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A REMARK ON THE LOXODROMIC MAPPING CONJECTURE

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The loxodromic mapping conjecture of J. Harrison is affirmed for diffeomorphisms of the 2-sphere that embed in flows.

Let f be a diffeomorphism of the 2-sphere S^2 onto itself. The Birkhoff mapping conjecture as posed by Birkhoff [**1**] asked if

$$\text{(a)}\quad \begin{array}{l} f \text{ has exactly} \\ \text{two periodic points} \\ \text{and } f \text{ preserves area} \end{array} \quad \overset{?}{\Longrightarrow} \quad \text{(b)}\quad \begin{array}{l} f \text{ is topologically conjugate} \\ \text{to an irrational rotation} \\ \text{of } S^2 \text{ along the latitudes.} \end{array}$$

Using work of Handel, Jerrard constructed a counter-example to Birkhoff's conjecture [**5**]. The loxodromic mapping conjecture as posed by Harrison [**2**] asks if

$$\text{(c)}\quad \begin{array}{l} f \text{ is of class } C^r,\ r \geqslant 3, \\ f \text{ has exactly two periodic points,} \\ \text{one a source and the other a sink,} \\ \text{and some orbit connects them.} \end{array} \quad \overset{?}{\Longrightarrow} \quad \text{(d)}\quad \begin{array}{l} f \text{ is topologically conjugate} \\ \text{to a Northpole-Southpole} \\ \text{diffeomorphism of } S^2. \end{array}$$

In a Northpole-Southpole diffeomorphism points slide downward along the longitudes. The Birkhoff conjecture was part of conservative dynamics and arose from celestial mechanics. The loxodromic conjecture is dissipative and arose from Harrison's work on the Seifert Conjecture. (Her counter-example to the C^2 loxodromic conjecture leads to a C^2 counter-example to the Seifert Conjecture [**3**]. The same thing would carry over in the C^r case, $r \geqslant 3$.)

Markus affirms a weakened form of the Birkhoff conjecture [**6**]. Namely, if in addition to (a) one assumes that f is the time-one map of a flow on S^2 then (b) does follow. Here we remark that the same is true in the loxodromic case. If in addition to (c) one assumes that f is the time-one map of a flow on S^2 then (d) does follow. In fact, we prove a little more.

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THEOREM. *Suppose that φ is a flow on S^2 whose time-one map f has exactly two periodic points, N and S. (They need not be a source and sink.) If for some $x \in S^2$, $\alpha(x, f) = N$ and $\omega(x, f) = S$, then each φ-trajectory connects N to N, N to S, or S to S.*

PROOF: Note that fixed points of φ are fixed points of f, so φ has at most two fixed points. Also, the period of any periodic orbit of φ must be irrational. Since N is periodic under f, so are all points $\varphi_t(N)$ on its φ-orbit, $\vartheta(N)$. Since f has only two periodic points, $\vartheta(N)$ is the single point N. That is, N is a fixed point of φ. The same is true for S. As $t \to \infty$, express $t = n + s$ where $n = [t]$ and $0 \leqslant s < 1$. Since φ is continuous and S is a fixed point of φ to which $f^n x$ converges,

$$\varphi_t(x) = \varphi_s(\varphi_n x) = \varphi_s(f^n x) \to S \quad \text{as} \quad n \to \infty.$$

The ω-limit set of x under the flow and its time-one map are the same: $\omega(x, \varphi) = \omega(x, f) = S$. Similarly for N, and we see that the φ-orbit of x, $\vartheta(x)$, connects N to S.

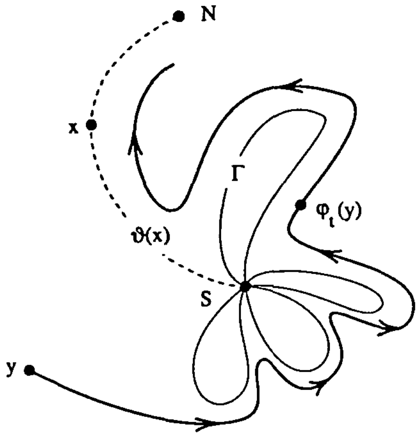
Now let $y \in S^2 \setminus \{N, S\}$ be given. Consider $\omega(y) = \omega(y, \varphi)$. By the singular Poincaré-Bendixson Theorem (see Hartman [4] for example) either

(i) $\omega(y)$ is a fixed point, or
(ii) $\omega(y)$ is a closed orbit of φ, or
(iii) $\omega(y)$ is a separatrix cycle towards which $\varphi_t(y)$ spirals as $t \to \infty$.

A separatrix cycle is a loop of trajectories connecting fixed points of φ that crosses itself only at the fixed points. It is the frontier of the complementary region in S^2 containing the orbit $\vartheta(y)$.

If $\omega(y)$ is a closed orbit γ then γ divides S^2 into two regions and by the Index Theorem each contains a fixed point of φ. Since $N \cup \vartheta(x) \cup S$ is a connected non-periodic set it cannot meet γ, so both N and S lie in one of the two regions. Therefore φ has at least three fixed points, a contradiction to the hypothesis on f; $\omega(y)$ cannot be a closed orbit.

Suppose that $\omega(y)$ is a separatrix cycle Γ toward which $\varphi_t(y)$ spirals as $t \to \infty$. If Γ contains S but not N then Γ consists of a bouquet of mutually exterior loops at S. (There are at most countably many of them.) But there is no way that $\varphi_t(y)$ can spiral toward Γ and avoid the arc $N \cup \vartheta(x) \cup S$. Similarly for N, and we see that Γ must contain both N and S. Since $\varphi_t(y)$ spirals toward Γ as $t \to \infty$, we see that the α-limit of y, $\alpha(y)$, is disjoint from Γ. It contains no fixed points of φ because both N and S lie on Γ. By the Poincaré-Bendixson Theorem, $\alpha(y)$ is a closed orbit. Again this leads to a third fixed point of φ in one of the complementary regions of $\alpha(y)$ and contradicts the hypothesis on f.



The only remaining possibility is that $\omega(y)$ is a fixed point for all $y \in S^2$. The same thing applies to $\alpha(y)$ and we see that each φ-trajectory connects N to N, N to S, or S to S. ∎

COROLLARY. *If f is the time-one map of a flow on S^2 then the loxodromic problem is solved — f is topologically conjugate to a Northpole-Southpole diffeomorphism provided that f has only two periodic points, a source and a sink, and has at least one orbit connecting the two.*

PROOF: Since a source for f is also a source for φ, no non-trivial trajectory can connect the source to itself. Likewise for the sink. By the theorem, all the other trajectories connect the source to the sink, so φ is a Northpole-Southpole flow and f is a Northpole-Southpole diffeomorphism. ∎

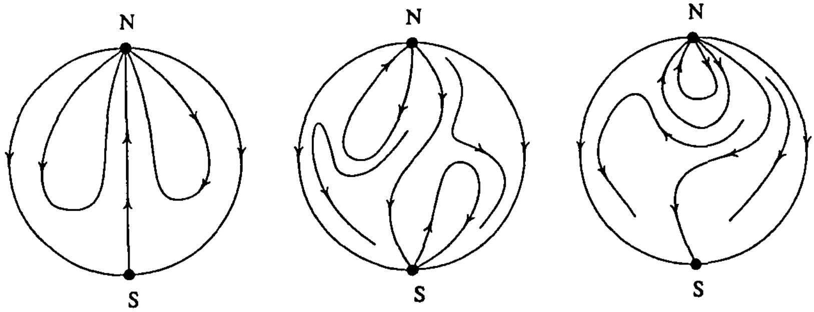
REMARK 1. The hypothesis that f be the time-one map of a flow is non-generic. See Palis [**7**].

2. Generically in the C^1 topology, a diffeomorphism f of S^2 with only two periodic points is conjugate to a Northpole-Southpole diffeomorphism. For generically the periodic points of f are hyperbolic and dense in its non-wandering set. See Pugh [**8**]. It follows that the periodic points must be a fixed-point source and a fixed-point sink which are the α- and ω-limit sets of all the f-orbits. Thus, the loxodromic problem is solved (positively) in the generic C^1 case.

3. Harrison's paper [**3**] shows that there exist C^2 counter-examples to the loxodromic conjecture. They are not C^1 structurally stable and appear not to be C^2 structurally stable either. So they seem to be non-generic also. It is possible that C^r counter-examples to the loxodromic conjecture exist, $r \geqslant 3$, and even that they are structurally stable, $r \geqslant 2$. They would give C^r structurally stable counter-examples to the C^r Seifert Conjecture and to the C^r closing lemma.

4. Without the source-sink hypothesis the corollary fails. The figures below illus-

trate three of the possibilities. They are drawn on a disc D; the orbit $\vartheta(x)$ connecting S to N has been split along itself and its two copies form the boundary of the D.



REFERENCES

[1] G.D. Birkhoff, 'Some unsolved problems in theoretical dynamics', *Science* **94** (1941), 598–600.

[2] J. Harrison, 'Continued fractals and the Seifert conjecture', *Bull. Amer. Math. Soc.* **13** (1985), 147–153.

[3] J. Harrison, '$C^{2+\epsilon}$ counter-example to the Seifert conjecture', *Topology* **27** (1988), 249–278.

[4] P. Hartman, *Ordinary differential equations* (Wiley, 1964).

[5] R. Jerrarrd, 'On the sphere conjecture of Birkhoff', *Proc. Amer. Math. Soc.* **102** (1988), 193–201.

[6] L. Markus, 'Three unresolved problems of dynamics on surfaces', University of Warwick, 1985 (preprint).

[7] J. Palis, 'Vector fields generate few diffeomorphisms', *Bull. Amer. Math. Soc,* **80** (1974), 503–505.

[8] C. Pugh, 'An improved closing lemma and general density theorem', *Amer. J. Math.* **89** (1967), 1010–1021.

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