ by disks B_i such that $\Sigma|B_i|^{1/\gamma} < \delta$ for any δ. Notice that $\Sigma|K_1 B_i|^{1/\gamma} < K_1^{1/\gamma}\delta$. Consider the cover $\{K_1 B_i\}$. Using (3) for each i we pick I_i such that

$$g^{-1}(B_i) \subset \Gamma_i \subset cl(g^{-1}(K_1 B_i)).$$

Therefore by (2) we have

$$A_3|I_i|^\gamma < |g(I_i)| \leq K_1|B_i|.$$

Hence $|I_i| < (K_1|B_i|/A_3)^{1/\gamma}$. Since $\cup I_i$ covers S we have

$$1 \leq \Sigma|I_i| \leq (K_1/A_3)^{1/\gamma}\Sigma|B_i|^{1/\gamma}.$$

This contradicts the assumption that $\mu_{1/\gamma} = 0$. From the preceding paragraph we have $0 < \mu_{1/\gamma}(g(S)) < \infty$. The theorem follows since $cl(g(S)) = \Gamma$.

Q.E.D.

The author thanks Curt McMullen for his assistance in the preceding proof.

§4. A $C^{2+\delta}$ DIFFEOMORPHISM $F: A \to A$

Recall Γ, the Cantor set $Q \backslash \cup\{\Delta_n\}$. Define a homeomorphism $f_\Gamma: \Gamma \to \Gamma$ by $f_\Gamma(y_n) = y_{n+2}$, $f_\Gamma(z_n) = z_{n+2}$ and if $x \notin \cup cl(\Delta_n)$

$$f_\Gamma(x) = h\rho^{-1}R_{2\alpha}\rho h^{-1}(x).$$

LEMMA 4.1. *There exists $C > 0$ such that if* (i) *x and y are the endpoints of a diagonal $\Delta_n \subset Q$ or* (ii) *x and y are the endpoints of $h\rho^{-1}$ (I) where I is a half closed W_k-interval with $\tau(j) \geq k-1$ for any $\langle j\alpha \rangle \in I$ then*

$$\|f_\Gamma(x) - f_\Gamma(y) - (x-y)\| < C\|x-y\|^{1+1/\gamma}.$$

Proof. Note that $1/n^{\gamma}-1/(n+2)^{\gamma}<2/n^{1+\gamma}$ for all $n\geq 1$. (i) If x and y are the endpoints of Δ_n, then,

$$\begin{aligned}\|f_\Gamma(x)-f_\Gamma(y)-(x-y)\| &= \sqrt{2}(g(n)-g(n+2)) = \sqrt{2}a(1/n^{\gamma}-1/(n+2)^{\gamma})\\ &< 2\sqrt{2}a/n^{1+\gamma}\\ &= A_1\|x-y\|^{1+1/\gamma} \text{ for some } A_1>0.\end{aligned}$$

(ii) Suppose x and y are the endpoints of $h\rho^{-1}(I)$ and $I=[\langle p\alpha\rangle,\langle q\alpha\rangle)$ is a W_k-interval of length r_k, $\tau(p)\geq k-1$. Let $U=\rho^{-1}(I)$, $\quad U'=\rho^{-1}(R_{2\alpha}(I))$. Then

$$\begin{aligned}\|\pi_1 f_\Gamma(x)-\pi_1 f_\Gamma(y)-\pi_1(x-y)\| &= |H(U)-H(U')|\\ &= \sum_{|n|=p}^{\infty}\chi_I\langle n\alpha\rangle(g(n)-g(n+2)) < 2a\sum_{|n|=p}^{\infty}\chi_I\langle n\alpha\rangle/n^{1+\gamma}\\ &< 2a|I|\sum_{|n|=p}^{\infty}1/n^{1+\gamma}+8a/p^{1+\gamma} \quad \text{(by 2.4)(i)).}\end{aligned}$$

Since $k-1\leq\tau(p)\leq k$ apply (1.7) to get $q_k/10<p<q_k$; also $\frac{1}{2}<q_k|I|<1$. It follows that there exists $A_2>0$ with

$$\|\pi_1 f_\Gamma(x)-\pi_1 f_\Gamma(y)-\pi_1(x-y)\|<A_2|I|^{1+\gamma}.$$

On the other hand,

$$\begin{aligned}&\|\pi_2 f_\Gamma(x)-\pi_2 f_\Gamma(y)-\pi_2(x-y)\| = |V(U)-V(U')|\\ &= \sum_{n=[p/2]}^{\infty}\left(\chi_I\langle 2n\alpha\rangle(g(2n)-g(2n+2))-\chi_I\langle(2n+1)\alpha\rangle(g(2n+1)-g(2n+3))\right)\\ &< 2a\sum_{n=[p/2]}^{\infty}\left(\chi_I\langle 2n\alpha\rangle/(2n)^{1+\gamma}+\chi_I\langle(2n+1)\alpha\rangle/(2n+1)^{1+\gamma}\right)\\ &< 4a|I|\sum_{n=[p/2]}^{\infty}1/(2n)^{1+\gamma}+16a/p^{1+\gamma}\\ &< A_3|I|^{1+\gamma} \text{ as above.}\end{aligned}$$

Hence

$$\|f_\Gamma(x)-f_\Gamma(y)-(x-y)\|<(A_2+A_3)|I|^{1+\gamma}. \tag{4.2}$$

By (3.7)(i) $C_1|I|^{\gamma}<H(\rho^{-1}(I))\leq\|x-y\|$. Let $C=(A_1+A_2+A_3)/C_1^{1+1/\gamma}$. If x and y satisfy either (i) or (ii),

$$\|f_\Gamma(x)-f_\Gamma(y)-(x-y)\|<C\|x-y\|^{1+1/\gamma}.$$

Q.E.D.

LEMMA 4.3. *There exists $C_\Gamma>0$ such that if x, $y\in\Gamma$ then*

$$\|f_\Gamma(x)-f_\Gamma(y)-(x-y)\|<C_\Gamma\|x-y\|^{1+1/\gamma}.$$

Proof. Let K be the quasi-circle constant for Q. Suppose x, $y\in\Gamma$. At least one of the arcs $A(x,y)$ connecting x and y is contained in a disk of diameter $K\|x-y\|$. Assume $x<y$ in $A(x,y)$. If $x\in\Delta_n$ define $x'=z_n$, otherwise let $x'=x$. If $y\in\Delta_m$ define $y'=y_m$, otherwise let $y'=y$. Then,

$$\begin{aligned}\|f_\Gamma(x)-f_\Gamma(y)-(x-y)\| &\leq \|f_\Gamma(x)-f_\Gamma(x')-(x-x')\|+\|f_\Gamma(y)-f_\Gamma(y')-(y-y')\|\\ &\quad+\|f_\Gamma(x')-f_\Gamma(y')-(x'-y')\|.\end{aligned}$$

The first two terms are bounded above by $C(\|x-x'\|^{1+1/\gamma}+\|y-y'\|^{1+1/\gamma})$ by (4.1)(i).

We use (4.2) to estimate the third term for $x' \neq y'$. Let U be the open interval $\rho h^{-1}(A(x', y'))$. Let $I_l \subset U$ be a half closed W_l-interval with length r_l and l minimal. I_l must satisfy $\tau(j) \geq l-1$ for any $\langle j\alpha \rangle \in I_l$. Beginning with I_l we may inductively write U as an infinite union of no more than 8 half closed W_k-intervals, $k \geq l$, satisfying the hypothesis of (3.7). Then by (4.2)

$$\begin{aligned}\|f_\Gamma(x') - f_\Gamma(y') - (x'-y')\| &\leq 8(A_2 + A_3)\Sigma r_k^{1+\gamma}, \ k \geq l \\ &\leq 16(A_2+A_3) r_l^{1+\gamma} \quad \text{since } r_k = \alpha^k \quad (1.4) \\ &= 16(A_2+A_3)|I_l|^{1+\gamma} \\ &< 16CH(\rho^{-1}(I_l))^{1+1/\gamma} \ \text{by (3.7)(i) and the definition of } C \text{ (4.1)}\end{aligned}$$

Since $h(\rho^{-1}(I_l)) \subset A(x, y)$ and $x', y' \in A(x, y)$ we have

$$\|x-x'\|, \ \|y-y'\|, \quad H(\rho^{-1}(I_l)) < 2K\|x-y\|.$$

Hence there exists C_Γ such that

$$\|f_\Gamma(x) - f_\Gamma(y) - (x-y)\| \leq C_\Gamma \|x-y\|^{1+1/\gamma}.$$

Q.E.D.

Definition of f_n: $R_n \to R_{n+2}$. For $n \in Z$ define $G_n(x) = (g(n+2)/g(n))(x - z_n) + z_{n+2}$ for $x \in R^2$. (Recall $g(n) = \|y_n - z_n\|/\sqrt{2}$.) Define

$$H_n(x) = \begin{cases} x - z_n + z_{n+2} & \text{if } x \in D(z_n, g(n)/4) \\ x - y_n + y_{n+2} & \text{if } x \in D(y_n, g(n)/4). \end{cases}$$

Extend H_n to $\mathbf{R}^2$ arbitrarily.

Let ϕ be a smooth bump function on $\mathbf{R}^2$ which is identically one on $D(0, 1/5)$ and vanishes off $D(0, 1/4)$. Let

$$\phi_n(x) = \begin{cases} \phi((x - z_n)/g(n)) & \text{if } x \in D(z_n, g(n)/4) \text{ and} \\ \phi((x - y_n)/g(n)) & \text{if } x \in D(y_n, g(n)/4). \end{cases}$$

Let $\phi_n(x) = 0$ elsewhere.

Define

$$f_n(x) = \phi_n(x) H_n(x) + (1 - \phi_n(x)) G_n(x).$$

Recall the constant C_Γ of (4.3).

LEMMA 4.4. *There exists $C' > 0$ such that if $\|y_n - z_n\| < (3C'C_\Gamma)^{-\gamma}$ then,*

(i) *$Df_n(p_n) = Id$ and $D^2 f_n(p_n) = 0$ for $p = y$ or z;*

(ii) *$\|D^r f_n\| < C'(g(n) - g(n+2))/g(n)^r$ for $1 \leq r \leq 3$;*

(iii) *f_n maps $\Delta_n \subset R_n$ diffeomorphically onto $\Delta_{n+2} \subset R_{n+2}$.*

Proof. (i) Simply observe $f_n = H_n$ in neighborhoods of y_n and z_n. (ii) This is a straightforward calculation from the definitions. (iii) G_n preserves boundary components since it is affine and R_{n+2} is a homothetic replica of R_n. Inside $D(p_n, g(n)/4)$, $p = y$ or z, H_n preserves boundary components since it is a translation. Thus any convex combination of the two preserves boundary components.

Let $x \in R_n$. By (i) and (ii) and since R_n is convex,

$$\begin{aligned}\|Df_n(x) - Id\| &= \|Df_n(x) - Df_n(z_n)\| < C'\|x - z_n\| \, \|D^2 f_n\| \\ &< 3C'(g(n) - g(n+2))/g(n) < 3C'C_\Gamma \|y_n - z_n\|^{1/\gamma} \text{ by (4.3).}\end{aligned}$$

If $\|y_n - z_n\| < (3C'C_\Gamma)^{-\gamma}$ then $\|Df_n(x) - ID\| < 1$.

The open unit ball of linear functions centered at the identity are all invertible. Thus $Df_n(x)$ is invertible for $\|y_n - z_n\| < (3C'C_\Gamma)^{-\gamma}$.

Hence f_n sends Δ_n and the boundary of R_n diffeomorphically onto Δ_{n+2} and the boundary of R_{n+2}. Any smooth mapping which sends the boundary of a disk D_1 diffeomorphically onto the boundary of another disk D_2 and which is locally invertible, is a diffeomorphism from D_1 to D_2.

Q.E.D.

For $\|y_n - z_n\| \geq (3C'C_\Gamma)^{-\gamma}$ redefine f_n to satisfy (i) and (iii) above. They will satisfy (ii) automatically for some constant $C > C'$. Hence

LEMMA 4.5. *There exists a constant $C > 0$ such that, if $n \in Z$,*
(i) *$Df_n(p_n) = Id$ and $D^2 f_n(p_n) = 0$ for $p = y$ or z;*
(ii) *$\|D^r f_n\| < C(g(n) - g(n+2))/g(n)^r$ for $1 \leq r \leq 3$;*
(iii) *f_n maps $\Delta_n \subset R_n$ diffeomorphically onto $\Delta_{n+2} \subset R_{n+2}$.*

Q.E.D.

If $x \in \Gamma \cup \{R_n\}$, define

$$\theta_0(x) = \begin{cases} f_\Gamma(x), & x \in \Gamma \\ f_n(x), & x \in R_n \end{cases}$$

$$\theta_1(x) = \begin{cases} Id, & x \in \Gamma \\ Df_n(x), & x \in R_n \end{cases} \qquad \theta_2(x) = \begin{cases} 0, & x \in \Gamma \\ D^2 f_n(x), & x \in R_n \end{cases}$$

THEOREM 4.6. *Let $\delta = 1/\gamma - 1$. There exists $B > 0$ such that for $x, y \in Q \cup \{R_n\}$*

(i) $\|\theta_2(x) - \theta_2(y)\| < B\|x - y\|^\delta$,

(ii) $\dfrac{\|\theta_1(x) - \theta_1(y) - \theta_2(y)(x-y)\|}{\|x-y\|} < B\|x-y\|^\delta$,

(iii) $\dfrac{\|\theta_0(x) - \theta_0(y) - \theta_1(y)(x-y) - \theta_2(y)(x-y)^2/2\|}{\|x-y\|^2} < B\|x-y\|^\delta$.

Proof. Suppose $x, y \in R_n$. Since R_n is convex, (i)–(iii) are all estimated by $\|x-y\|\,\|D^3 f_n\|$ by Taylor's theorem. By (4.5)(ii) this is bounded by $\|x-y\|\,C(g(n)-g(n+2))/g(n)^3 < C_R\,\|x-y\|^\delta$ for some $C_R > 0$ since $\gamma < 1$.

Suppose $x, y \in \Gamma$. By (4.3)

$$\|f_\Gamma(x) - f_\Gamma(y) - (x-y)\| < C_\Gamma\|x-y\|^{1+1/\gamma} = C_\Gamma\|x-y\|^{2+\delta}.$$

Since $\theta_1|\Gamma = Id$ and $\theta_2|\Gamma = 0$ (i)–(iii) are valid for $B > C_\Gamma$.

The general case is proved with the triangle inequality and the fact that Q is a quasi-circle.

Let $x, y \in Q \cup R_n$ and not in the same disk R_n and not both in Γ. Let K be the quasi-constant for $Q \cup \{R_n\}$. There exists an arc $A(x, y) \subset Q \cup \{R_n\}$ connecting x and y and contained in a disk of diameter $< K\|x-y\|$. If $x \in R_n$, let x' be a boundary point of $R_n \cap A(x, y)$. Otherwise let $x = x'$. Similarly define y'. Then $x', y' \in \Gamma$.

Since x, y, x', y' are all in the arc $A(x, y)$,

$$\|x' - y'\|,\ \|x - x'\|,\ \|y - y'\| \leq K\|x-y\|.$$

(i) $\|\theta_2(x)-\theta_2(y)\| \le \|\theta_2(x)-\theta_2(x')\| + \|\theta_2(x')-\theta_2(y')\| + \|\theta_2(y')-\theta_2(y)\|$

$$\le C_R(\|x-x'\|^\delta + \|y-y'\|^\delta) + C_\Gamma \|x'-y'\|^\delta$$
$$\le (2C_R + C_\Gamma)K^\delta \|x-y\|^\delta$$
$$= B_1 \|x-y\|^\delta.$$

(ii) $$\frac{\|\theta_1(x)-\theta_1(y)-\theta_2(y)(x-y)\|}{\|x-y\|} \le \frac{\|\theta_1(x)-\theta_1(y)\|}{\|x-y\|} + \|\theta_2(y)\|$$

$$\le \frac{\|\theta_1(x)-\theta_1(x')\| + \|\theta_1(x')-\theta_1(y')\| + \|\theta_1(y')-\theta_1(y)\|}{\|x-y\|} + \|\theta_2(y)-\theta_2(y')\|$$

$$\le K\frac{\|\theta_1(x)-\theta_1(x')\|}{\|x-x'\|} + K\frac{\|\theta_1(y')-\theta_1(y)\|}{\|y'-y\|} + \|\theta_2(y)-\theta_2(y')\|$$

$$\le KC_R\|x-x'\|^\delta + KC_R\|y'-y\|^\delta + C_R\|y-y'\|^\delta$$
$$\le 3K^{1+\delta}C_R\|x-y\|^\delta = B_2\|x-y\|^\delta.$$

(iii) $$\frac{\|\theta_0(x)-\theta_0(y)-\theta_1(y)(x-y)-\theta_2(y)(x-y)^2/2\|}{\|x-y\|^2}$$

$$\le \frac{\|\theta_0(x)-\theta_0(y)-(x-y)\|}{\|x-y\|^2} + \frac{\|\theta_1(y)-Id\|}{\|x-y\|} + \tfrac{1}{2}\|\theta_2(y)\| = *.$$

The first term of * is bounded by

$$K^2\left(\frac{\|\theta_0(x)-\theta_0(x')-(x-x')\|}{\|x-x'\|^2} + \frac{\|\theta_0(x')-\theta_0(y')-(x'-y')\|}{\|x'-y'\|^2} + \frac{\|\theta_0(y')-\theta_0(y)-(y'-y)\|}{\|y'-y\|^2}\right)$$

$$\le K^2(C_R\|x-x'\|^\delta + C_\Gamma\|x'-y'\|^\delta + C_R\|y'-y\|^\delta) \le B_3\|x-y\|^\delta \text{ for some } B_3>0.$$

The last two terms of * are non-zero only if $y \in R_n$. In this case

$$\frac{\|\theta_1(y)-Id\|}{\|x-y\|} \le K\frac{\|\theta_1(y)-\theta_1(y')\|}{\|y-y'\|} \le KC_R\|y-y'\|^\delta \le K^{1+\delta}C_R\|x-y\|^\delta = B_4\|x-y\|^\delta.$$

And

$$\|\theta_2(y)\| = \|\theta_2(y)-\theta_2(y')\| \le C_R\|y-y'\|^\delta \le C_R K^\delta\|x-y\|^\delta = B_5\|x-y\|^\delta.$$

Let

$$B = B_1 + B_2 + B_3 + B_4 + B_5.$$

Q.E.D.

The θ_i are continuous on R_n. Since $\theta_1|\Gamma = Id$ and $\theta_2|\Gamma = 0$ it follows easily from (4.6) that the θ_i are continuous at Γ. Therefore the conditions of the Whitney extension theorem have been met. (See [1] or [20].)

COROLLARY 4.7. *There exists a $C^{2+\delta}$ mapping $F: S^1 \times R^1 \to S^1 \times \mathbf{R}^1$, $\delta = 1/\gamma_{-1}$, such that if $x \in Q \cup \{R_n\}$ then $F(x) = \theta_0(x)$, $DF_x = \theta_1(x)$ and $D^2F_x = \theta_2(x)$.*

Q.E.D.

In summary we have proved

PROPOSITION 4.8. *Suppose $f: \Gamma \to \Gamma$ is a homeomorphism and*

$$\|f(x)-f(y)-(x-y)\| < C\|x-y\|^{2+\delta}$$

for all $x, y \in \Gamma$ where x and y are endpoints of Δ_n or x and y are the endpoints of $h\rho^{-1}(I)$ where I is a half closed W_k-interval with $\tau(j) \ge k-1$ for any $\langle j\alpha\rangle \in I$. There exists a $C^{2+\delta}$ embedding F of a *neighborhood of* $Q \cup \{R_n\}$ *into* U *such that,*

(i) $F|\Gamma = f$, $DF|\Gamma = Id$, $D^2F|\Gamma = 0$;

(ii) F *maps each* $\Delta_n \subset R_n$ *diffeomorphically onto* $\Delta_{n+2} \subset R_{n+2}$.

Q.E.D.

Since F is a diffeomorphism of $Q \cup \{R_n\}$, there exists a smooth neighborhood N of $Q \cup \{R_n\}$ on which F is a $C^{2+\delta}$ embedding into $S^1 \times \mathbf{R}^1$.

Since $Q \cup \{R_n\}$ is F-invariant and Q has winding number 1, (see (3.14)) we may assume that N is a smooth deformation retract of the annulus A and that its image $F(N)$ is contained in A. $F(N)$ is therefore a smooth deformation retract of A. Denote these two retractions by $d_N: A \to N$ and $d_{F(N)}: A \to F(N)$. Then define $f: A \to A$ by $f(x) = d_{F(N)}^{-1}(F(d_N(x)))$. This is a $C^{2+\delta}$ diffeomorphism of the annulus extending $f: Q \cup \{R_n\} \to Q \cup \{R_n\}$. This establishes Theorem A.

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