THE HIRZEBRUCH–RIEMANN–ROCH THEOREM IN TRUE GENUS-0 QUANTUM K-THEORY

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Abstract. We completely characterize genus-0 K-theoretic Gromov–Witten invariants of a compact complex algebraic manifold in terms of cohomological Gromov–Witten invariants of this manifold. This is done by applying (a virtual version of) the Kawasaki–Hirzebruch–Riemann–Roch formula for expressing holomorphic Euler characteristics of orbibundles on moduli spaces of genus-0 stable maps, analyzing the sophisticated combinatorial structure of inertia stacks of such moduli spaces, and employing various quantum Riemann–Roch formulas from *fake* (i.e. orbifold–ignorant) quantum K-theory of manifolds and orbifolds (formulas, either previously known from works of Coates–Givental, Tseng, and Coates–Corti–Iritani–Tseng, or newly developed for this purpose by Tonita). The ultimate formulation combines properties of overruled Lagrangian cones in symplectic loop spaces (the language that has become traditional in description of generating functions of genus-0 Gromov-Witten theory) with a novel framework of *adelic characterization* of such cones. As an application, we prove that tangent spaces of the overruled Lagrangian cones of quantum K-theory carry a natural structure of modules over the algebra of finite-difference operators in Novikov's variables. As another application, we compute one of such tangent spaces for each of the complete intersections given by equations of degrees $l_1, \dots, l_k$ in a complex projective space of dimension $\geq l_1^2 + \cdots + l_k^2 - 1$.

0. Motivation

K-theoretic Gromov–Witten invariants of a compact complex algebraic manifold X are defined as holomorphic Euler characteristics of various interesting vector bundles over moduli spaces of stable maps of compact complex curves to X. They were first introduced in [10] (albeit, in a limited generality of genus-0 curves mapped to homogeneous Kähler spaces), where it was shown that such invariants define

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on $K^0(X)$ a geometric structure resembling Frobenius structures of quantum cohomology theory.

At about the same time, it was shown [14] that simplest genus-0 K-theoretic GW-invariants of the manifold X of complete flags in $\mathbb{C}^{n+1}$ are governed by the finite-difference analogue of the quantum Toda lattice. More precisely, a certain generating function for K-theoretic GW-invariants, dubbed in the literature the *J-function* (and depending on n variables, namely, Novikov's variables $Q_1, \dots, Q_n$ introduced to separate contributions of complex curves in X by their degrees) turns out to be a common eigenfunction (known in representation theory as Whittaker's function) of n commuting finite-difference operators, originating from the center of the *quantized* universal enveloping algebra $U_q sl_{n+1}$. In quantum cohomology theory, the corresponding fact was established by B. Kim [18], who showed that the cohomological J-function of the flag manifold $X = G/B$ of a complex simple Lie algebra $\mathfrak{g}$ is a Whittaker function of the Langlands-dual Lie algebra $\mathfrak{g}'$. The K-theoretic generalization involving quantized universal enveloping algebras $U_q\mathfrak{g}'$ was stated in [14] as a conjecture (confirmed recently in [2]).

Foundations for K-theoretic counterpart of GW-theory were laid down by Y.-P. Lee [20] in the reasonable generality of arbitrary complex algebraic target spaces X (and holomorphic curves of arbitrary genus). While the general structure and universal identities (such as the string equation, or topological recursion relations) of quantum cohomology theory carry over to case of quantum K-theory, the latter is still lacking certain computational tools of the former one, and for the following reason.

The so-called *divisor equations* in quantum cohomology theory tell that the number of holomorphic curves of certain degree d with an additional constraint, that a certain marked point is to lie on a certain divisor p, is equivalent to (more precisely, differs by the factor (p, d) from) the number of such curves without the marked point and without the constraint. Consequently, the dependence of J-functions on Novikov's variables is redundant to their behavior as functions on $H^2(X)$ introduced through constraints at marked points. In particular, differential equations satisfied by the J-function in Novikov's variables (e.g. the Toda equations in the case of flag manifolds) are directly related to the quantum cup-product on $H^*(X)$.

In K-theory, however, any analogue of the divisor equation is seemingly missing, and respectively the K-theoretic analogue of the quantum cup-product, and differential equations of the Frobenius-like structure on $K^0(X)$ are completely detached from the way the J-functions depend on Novikov's variables. Because of this lack of structure with respect to Novikov's variables, it appears even more perplexing that in examples (such as projective spaces, or flag manifolds) the J-functions of quantum K-theory turn out to satisfy interesting finite-difference equations.

The idea of computing K-theoretic GW-invariants in cohomological terms is naturally motivated by the classical Hirzebruch–Riemann–Roch formula [15]

$$\chi(M,V) := \sum_k (-1)^k \dim H^k(M,V) = \int_M \mathrm{ch}(V)\,\mathrm{td}(T_M).$$

The problem (which is at least a decade old) of putting this idea to work encounters the following general difficulty. The HRR formula needs to be applied to the base M which, being a moduli space of stable maps, behaves as a virtual *orbifold* (rather than virtual manifold). The HRR formula for orbibundles V on orbifolds M was established by Kawasaki [17] and expresses the holomorphic Euler characteristic (which is an integer) as an integral over the *inertia orbifold* IM (rather than M itself). The latter is a disjoint union of *strata* corresponding to points with various types of local symmetry (and M being one of the strata corresponding to the trivial symmetry).

When M is a moduli space of stable maps, the strata of the inertia stack IM parametrize *stable maps with prescribed automorphisms.* It is reasonable to expect that individual contributions of such strata can be expressed as integrals over moduli spaces of stable maps from quotient curves, and thus in terms of traditional GW-invariants. However, the mere combinatorics of possible symmetries of stable maps appears so complicated (not mentioning the complexity of the integrands required by Kawasaki's theorem), that obtaining a "quantum HRR formula" expressing K-theoretic GW-invariants via cohomological ones didn't seem feasible.

In the present paper, we give a complete solution in genus-0 to the problem of expressing K-theoretic GW-invariants of a compact complex algebraic manifold in terms of its cohomological GW-invariants. The solution turned out to be technology-consuming, and we would like to list here those developments of the last decade that made it possible.

One of them is the Quantum HRR formula [7, 4] in *fake* quantum K-theory. One can take the right-hand side of the classical Hirzebruch–Riemann–Roch formula for the definition of $\chi^{\mathrm{fake}}(M,V)$ on an *orbifold* M. Applying this idea systematically to moduli spaces of stable maps, one obtains fake K-theoretic GW-invariants, whose properties are similar to those of true ones, but the values (which are rational, rather than integer) are different. The formula expresses fake K-theoretic GW-invariants in terms of cohomological ones.

Another advance is the Chen–Ruan theory [1, 3] of GW-invariants of *orbifold target spaces*, and the computation by Jarvis–Kimura of such invariants in the case when the target is the quotient of a point (or more generally a manifold) by the trivial action of a finite group.

Next is the theorem of Tseng [24] expressing *twisted* GW-invariants of orbifold target spaces in terms of untwisted ones.

Yet, two more "quantum Riemann–Roch formulas" of [4] had to be generalized to the case of orbifold targets. This is done in [21, 23].

Finally, our formulation of the Quantum HRR Theorem in true quantum K-theory is based on a somewhat novel form of describing generating functions of GW-theory, which we call *adelic characterization*. For a general and precise formulation of the theorem, the reader will have to wait until Section 6, but here we would like to illustrate the result with an example that was instrumental in shaping our understanding.

Let

$$J = (1-q)\sum_{d\geq 0} \frac{Q^d}{(1-Pq)^n(1-Pq^2)^n\cdots(1-Pq^d)^n}.$$

Here P is unipotent, and stands for the Hopf bundle on $\mathbb{C}P^{n-1}$, satisfying the relation $(1-P)^n = 0$ in $K^0(\mathbb{C}P^{n-1})$. It is a power series in Novikov's variable Q with vector coefficients which are rational functions of q, and take values in $K^0(\mathbb{C}P^{n-1})$. It was shown[1] in [14] that J represents (one value of) the *true* K-theoretic J-function of $\mathbb{C}P^{n-1}$.

On the other hand, one can use quantum Riemann–Roch and Lefschetz theorems of [4] and [5] to compute, starting from the cohomological J-function of $\mathbb{C}P^{n-1}$, a value of the J-function of the *fake* quantum K-theory. The result (see Section 10) turns out to be the same: J. This sounds paradoxical, since — one can check this directly for $\mathbb{C}P^1$ in low degrees! — contributions of non-trivial Kawasaki strata neither vanish nor cancel out.

In fact this is not a contradiction, for as it turns out, coefficients of the series J do encode fake K-theoretic GW-invariants, *when J is expanded into a Laurent series* near the pole $q = 1$. Furthermore, when J is expanded into a Laurent series near the pole $q = \zeta^{-1}$, where ζ is a primitive m-th root of unity, the coefficients represent certain

[1]Using birational invariance of holomorphic Euler characteristcs replacing certain moduli spaces of stable maps to $\mathbb{C}P^{n-1}$ with toric compactifications.

fake K-theoretic GW-invariants of the *orbifold target space* $\mathbb{C}P^{n-1}/\mathbb{Z}_m$. Moreover, according to our main result, these properties altogether completely characterize those Q-series (whose coefficients are vector-valued rational functions of q) which represent true genus-0 K-theoretic GW-invariants of a given target manifold.

This fact is indeed the result of application of Kawasaki's HRR formula to moduli spaces of stable maps. Namely, the complicated combinatorics of strata of the inertia stacks can be interpreted as a certain identity which, recursively in degrees, governs the decomposition of the J-function into the sum of elementary fractions of q with poles at all roots of unity. The theorem is stated in Section 6 (after the general notations, properties of quantum K-theory, Kawasaki's HRR formula, and results of fake quantum K-theory are described in Sections 1–5), and proved in Sections 7 and 8.

In Section 10, we develop a technology that allows one to extract concrete results from this abstract characterization of quantum K-theory. In particular, we prove (independently of [14]) that the function J is indeed the J-function of $\mathbb{C}P^{n-1}$, as well as similar results for codimension-k complete intersections of degrees $l_1, \dots, l_k$ satisfying $l_1^2 + \cdots + l_k^2 \leq n$.

Let $q^{Q\partial_Q}$ denote the operator of *translation* through $\log q$ of the variable $\log Q$. It turns out that for every $s \in \mathbb{Z}$,

$$\left(Pq^{Q\partial_Q}\right)^s J = (1-q)P^s \sum_{d\geq 0} \frac{Q^d q^{sd}}{(1-Pq)^n(1-Pq^2)^n\cdots(1-Pq^d)^n}$$

also represent genus-0 K-theoretic GW-invariants of $\mathbb{C}P^{n-1}$. This example illustrates a general theorem of Section 9, according to which J-functions of quantum K-theory are organized into modules over the algebra $\mathcal{D}_q$ of finite-difference operators in Novikov's variables. This turns out to be a consequence of our adelic characterization of quantum K-theory in terms of quantum cohomology theory, and of the $\mathcal{D}$-module structure (and hence of the divisor equation) present in quantum cohomology theory.

1. K-theoretic Gromov–Witten invariants

Let X be a *target* space, which we assume to be a nonsingular complex projective variety. Let $\overline{\mathcal{M}}_{g,n}^{X,d}$ denote Kontsevich's moduli space of degree-d stable maps to X of complex genus-g curves with n marked points. Denote by $L_1, \dots, L_n$ the line (orbi)bundles over $\overline{\mathcal{M}}_{g,n}^{X,d}$ formed by the cotangent lines to the curves at the respective marked points. When $a_1, \dots, a_n \in K^0(X)$, and $d_1, \dots, d_n \in \mathbb{Z}$, we use the correlator notation

$$\langle a_1 L^{d_1}, \dots, a_n L^{d_n} \rangle_{g,n}^{X,d}$$

for the holomorphic Euler characteristic over $\overline{\mathcal{M}}_{g,n}^{X,d}$ of the following sheaf:

$$\mathrm{ev}_1^*(a_1) L_1^{d_1} \dots \mathrm{ev}_n^*(a_n) L_n^{d_n} \otimes \mathcal{O}^{vir}.$$

Here $\mathrm{ev}_i : \overline{\mathcal{M}}_{g,n}^{X,d} \to X$ are the *evaluation* maps, and $\mathcal{O}^{vir}$ is the *virtual structure sheaf* of the moduli spaces of stable maps. The sheaf $\mathcal{O}^{vir}$ was introduced by Yuan-Pin Lee [20]. It is an element of the Grothendieck group of coherent sheaves on the *stack* $\overline{\mathcal{M}}_{g,n}^{X,d}$, and plays a role in K-theoretic version of GW-theory of X pretty much similar to the role of the virtual fundamental cycle $[\overline{\mathcal{M}}_{g,n}^{X,d}]^{vir}$ in cohomological GW-theory of X. According to [20], the collection of virtual structure sheaves on the spaces $\overline{\mathcal{M}}_{g,n}^{X,d}$ satisfies K-theoretic counterparts of Kontsevich–Manin's axioms [19] for Gromov–Witten invariants.

Note that, in contrast with cohomological GW-theory, where the invariants are rational numbers, *K-theoretic GW-invariants are integers.*

The following generating function for K-theoretic GW-invariants is called the *genus-0 descendant potential* of X:

$$\mathcal{F} := \sum_{n,d} \frac{Q^d}{n!} \langle t(L), \dots, t(L) \rangle_{0,n}^{X,d}.$$

Here Q^d denotes the monomial in the *Novikov ring*, the formal series completion of the semigroup ring of the *Mori cone* of X, where the monomial represents the degree d of rational curves in X, and t stands for any Laurent polynomial of one variable, L, with vector coefficients in $K^0(X)$. Thus, $\mathcal{F}$ is a formal function of t with Taylor coefficients in the Novikov ring.

2. The symplectic loop space formalism

Let $\mathbb{C}[[Q]]$ be the Novikov ring. Introduce the *loop space*

$$\mathcal{K} := \left[K^0(X) \otimes \mathbb{C}(q, q^{-1}) \right] \otimes \mathbb{C}[[Q]].$$

By definition, elements of K are Q-series whose coefficients are vector-valued rational functions on the complex circle with the coordinate q. It is a $\mathbb{C}[[Q]]$-module, but we often suppress Novikov's variables in our notation and refer to $\mathcal{K}$ as a linear "space." Moreover, abusing notation, we write $\mathcal{K} = K(q, q^{-1})$, where $K = K^0(X) \otimes \mathbb{C}[[Q]]$. We call elements of $\mathcal{K}$ "rational functions of q with coefficients in K," meaning that they are rational functions in the Q-adic sense, i.e. modulo any power of the maximal ideal in the Novikov ring.

We endow $\mathcal{K}$ with *symplectic form* Ω, which is a $\mathbb{C}[[Q]]$-valued non-degenerate anti-symmetric bilinear form:

$$\mathcal{K} \ni f, g \mapsto \ \Omega(f,g) = [\mathrm{Res}_{q=0} + \mathrm{Res}_{q=\infty}] \ \left(f(q), g(q^{-1})\right) \ \frac{dq}{q}.$$

Here $(\cdot,\cdot)$ stands for the K-theoretic intersection pairing on K:

$$(a,b) := \chi(X; a\otimes b) = \int_X \mathrm{td}(T_X)\,\mathrm{ch}(a)\,\mathrm{ch}(b).$$

It is immediate to check that the following subspaces in $\mathcal{K}$ are Lagrangian and form a *Lagrangian polarization*, $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$:

$$\mathcal{K}_+ = K[q,q^{-1}], \ \ \mathcal{K}_- = \{f \in \mathcal{K} \mid f(0) \neq \infty, \ f(\infty) = 0\},$$

i.e. $\mathcal{K}_+$ is the space of Laurent polynomials in q, and $\mathcal{K}_-$ consists of rational functions vanishing at $q = \infty$ and regular at $q = 0$.

The following generating function for K-theoretic GW-invariants is defined as a map $\mathcal{K}_+ \to \mathcal{K}$ and is nicknamed the *big*[2] *J-function* of X:

$$\mathcal{J}(t) := (1-q) + t(q) + \sum_a \Phi^a \sum_{n,d} \frac{Q^d}{n!} \langle \frac{\Phi_a}{1-qL}, t(L), \dots, t(L) \rangle_{0,n+1}^{X,d}.$$

The first summand, $1-q$, is called the *dilaton shift*, the second, $t(q)$, the *input*, and the sum of the two lies in $\mathcal{K}_+$. The remaining part consists of GW-invariants, with $\{\Phi_a\}$ and $\{\Phi^a\}$ being any Poincaré-dual bases of $K^0(X)$. It is a formal vector-valued function of $t \in \mathcal{K}_+$ with Taylor coefficients in $\mathcal{K}_-$.

Indeed, the moduli space $\overline{\mathcal{M}}_{0,n+1}^{X,d}$ is a "virtual orbifold" of finite dimension. In particular, in the K-ring of it, the line bundle L_1^{-1} satisfies a polynomial equation, $P(L_1^{-1}) = 0$, with $P(0) \neq 0$.[3] From $P(q) - P(L) = F(q,L)(L-q)$, where $\deg F < \deg P$, we find (by putting $L = L_1^{-1}$) that $1/(1-qL_1) = L_1^{-1}F(q, L_1^{-1})/P(q)$. Thus each correlator is a reduced rational function of q with no pole at $q = 0$ and a zero at $q = \infty$.

`Proposition.` *The big J-function coincides with the differential of the genus-0 descendant potential, considered as the section of the cotangent bundle $T^*\mathcal{K}_+$ which is identified with the symplectic loop space*

[2]In our terminology, specializing the Laurent polynomial t to a constant yields the *J-function* (without the adjective "big"), while taking $t = 0$ makes it the *small J-function*.

[3]On a manifold of complex dimension $< D$ we would have: $(L-1)^D = 0$ for any line bundle L, i.e. L would be unipotent. This may be false on an orbifold, as the minimal polynomial of a line bundle can vanish at roots of 1, but it does not vanish at 0 since L^{-1} exists.

by the Lagrangian polarization $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$ and the dilaton shift $f \mapsto f + (1-q)$:

$$\mathcal{J}(t) = 1 - q + t(q) + d_t\mathcal{F}.$$

`Proof.` To verify the claim, we compute the symplectic inner product of the $\mathcal{K}_-$-part of $\mathcal{J}(t)$, with a variation, $\delta t \in \mathcal{K}_+$, of the input, and show that it is equal to the value of the differential $d_t\mathcal{F}$ on δt. Note that, since δt has no poles other than $q = 0$ or ∞, we have (think of L in this identity as a letter):

$$\Omega\left(\sum_a \Phi^a \otimes \frac{\Phi_a}{1-qL}, \delta t\right) = -\Omega\left(\delta t, \sum_a \Phi^a \otimes \frac{\Phi_a}{1-qL}\right) =$$

$$-\left[\mathrm{Res}_{q=0} + \mathrm{Res}_{q=\infty}\right] \frac{\sum_a \delta t^a(q)\Phi_a}{1-q^{-1}L}\frac{dq}{q} = \mathrm{Res}_{q=L}\frac{\delta t(q)}{q-L} = \delta t(L).$$

Therefore the symplectic inner product in question is equal to

$$\sum_{n,d} \frac{Q^d}{n!}\langle \delta t(L), t(L), \dots, t(L)\rangle_{0,n+1}^{X,d} = (d_t\mathcal{F})(\delta t),$$

as claimed.

3. Overruled Lagrangian cones

A Lagrangian variety, $\mathcal{L}$, in the symplectic loop space $(\mathcal{K}, \Omega)$ is called an *overruled Lagrangian cone* if $\mathcal{L}$ is a cone with the vertex at the origin, and if for every regular point of $\mathcal{L}$, the tangent space, T, is tangent to $\mathcal{L}$ along the whole subspace $(1-q)T$. In particular: (i) tangent spaces are invariant with respect to multiplication by $q-1$, (ii) the subspaces $(q-1)T$ lie in $\mathcal{L}$ (so that $\mathcal{L}$ is ruled by a finite-parametric family of such subspaces), and (iii) the tangent spaces at all regular points in a ruling subspace $(q-1)T$ are the same and equal to T.

`Theorem` ([13]). *The range of the big J-function $\mathcal{J}$ of quantum K-theory of X is a formal germ at $\mathcal{J}(0)$ of an overruled Lagrangian cone.*

`Proof.` As explained in [13], this is a consequence of the relation between descendants and *ancestors*.

The *ancestor correlators* of quantum K-theory

$$K^0(X) \ni \tau \;\mapsto\; \langle a_1\bar{L}^{d_1}, \dots, a_n\bar{L}^{d_n}\rangle_{g,n}^{X,d}(\tau),$$

are defined as formal power series of holomorphic Euler characteristics

$$\sum_{l=0}^{\infty}\frac{1}{l!}\chi\left(\overline{\mathcal{M}}_{g,n+l}^{X,d}; \mathcal{O}^{vir}\,\mathrm{ev}_1^*(a_1)\bar{L}_1^{d_1}\cdots \mathrm{ev}_n^*(a_n)\bar{L}_n^{d_n}\,\mathrm{ev}_{n+1}^*(\tau)\cdots \mathrm{ev}_{n+l}^*(\tau)\right),$$

where $\bar{L}_i$, the "ancestor" bundles, are pull-backs of the universal cotangent line bundles L_i on the Deligne-Mumford space $\overline{\mathcal{M}}_{g,n}$ by the *contraction* map $\mathrm{ct} : \overline{\mathcal{M}}^{X,d}_{g,n+l} \to \overline{\mathcal{M}}_{g,n}$. The latter map involves forgetting the map of holomorphic curves to the target space as well as the last l marked points.

The genus-0 *ancestor potential* is defined by

$$\overline{\mathcal{F}}_\tau := \sum_{n,d} \frac{Q^d}{n!} \langle \bar{t}(\bar{L}), \dots, \bar{t}(\bar{L}) \rangle^{X,d}_{0,n}(\tau)$$

and depends on $\bar{t}$ and τ. The graph of its differential is identified in terms of the ancestor version of the big J-function:

$$\overline{\mathcal{J}} = 1 - q + \bar{t}(q) + \sum_{a,b} \Phi_a G^{ab}(\tau) \sum_{n,d} \frac{Q^d}{n!} \langle \frac{\Phi_b}{1 - q\bar{L}}, \bar{t}(\bar{L}), \dots, \bar{t}(\bar{L}) \rangle^{X,d}_{0,n+1}(\tau).$$

Here $\left(G^{ab}\right) = (G_{ab})^{-1}$, and

$$G_{ab}(\tau) := (\Phi_a, \Phi_b) + \sum_{n,d} \frac{Q^d}{n!} \langle \Phi_a, \tau, \dots, \tau, \Phi_b \rangle^{X,d}_{0,2+n}.$$

In the ancestor version of the symplectic loop space formalism, the loop space and its polarization $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$ are the same as in the theory of descendants, but the symplectic form Ω_τ is based on the pairing tensor (G_{ab}) rather than the constant Poincaré pairing (Φ_a, Φ_b).

Let $\mathcal{L} \subset (\mathcal{K}, \Omega)$ and $\overline{\mathcal{L}}_\tau \subset (\mathcal{K}, \Omega_\tau)$ be Lagrangian submanifolds defined by the descendant and ancestor J-functions $\mathcal{J}$ and $\overline{\mathcal{J}}$. Then

$$\overline{\mathcal{L}}_\tau = S_\tau \mathcal{L},$$

where $S_\tau : \mathcal{K} \to \mathcal{K}_\tau$ is an isomorphism of the symplectic loop spaces, defined by the following matrix $S_\tau = (S^a_b)$:

$$S^a_b = \delta^a_b + \sum_{l,d} \frac{Q^d}{l!} \sum_\mu g^{a\mu} \langle \Phi_\mu, \tau, \dots, \tau, \frac{\Phi_b}{1 - qL} \rangle^{X,d}_{0,2+n}.$$

It is important that the genus-0 Deligne-Mumford spaces $\overline{\mathcal{M}}_{0,n}$ are manifolds (of dimension $n-3$). Consequently, the line bundles $\bar{L}_i$ are unipotent. Moreover, at the points $\bar{t} \in \mathcal{K}_+$ with $\bar{t}(1) = 0$ the ancestor potential $\overline{\mathcal{F}}_\tau$ has all partial derivatives of order < 3 equal to 0. In geometric terms, the cone $\overline{\mathcal{L}}_\tau$ is tangent to $\mathcal{K}_+$ along $(1-q)\mathcal{K}_+$. This means that the cone $\mathcal{L}$ is swept by ruling subspaces $(1-q)S_\tau^{-1}\mathcal{K}_+$ parametrized by $\tau \in K$, and that each Lagrangian subspace $S_\tau \mathcal{K}_+$ is tangent to $\mathcal{L}$ along the corresponding ruling subspace. The theorem follows.

The proof of the relationship $\overline{\mathcal{L}} = S_\tau \mathcal{L}$ is based on comparison of the bundles L_i and $\bar{L}_i$, and is quite similar to the proof of the corresponding cohomological theorem given in Appendix 2 of [6]. It uses the K-theoretic version of the WDVV-identity introduced in [10], as well as the *string* and *dilaton* equations.

The genus-0 dilaton equation can be derived from the geometric fact $(\mathrm{ft}_1)_*(1 - L_1) = 2 - n$ about the K-theoretic push-forward along the map $\mathrm{ft}_1 : \overline{\mathcal{M}}_{0,n+1}^{X,d} \to \overline{\mathcal{M}}_{0,n}^{X,d}$ forgetting the first marked point. It leads to the relation

$$\langle 1 - L, t(L), \dots, t(L)\rangle_{0,n+1}^{X,d} = (2 - n)\langle t(L), \dots, t(L)\rangle_{0,n}^{X,d}.$$

The latter translates into the degree-2 homogeneity of $\mathcal{F}$ with respect to the dilaton-shifted origin, and respectively to the conical property of $\mathcal{L}$.

The string equation is derived from $(\mathrm{ft}_1)_* 1 = 1$ (thanks to rationality of the fibers of the forgetting map) and relationships between $\mathrm{ft}_1^*(L_i)$ and L_i for $i > 1$ (see for instance [10]). It can be stated as the tangency to the cone $\mathcal{L}$ of the linear vector field in $\mathcal{K}$ defined by the operator of multiplication by $1/(1 - q)$. The operator of multiplication by

$$\frac{1}{1-q} - \frac{1}{2} = \frac{1}{2}\,\frac{1+q}{1-q}$$

is anti-symmetric with respect to Ω and thus defines a linear *Hamiltonian* vector field. Since $\mathcal{L}$ is a cone, this vector field is also tangent to $\mathcal{L}$, which lies therefore on the zero level of its quadratic Hamilton function. This gives another, *Hamilton-Jacobi form* of the string equation.

4. Hirzebruch–Riemann–Roch formula for orbifolds

Given a compact complex manifold M equipped with a holomorphic vector bundle E, the *Hirzebruch–Riemann–Roch formula* [15] provides a cohomological expression for the *super-dimension* (i.e. Euler characteristic) of the sheaf cohomology:

$$\chi(M, E) := \dim H^\bullet(M, E) = \int_M \mathrm{td}(T_M)\ \mathrm{ch}(E).$$

The generalization of this formula to the case when M is an orbifold and E an orbibundle is due to T. Kawasaki [17]. It expresses $\chi(M, E)$ as an integral over the *inertia orbifold* IM of M:

$$\chi(M, E) = \int_{[IM]} \mathrm{td}(T_{IM})\ \mathrm{ch}\left(\frac{\mathrm{Tr}(E)}{\mathrm{Tr}(\bigwedge^\bullet N_{IM}^*)}\right).$$

By definition, the structure of an n-dimensional complex orbifold on M is given by an atlas of local charts $U \to U/G(x)$, the quotients of neighborhoods of the origin in $\mathbb{C}^n$ by (linear) actions of finite *local symmetry groups* (one group $G(x)$ for each point $x \in M$).

By definition, charts on the inertia orbifold IM have the form $U^g \to U^g/Z_g(x)$, where U^g is the fixed point locus of $g \in G(x)$, and $Z_g(x)$ is the centralizer of g in $G(x)$. For elements g from the same conjugacy class, the charts are canonically identified by the action of $G(x)$. Thus, locally near $x \in M$, connected components of the inertia orbifold are labeled by conjugacy classes, $[g]$, in $G(x)$. Integration over the fundamental class $[IM]$ involves the division by the order of the stabilizer of a typical point in U^g (and hence by the order of g at least).

Near a point $(x, [g]) \in IM$, the *tangent* and *normal* orbibundles T_{IM} and N_{IM} are identified with the tangent bundle to U^g and normal bundle to U^g in U respectively.

The Kawasaki's formula makes use of the obvious lift to IM of the orbibundle E on M. By $\bigwedge^\bullet N_{IM}^*$, we have denoted the *K-theoretic Euler class* of N_{IM}, i.e. the exterior algebra of the dual bundle, considered as a $\mathbb{Z}_2$-graded bundle (the "Koszul complex").

The fiber F of an orbibundle on IM at a point $(x, [g])$ carries the direct decomposition into the sum of eigenspaces F_λ of g. By $\mathrm{Tr}(F)$ we denote the *trace bundle*[4], the virtual orbibundle

$$\mathrm{Tr}(F) := \sum_\lambda \lambda F_\lambda.$$

The denominator in Kawasaki's formula is invertible because g does not have eigenvalue 1 on the normal bundle to its fixed point locus.

Finally, td and ch denote the *Todd class* and *Chern character*.

When M is a global quotient, $\widetilde{M}/G$, of a manifold by a finite group, and E is a G-equivariant bundle over $\widetilde{M}$, Kawasaki's result reduces to Lefschetz' holomorphic fixed point formula for super-traces in the sum

$$\chi(M, E) = \dim H^\bullet(\widetilde{M}, E)^G = \frac{1}{|G|} \sum_{g \in G} \mathrm{tr}\left(g \mid H^\bullet(\widetilde{M}, E)\right).$$

The orbifold M is contained in its inertia orbifold IM as the component corresponding to the identity elements of local symmetry groups. The corresponding term of Kawasaki's formula is

$$\chi^{\mathrm{fake}}(M, E) := \int_M \mathrm{td}(T_M)\ \mathrm{ch}(E).$$

[4]In fact, *super-trace*, if the bundle is $\mathbb{Z}_2$-graded.

We call it the *fake holomorphic Euler characteristic* of E. It is generally speaking a rational number, while the "true" holomorphic Euler characteristic $\chi(M, E)$ is an integer.

Note that *the right hand side of Kawasaki's formula is the fake holomorphic Euler characteristic of an orbibundle,* $\operatorname{Tr}(E)/\operatorname{Tr}(\bigwedge^\bullet(N^*_{IM}))$, *on the inertia orbifold.*

Our goal in this paper is to use Kawasaki's formula for expressing genus-0 K-theoretic GW-invariants in terms of cohomological ones. We refer to [22] (see also the thesis [21]) for the *virtual* version of Kawasaki's theorem, which justifies application of the formula to moduli spaces of stable maps.

The moduli spaces of stable maps are Deligne–Mumford stacks, i.e. locally are quotients of spaces by finite groups. The local symmetry groups $G(x)$ are automorphism groups of stable maps. A point in the inertia stack $I\overline{\mathcal{M}}_{0,n}^{X,d}$ is specified by a pair: a stable map to the target space and an automorphism of the map. In a sense, a component of the inertia stack parametrizes stable maps with prescribed symmetry.

The components themselves are moduli spaces naturally equipped with virtual fundamental cycles and virtual structure sheaves. In fact, they are glued from moduli spaces of stable maps of smaller degrees — quotients of symmetric stable maps by the symmetries. Thus the individual integrals of Kawasaki's formula can be set up as certain invariants of *fake* quantum K-theory, i.e. fake holomorphic Euler characteristics of certain orbibundles on spaces glued from usual moduli spaces of stable maps.

Our plan is to identify these invariants in terms of conventional ones and express them — and thereby the "true" genus-0 K-theoretic Gromov-Witten theory — in terms of cohomological GW-invariants.

For this, a summary of relevant results about fake quantum K-theory, including the Quantum Hirzebruch–Riemann–Roch Theorem of Coates–Givental [4, 7], will be necessary.

5. The fake quantum K-theory

Fake K-theoretic GW-invariants are defined by

$$\langle a_1 L^{d_1}, \dots, a_n L^{d_n}\rangle_{g,n}^{X,d} := \int_{\left[\overline{\mathcal{M}}_{g,n}^{X,d}\right]^{vir}} \operatorname{td}\left(T_{\overline{\mathcal{M}}_{g,n}^{X,d}}\right)\ \operatorname{ch}\left(\operatorname{ev}_1^*(a_1)L_1^{d_1}\dots \operatorname{ev}_n^*(a_n)L_n^{d_n}\right),$$

i.e. as cohomological GW-invariants involving the Todd class of the *virtual tangent bundle* to the moduli spaces of stable maps.

The Chern characters $\mathrm{ch}(L_i)$ are unipotent, and as a result, generating function for the fake invariants are defined on the space of formal power series of $L-1$. In particular, the big J-function

$$\mathcal{J}^{\text{fake}} := 1-q+t(q)+\sum_a \Phi^a \sum_{n,d} \frac{Q^d}{n!}\langle \frac{\Phi_a}{1-qL}, t(L),\dots,t(L)\rangle_{0,n+1}^{X,d}$$

takes an input t[5] from the space $\mathcal{K}_+^{\text{fake}} = K[[q-1]]$ of power series in $q-1$ with vector coefficients, and takes values in the loop space

$$\mathcal{K}^{\text{fake}} := \left\{ \begin{array}{l} Q\text{-series whose coefficients} \\ \text{are Laurent series in } q-1 \end{array} \right\}.$$

The symplectic form is defined by

$$\Omega^{\text{fake}}(f,g) := -\operatorname{Res}_{q=1}\big(f(q), g(q^{-1})\big)\frac{dq}{q}.$$

Expand $1/(1-qL)$ into a series of powers of $L-1$:

$$\frac{1}{1-qL} = \sum_{k\geq 0}(L-1)^k \frac{q^k}{(1-q)^{k+1}}.$$

According to [7], we obtain a Darboux basis:

$$\Phi_a(q-1)^k,\ \Phi^a q^k/(1-q)^{k+1},\ a=1,\dots,\dim K^0(X),\ k=0,1,2,\dots$$

Taking $\mathcal{K}_-^{\text{fake}}$ to be spanned over K by $q^k/(1-q)^{k+1}$, we obtain a Lagrangian polarization of $(\mathcal{K}^{\text{fake}}, \Omega^{\text{fake}})$. As before, the big J-function coincides, up to the dilaton shift $1-q$, with the graph of the differential of the genus-0 descendant potential: $\mathcal{J}^{\text{fake}}(t) = 1-q+t(q)+d_t\mathcal{F}^{\text{fake}}$.

The range of the function $\mathcal{J}^{\text{fake}}$ forms (a formal germ at $\mathcal{J}(0)$ of) an overruled Lagrangian cone, $\mathcal{L}^{\text{fake}}$. The proof is based on the relationship [13] between gravitational descendants and ancestors of fake quantum K-theory, which looks identical to the one in "true" K-theory (although the values of fake and true GW-invariants disagree).

In fact the whole setup for fake GW-invariants can be made purely topological, extended to include $K^1(X)$, and moreover, generalized to all complex-orientable extraordinary cohomology theories (i.e. complex cobordisms). In this generality, the quantum Hirzebruch–Riemann–Roch theorem of [4, 7] expresses the fake GW-invariants (of all genera) in terms of the cohomological gravitational descendants. The special

[5] Note that we still treat our generating functions as formal in t. In particular, an input here is a series in $q-1$ whose coefficients can be arbitrary as long as they remain "small". In practice they will be the sums of indeterminates (like t, which are small in their own, t-adic topology) with constants taken from the maximal ideal of Novikov's ring (and thus small in the Q-adic sense).

case we need is stated below, after a summary of the symplectic loop space formalism of quantum cohomology theory.

Take $H = H^{even}(X) \otimes \mathbb{Q}[[Q]]$, and $(a,b)^H = \int_X ab$. Let $\mathcal{H}$ denote the space of power Q-series whose coefficients are Laurent series in one indeterminate, z. Abusing notation we write: $\mathcal{H} = H((z))$, (remembering that elements of $\mathcal{H}$ are Laurent series only modulo any power of Q). Define in $\mathcal{H}$ the symplectic form

$$\Omega^H(f,g) = \mathrm{Res}_{z=0}\,(f(-z),g(z))^H\,dz,$$

and Lagrangian polarization

$$\mathcal{H}_+ = H[[z]], \quad \mathcal{H}_- = z^{-1}H[z^{-1}].$$

Using Poincaré-dual bases of H, and the notation $\psi = c_1(L)$, we define the big J-function of cohomological GW-theory

$$\mathcal{J}^H = -z + t(z) + \sum_a \phi^a \sum_{n,d} \frac{Q^d}{n!}\langle \frac{\phi_a}{-z-\psi}, t(\psi),\dots,t(\psi)\rangle_{0,n+1}^{X,d}.$$

It takes inputs t from $\mathcal{H}_+$, takes values[6] in $\mathcal{H}$, and coincides with the graph of differential of the cohomological genus-0 descendant potential, $\mathcal{F}^H$, subject to the dilaton shift $-z$: $\mathcal{J}^H(t) = -z + t(z) + d_t\mathcal{F}^H$. Here

$$\mathcal{F}^H := \sum_{n,d} \frac{Q^d}{n!}\langle t(\psi),\dots,t(\psi)\rangle_{0,n}^{X,d},$$

where for $a_i \in H$ and $d_i \geq 0$, we have:

$$\langle a_1\psi^{d_1},\dots,a_n\psi_n^{d_n}\rangle_{0,n}^{X,d} := \int_{\left[\overline{\mathcal{M}}_{0,n+1}^{X,d}\right]} \mathrm{ev}_1^*(a_1)c_1(L_1)^{d_1}\cdots \mathrm{ev}_n^*(a_n)c_1(L_n)^{d_n}.$$

The range of the function $\mathcal{J}^H$ is a Lagrangian cone, $\mathcal{L}^H \subset \mathcal{H}$, overruled in the sense that its tangent spaces, T, are tangent to $\mathcal{L}^H$ along zT (see Appendix 2 in [6]).

`Theorem` ([7], see details in [4]). *Denote by $\triangle$ the Euler–Maclaurin asymptotic of the infinite product*

$$\triangle \sim \prod_{\text{Chern roots } x \text{ of } T_X} \prod_{r=1}^{\infty} \frac{x - rz}{1 - e^{-x+rz}}.$$

[6]The previous footnote about fake K-theory applies here too. In particular, for the formal function, to assume *values* in $\mathcal{H}$ merely means that the *coefficients* of it as a formal t-series become Laurent series in z when reduced modulo a power of Q.

Identify $\mathcal{K}^{fake}$ with $\mathcal{H}$ using the Chern character isomorphism $\mathrm{ch}: K \to H$ *and* $\mathrm{ch}(q) = e^z$. *Then $\mathcal{L}^{\text{fake}}$ is obtained from $\mathcal{L}^H$ by the pointwise multiplication on $\mathcal{H}$ by $\triangle$:*

$$\mathrm{ch}\left(\mathcal{L}^{\text{fake}}\right) = \triangle \mathcal{L}^H.$$

`Remarks.` (1) Given a function $x \mapsto s(x)$, the *Euler–Maclaurin asymptotics* of $\prod_{r=1}^{\infty} e^{s(x-rz)}$ is obtained by the formal procedure:

$$\sum_{r=1}^{\infty} s(x-rz) = \left(\sum_{r=1}^{\infty} e^{-rz\partial_x}\right) s(x) = \frac{z\partial_x}{e^{z\partial_x}-1}(z\partial_x)^{-1}s(x)$$
$$= \frac{s^{(-1)}(x)}{z} - \frac{s(x)}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} s^{(2k-1)}(x) z^{2k-1},$$

where $s^{(k)} = d^k s/dx^k$, $s^{(-1)}$ is the anti-derivative $\int_0^x s(\xi)d\xi$, and B_{2k} are Bernoulli numbers. Taking $e^{s(x)}$ to be the Todd series, $x/(1-e^{-x})$, and summing over the Chern roots x of the tangent bundle T_X, we get:

$$\triangle = \frac{1}{\sqrt{\mathrm{td}(T_X)}} \exp\left\{\sum_{k\geq 0}\sum_{l\geq 0} s_{2k-1+l} \frac{B_{2k}}{(2k)!} \mathrm{ch}_l(T_X) z^{2k-1}\right\},$$

where the coefficients s_l hide another occurrence of Bernoulli numbers:

$$e^{\sum_{l\geq 0} s_l x^l/l!} = \frac{x}{1-e^{-x}} = 1 + \frac{x}{2} + \sum_{l=1}^{\infty} \frac{B_{2l}}{(2l)!} x^{2l}.$$

(2) Note that neither $\mathrm{ch}: \mathcal{K} \to \mathcal{H}$ nor $\triangle: \mathcal{H} \to \mathcal{H}$ is symplectic: the former because $(a,b)^{\text{fake}} = (\mathrm{ch}(a), \mathrm{td}(T_X)\,\mathrm{ch}(b))^H \neq (\mathrm{ch}(a), \mathrm{ch}(b))^H$, the latter because of the factor $\mathrm{td}(T_X)^{-1/2}$. However the composition $\mathrm{ch}^{-1} \circ \triangle : \mathcal{H} \to \mathcal{K}^{\text{fake}}$ is symplectic.

(3) The transformation between cohomological and K-theoretic J-functions (or descendant potentials) encrypted by the theorem, involves three aspects. One is the transformation $\triangle$, while the other two are the changes of the polarization and dilaton shift. Namely, $\mathrm{ch}^{-1}: \mathcal{H} \to \mathcal{K}^{\text{fake}}$ maps $\mathcal{H}_+$ to $\mathcal{K}_+$ but does *not* map $\mathcal{H}_-$ to $\mathcal{K}_-^{\text{fake}}$, and there is a discrepancy between the dilaton shifts: $\mathrm{ch}^{-1}(-z) = \log q^{-1} \neq 1-q$.

(4) Since $\mathcal{L}^{\text{fake}}$ is an overruled cone, it is invariant under the multiplication by the ratio $(1-q)/\log q^{-1}$. This shows one way of correcting for the discrepancy in dilaton shifts.

(5) The proof of the theorem does not exploit any properties of overruled cones. One uses the family $\mathrm{td}_\epsilon(x) := \epsilon x/(1-e^{-\epsilon x})$ of "extraordinary" Todd classes to interpolate between cohomology and K-theory, and establishes an infinitesimal version of the theorem. For this, the

twisting classes $\mathrm{td}_\epsilon(T_{\overline{\mathcal{M}}^{X,d}_{g,n}})$ of the moduli spaces are expressed in terms of the descendant classes by applying the Grothendieck–Riemann–Roch formula to the fibrations $\mathrm{ft}_{n+1} : \overline{\mathcal{M}}^{X,d}_{g,n+1} \to \overline{\mathcal{M}}^{X,d}_{g,n}$.

We refer for all details to the dissertation [4]. However, in Section 8, we indicate geometric origins of the three changes described by the theorem: the change in the position of the cone, in the dilaton shift, and in the polarization.

6. Adelic characterization of quantum K-theory

Recall that point $f \in \mathcal{K}$ is a series in the Novikov variables, Q, with vector coefficients which are rational functions of $q^{\pm 1}$. For each $\zeta \neq 0, \infty$, we expand (coefficients of) f in a Laurent series in $1 - q\zeta$ and thus obtain the *localization* f_ζ near $q = \zeta^{-1}$. Note that for $\zeta = 1$, the localization lies in the loop space $\mathcal{K}^{\text{fake}}$ of fake quantum K-theory. The main result of the present paper is the following theorem, which provides a complete characterization of the true quantum K-theory in terms of the fake one.

`Theorem.` *Let $\mathcal{L} \subset \mathcal{K}$ be the overruled Lagrangian cone of quantum K-theory of a target space X. If $f \in \mathcal{L}$, then the following conditions are satisfied:*

(i) *f has no pole at $q = \zeta^{-1} \neq 0, \infty$ unless ζ is a root of* 1.

(ii) *When $\zeta = 1$, the localization f_ζ lies in $\mathcal{L}^{\text{fake}}$.*

In particular, the localization $\mathcal{J}(0)_1$ at $\zeta = 1$ of the value of the J-function with the input $t = 0$ lies in $\mathcal{L}^{\text{fake}}$. In the tangent space to $\mathcal{L}^{\text{fake}}$ at the point $\mathcal{J}(0)_1$, make the change $q \mapsto q^m$, $Q^d \mapsto Q^{md}$, and denote by $\mathcal{T}$ the resulting subspace in $\mathcal{K}^{\text{fake}}$. Let ∇_ζ denote the Euler–Maclaurin asymptotics as $q\zeta \to 1$ of the infinite product:

$$\nabla_\zeta \sim_{q\zeta \to 1} \prod_{\substack{\text{K-theoretic Chern} \\ \text{roots } P \text{ of } T_X^*}} \frac{\prod_{r=1}^{\infty}(1 - q^{mr}P)}{\prod_{r=1}^{\infty}(1 - q^r P)}.$$

(iii) *If $\zeta \neq 1$ be a primitive m-th root of 1, then $\left(\nabla_\zeta^{-1} f_\zeta\right)(q/\zeta) \in \mathcal{T}$.*

Conversely, if $f \in \mathcal{K}$ satisfies conditions (i),(ii),(iii), *then $f \in \mathcal{L}$.*

`Remarks.` (1) The cone $\mathcal{L}$ is a formal germ at $\mathcal{J}(0)$. The statements (direct and converse) about "points" $f \in \mathcal{L}$ are to be interpreted in the spirit of formal geometry: as statements about *families* based at $\mathcal{J}(0)$.

(2) K-theoretic Chern roots P are characterized by $\mathrm{ch}(P) = e^{-x}$ where x are cohomological Chern roots of T_X.

(3) After the substitution $q\zeta = e^z$ the infinite product becomes

$$\prod_{\text{Chern roots } x} \prod_{k=1}^{m-1} \prod_{r=0}^{\infty} (1-\zeta^{-k}e^{kz}e^{-x+mrz})^{-1}.$$

The Euler–Maclaurin expansion has the form

$$\log \nabla_\zeta = \frac{s^{(-1)}}{mz} + \frac{s}{2} + \sum_{k>0} \frac{B_{2k}}{(2k)!}(mz)^{2k-1} s^{(2k-1)},$$

where s also depends on z as a parameter:

$$s(x,z) = -\log \prod_x \prod_{k=1}^{m-1} (1-\zeta^{-k}e^{kz}e^{-x}).$$

Note that since x are nilpotent, $s(x,z)$ is polynomial in x with coefficients which expand into power series of z. The scalar factor of ∇_ζ is $e^{s(0,0)/2} = m^{-\dim X/2}$ since for each of $\dim X$ Chern roots,

$$\lim_{x\to 0} \prod_{k=1}^{m-1} (1-\zeta^{-k}e^{-x}) = \lim_{x\to 0} \frac{1-e^{-mx}}{1-e^{-x}} = m.$$

(4) The (admittedly clumsy) definition of subspace $\mathcal{T}$ can be clarified as follows. The tangent space to $\mathcal{L}^{\text{fake}}$ at the point $\mathcal{J}(0)_1$ is the range of the linear map $S^{-1} : \mathcal{K}_+^{\text{fake}} \to \mathcal{K}^{\text{fake}}$, where S^{-1} is a matrix Laurent series in $q-1$ with coefficients in the Novikov ring (see Section 3). Let $\widetilde{S}$ be obtained from S by the change $q \mapsto q^m$, $Q^d \mapsto Q^{md}$. Then $\mathcal{T} := \widetilde{S}^{-1}\mathcal{K}_+^{\text{fake}}$.

(5) The condition (iii) seems ineffective, since it refers to a tangent space to the cone $\mathcal{L}^{\text{fake}}$ at a yet unknown point $\mathcal{J}(0)_1$. However, we will see later that the three conditions together allow one, at least in principle, to compute the values $\mathcal{J}(t)$ for any input t, assuming that the cone $\mathcal{L}^{\text{fake}}$ is known, in a procedure recursive on degrees of stable maps. In particular, this applies to $\mathcal{J}(0)_1$. The cone $\mathcal{L}^{\text{fake}}$, in its turn, is expressed through $\mathcal{L}^H$, thanks to the quantum HRR theorem of the previous section, by a procedure which in principle has a similar recursive nature. Altogether, our theorem expresses all genus-0 K-theoretic gravitational descendants in terms of the cohomological ones. Thus this result indeed qualifies for the name: *the Hirzebruch–Riemann–Roch theorem of true genus-0 quantum K-theory.*

We describe here a more geometric (and more abstract) formulation of the theorem using the *adelic* version of the symplectic loop space formalism.

For each $\zeta \neq 0, \infty$, let $\mathcal{K}^\zeta$ be the space of power Q-series with vector Laurent series in $1 - q\zeta$ as coefficients. Define the symplectic form

$$\Omega^\zeta(f, g) = -\operatorname{Res}_{q=\zeta^{-1}} \left(f(q), g(q^{-1})\right) \frac{dq}{q},$$

and put $\mathcal{K}_+^\zeta := K[[1 - q\zeta]]$. The *adele space* is defined as the subset in the Cartesian product:

$$\widehat{\mathcal{K}} \subset \prod_{\zeta \neq 0, \infty} \mathcal{K}^\zeta$$

consisting of collections $f_\zeta \in \mathcal{K}^\zeta$ such that, modulo any power of Novikov's variables, $f_\zeta \in \mathcal{K}_+$ for all but finitely many values of ζ. The adele space is equipped with the product symplectic form:

$$\widehat{\Omega}(f, g) = -\sum \operatorname{Res}_{q=\zeta^{-1}} \left(f_\zeta(q), g_\zeta(q^{-1})\right) \frac{dq}{q}.$$

Next, there is a map $\mathcal{K} \to \widehat{\mathcal{K}} : f \mapsto \widehat{f}$, which to a rational function of $q^{\pm 1}$ assigns the collection (f_ζ) of its localizations at $q = \zeta^{-1} \neq 0, \infty$. Due to the residue theorem, the map is symplectic:

$$\Omega(f, g) = \widehat{\Omega}(\widehat{f}, \widehat{g}).$$

Given a collection $\mathcal{L}^\zeta \subset (\mathcal{K}^\zeta, \Omega^\zeta)$ of overruled Lagrangian cones such that modulo any power of Novikov's variables, $\mathcal{L}^\zeta = \mathcal{K}_+^\zeta$ for all but finitely many values of ζ, the product $\prod_{\zeta \neq 0, \infty} \mathcal{L}^\zeta \subset \widehat{\mathcal{K}}$ becomes an *adelic* overruled Lagrangian cone in the adele symplectic space.

In fact, "overruled" implies invariance of tangent spaces under multiplication by $1 - q$. Since $1 - q$ is invertible at $q = \zeta^{-1} \neq 1$, all $\mathcal{L}^\zeta$ with $\zeta \neq 1$ must be linear subspaces.

According to the theorem, *the image $\widehat{\mathcal{L}} \subset \widehat{\mathcal{K}}$ of the cone $\mathcal{L} \subset \mathcal{K}$ under the map* $\widehat{\ }: \mathcal{K} \to \widehat{\mathcal{K}}$ *followed by a suitable adelic (pointwise) completion, is an adelic overruled Lagrangian cone:*

$$\widehat{\mathcal{L}} = \prod_{\zeta \neq 0, \infty} \mathcal{L}^\zeta,$$

where $\mathcal{L}^\zeta = \mathcal{K}_+^\zeta$ unless ζ is a root of 1, $\mathcal{L}^\zeta = \mathcal{L}^{\text{fake}}$ when $\zeta = 1$, and $\mathcal{L}^\zeta = \nabla_\zeta \mathcal{T}^\zeta$ when $\zeta \neq 1$ is a root of 1, $\mathcal{T}^\zeta \subset \mathcal{K}^\zeta$ being obtained from the subspace $\mathcal{T} \subset \mathcal{K}^{\text{fake}}$ (described in the theorem) by the isomorphism $\mathcal{K}^{\text{fake}} \to \mathcal{K}^\zeta$ induced by the change $q \mapsto q\zeta$.

`Corollary.` *Two points $f, g \in \mathcal{L}$ lie in the same ruling space of $\mathcal{L}$ if and only if their expansions f_1, g_1 near $q = 1$ lie in the same ruling space of $\mathcal{L}^{\text{fake}}$.*

`Proof.` If f_1, g_1 lie in the same ruling space of $\mathcal{L}^{\text{fake}}$, then $\epsilon \widehat{f} + (1-\epsilon)\widehat{g} \in \widehat{\mathcal{L}}$ for each value of ϵ, and therefore, by the theorem, the whole line $\epsilon f + (1-\epsilon) g$ lies in $\mathcal{L}$. The converse is, of course, also true: if the line through f, g lies in $\mathcal{L}$ then the line through f_1, g_1 lies in $\mathcal{L}^{\text{fake}}$. It remains to notice that ruling spaces of $\mathcal{L}$ or $\mathcal{L}^{\text{fake}}$ are *maximal* linear subspaces of these cones (because this is true modulo Novikov's variables, i.e. in the classical K-theory). ■

7. Applying Kawasaki's formula

Here we begin our proof of the theorem formulated in the previous section. The big J-function (see Section 2) consists of the *dilaton shift* $1-q$, the *input* $t(q)$, and holomorphic Euler characteristics of bundles on virtual orbifolds $\overline{\mathcal{M}}_{0,n+1}^{X,d}$. The Euler characteristics can be expressed, by applying Kawasaki's formula, as sums of fake holomorphic Euler characteristics over various strata of the inertia stacks $I\overline{\mathcal{M}}_{0,n+1}^{X,d}$. A point in the inertia stack is represented by a stable map with symmetry (an automorphism, possibly trivial one). A stratum is singled out by the combinatorics of such a curve with symmetry. Figure 1 below is our book-keeping device for cataloging all the strata.

Let us call what is written in a given seat of a correlator the *content* of that seat. In the J-function, the content of the first marked point has the factor $1/(1-qL)$. We call this marked point the *horn*.

Given a stable map with symmetry, we focus our attention on the horn. The symmetry preserves the marked point and acts on the *co*tangent line at this point with an eigenvalue, which we denote ζ. In Figure 1, contributions of strata with $\zeta = 1$ are separated from those where $\zeta \neq 1$, in which case ζ is a primitive root of 1 of certain order $m \neq 1$.

When $\zeta = 1$, the symmetry is trivial on the irreducible component of the curve carrying the horn. In the curve, we single out the *maximal* connected subcurve containing the horn on which the symmetry is trivial, and call this subcurve (and the restriction to it of the stable map) the *head*.

The heads themselves are stable maps without symmetry, and are parametrized by moduli spaces $\overline{\mathcal{M}}_{0,n'+1}^{X,d'}$. Apart from the horn, the n' marked points are either marked points of the whole curve or the nodes where "arms" are attached. An *arm* is a stable map obtained as a connected component of what is left of the original curve when the head is removed. The arm has its own horn — the nodal point where it is attached to the head. An arm can be *any* stable map with

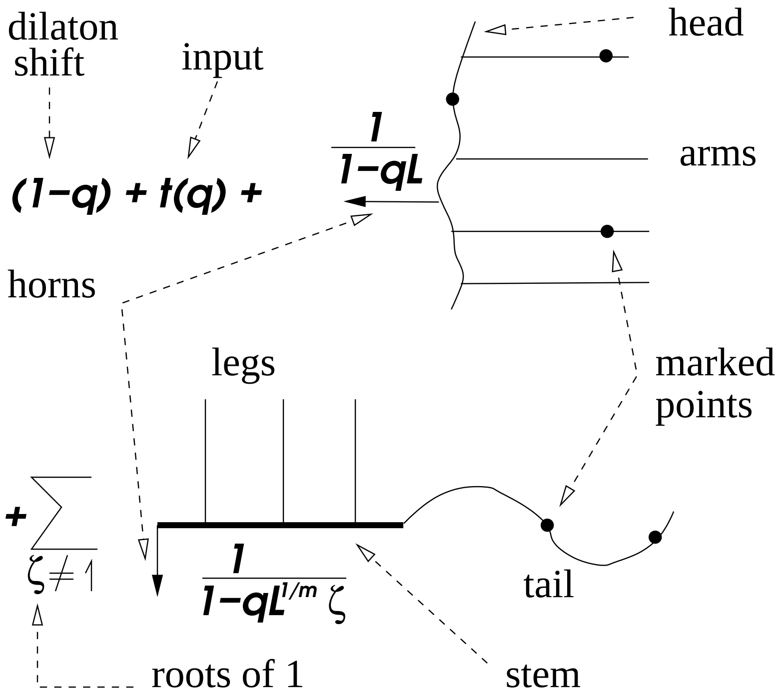

Figure 1. Cataloging the strata.

symmetry, with the only restriction: at its horn, the eigenvalue of the symmetry $\neq 1$ (because otherwise the head could be increased).

In Figure 1, contributions of the strata with the eigenvalues $\zeta \neq 1$ are appended into the sum $\sum$. If g denotes the symmetry of the stable map, and ζ is a primitive mth root of 1, then g^m acts trivially on the component carrying the horn. We single out the *maximal* connected subcurve on which g^m is trivial. Then the restriction of the stable map to it has g as a symmetry of order m. We call the quotient stable map (of the quotient curve) the *stem*. We will come back soon to a detailed discussion of "legs" and "tails" attached to the stems.

Let us denote by L the universal cotangent line (on the moduli space of stems) at the horn. The content in the fake holomorphic Euler characteristic represented by this term in the sum $\sum$ has the factor $1/(1-qL^{1/m}\zeta)$. Indeed, if L' denotes the universal cotangent line to the original stable map, restricted to the stratum in question, then (in the notation of Kawasaki's formula in Section 4) $\mathrm{ch}(\mathrm{Tr}\, L') = \zeta e^{c_1(L)/m}$.

Note that $c_1(L)$ is nilpotent on each of the stem or head spaces. Thus, Figure 1 provides the decomposition of $\mathcal{J}$ into the Laurent polynomial part $1-q+t(q)$ and elementary fractions $1/(1-q\zeta)^r$ at different poles $q=\zeta^{-1}\neq 0,\infty$. We are ready for our first conclusion.

`Proposition 1.` *The localization* $\mathcal{J}_\zeta$ *at* $\zeta=1$ *lies in the cone* $\mathcal{L}^{\text{fake}}$ *of fake quantum K-theory.*

`Proof.` Denote by $\widetilde{t}(q)$ the sum of $t(q)$ and of all the terms of $\sum$ with $\zeta\neq 1$. Note that in genus 0, stable maps of degree 0 have no non-trivial automorphisms. So all terms of the sum $\sum$ have non-zero degrees. This shows that (thanks to Novikov's variables) the whole sum $\widetilde{t}$ makes sense as a q-series lying in $\mathcal{K}_+^{\text{fake}}$, and is "small" in the (t,Q)-adic sense, hence qualifying on the role of an input of fake quantum K-theory of X. We claim that the whole sum shown on Figure 1 is the value of J-function of this fake theory with the input $\widetilde{t}$.

Indeed, examine contributions into the virtual Kawasaki formula [22] of the terms with $\zeta=1$. Denote by L_- the cotangent line at a marked point of the head. When the marked point of the head is that of the original curve, the content of it is $t(L_-)$. When this is a node where an arm is attached, denote by L_+ the cotangent line to the arm. The only ingredients that do not factor into separate contributions of the head and of the arms are

$$\frac{\sum_a \Phi_a\otimes\Phi^a}{1-L_-\,\mathrm{Tr}(L_+')}.$$

The top comes from the gluing of the arm to the head, and the bottom from the smoothing of the curve at the node, as a mode of perturbation *normal* to the stratum of the inertia orbifold. We conclude that the content of the marked point of the head correlator is exactly $\widetilde{t}(L_-)$.

Thus $\mathcal{J}(t)$ is represented as

$$1-q+\widetilde{t}(q)+\sum_a\Phi^a\sum_{n',d'}\frac{Q^{d'}}{n'!}\langle\frac{\Phi^a}{1-qL},\widetilde{t}(L),\dots,\widetilde{t}(L)\rangle_{0,n'+1}^{X,d'}=\mathcal{J}^{\text{fake}}(\widetilde{t}),$$

since the correlators come from the fake K-theory of X. ■

Let us return to the term with $\zeta\neq 1$. The stem curve itself is typically the quotient of $\mathbb{C}P^1$ by the rotation through ζ about two points: the horn and one more — let's call it the *butt* — where the eigenvalue of the symmetry on the cotangent line is ζ^{-1}. In fact the stem can degenerate into the quotient of a *chain* of several copies of $\mathbb{C}P^1$, with the same action of the symmetry on each of them, and connected "butt-to-horn" to each other (and even further, with other irreducible components attached on the "side" of the chain, see Figure 2 in the next section). In this case the butt of the stem is that of the last component of the chain. The butt can be a regular point of the whole curve, a marked point of it, or a node where the *tail* is attached (see Figure 1). The tail can be any stable map with any symmetry, except that at the point where it is attached to the stem, the eigenvalue of the symmetry cannot be equal to ζ. (Otherwise the stem chain could be prolonged.) In Figure 1, put $\delta t(q) = 1 - q + t(q) + \check{t}(q)$, where $\check{t}(q)$ is the sum of all remaining terms except the one with the pole at $q = \zeta^{-1}$ (with this particular value of ζ). We claim that the expansion $\mathcal{J}_\zeta$ of the big J-function near $q = \zeta^{-1}$ has the form

$$\delta t(q) + \sum_a \Phi_a \sum_{n,d} \frac{Q^{md}}{n!} \left[\frac{\Phi^a}{1 - qL^{1/m}\zeta}, T(L), \dots, T(L), \delta t(L^{1/m}/\zeta) \right]_{0,n+2}^{X,d},$$

where $[\dots]$ are certain correlators of "stem" theory, and $T(L)$ are leg contributions, both yet to be identified.

Indeed, let L_+ denote the cotangent line at the butt of the stem, and L_+' its counterpart on the m-fold cover. When the butt is a marked point, its content is $t(L_+^{1/m}\zeta)$, and when it is the node with a tail attached, then it is $\check{t}(L_+^{1/m}\zeta)$. This is because $\mathrm{ch}(\mathrm{Tr}\, L_+') = \zeta e^{c_1(L_+)/m}$. The case when the butt is a regular point on the original curve but a marked point on the stem, can be compared to the case when the butt is a marked point on the original curve as well. In the former case, the conormal bundle to the stratum of stable maps with symmetry is missing, comparing to the latter case, the line L_+'. In other words, one can replace the former contribution with the latter one, by taking the content at the butt to be $1 - L_+^{1/m}\zeta$, i.e. the K-theoretic Euler factor corresponding to the conormal line bundle L_+'. We summarize our findings.

`Proposition 2.` *The expansion $\mathcal{J}_\zeta$ of $\mathcal{J}$ near $q = \zeta^{-1}$ is a tangent vector to the range of the fake J-function of the "stem" theory at the "leg" point, T.*

Our next goal is to understand leg contributions $T(L)$.

`Proposition 3.` *Let $\widetilde{T}(L)$ denote the arm contribution $\widetilde{t}(L)$ computed when the input $t = 0$. Then*

$$T(L) = \Psi^m\left(\widetilde{T}(L)\right).$$

We recall that Adams' operations Ψ^m are additive and multiplicative endomorphisms of K-theory acting on a line bundle by $\Psi^m(L) = L^m$. In this proposition, Ψ^m acts not only on L and elements of $K^0(X)$, but also by $\Psi^m(Q^d) = Q^{md}$ on Novikov's variables.

`Proof.` The *legs* of a stable map with an automorphism, g, of order $m \neq 1$ on the cotangent line at the horn, are obtained by removing the stem (and the tail). Each leg shown in Figure 1 represents m copies of the same stable map glued to the m-fold cover of the stem and cyclically permuted by g. The automorphism g^m preserves each copy of the leg but acts non-trivially on the cotangent line at the horn of the leg (i.e. the point of gluing), since otherwise the stem could be extended. The only other restriction on what a leg could be is that it cannot carry (or be) a marked point of the original curve, since the numbering of the m copies of the marked point would break the symmetry. This identifies each of the m copies of a leg with an arm carrying no marked points.

As in the proof of Proposition 1, denote by L_- and L'_+ the cotangent lines at the point of gluing to the m-fold cover of the stem and to the leg respectively. Then the smoothing perturbation at the node of the curve with symmetry represents a direction normal to the stratum of symmetric curves. In the denominator of the virtual Kawasaki formula [22], it is represented by one Euler factor $1 - L_- \operatorname{Tr}(L'_+)$ for each copy of the leg. As in the case of arms, the gluing factor has the form

$$\frac{\sum_a \Phi_a \otimes \Phi^a}{1 - L_- \operatorname{Tr}(L'_+)}.$$

Then $\operatorname{ch}(\Phi^a)$ and $\operatorname{ch}\left(\operatorname{Tr}(L'_+)\right)$ are integrated out over the moduli space of legs, and the leg contributes into the fake Euler characteristics over the space of stems through Φ_a and L_-. We claim however that the contribution of the gluing factor into the stem correlator has the form

$$\frac{\Psi^m(\Phi_a) \otimes \Phi^a}{1 - L_-^m \operatorname{Tr}(L'_+)}.$$

This follows from the following general lemma.

`Lemma.` *Let V be a vector bundle, and g the automorphism of $V^{\otimes m}$ acting by the cyclic permutation of the factors. Then*

$$\operatorname{Tr}(g \mid V^{\otimes m}) = \Psi^m(V).$$

We conclude that the contribution of the leg into stem correlators is obtained from $\widetilde{t}(L_-)$ (the contribution of the arm into head correlators) by computing it at the input $t = 0$ (this eliminates those arms that carry marked points), then applying Ψ^m, and also replacing Q^d with Q^{md}, because the total degree of the m copies of a leg is m times the degree of each copy. ■

`Proof of Lemma.` It suffices to prove it for the universal U_N-bundle, or equivalently, for the vector representation $V = \mathbb{C}^N$ of U_N. Computing the value at $h \in U_N$ of the character of $\operatorname{Tr}(g \mid V^{\otimes m})$, considered as a representation of U_N, we find that it is equal to $\operatorname{tr}(gh^{\otimes m})$, because g and $h^{\otimes m}$ commute. Let e_i denote eigenvectors of h with eigenvalues x_i. A column of the matrix of $gh^{\otimes m}$ in the basis $e_{i_1} \otimes \cdots \otimes e_{i_N}$ has zero diagonal entry unless $i_1 = \cdots = i_N$. Thus, $\operatorname{tr}(gh^{\otimes N}) = x_1^m + \cdots + x_N^m$. This is the same as the trace of h on $\Psi^m(V)$. ■

`Remark.` The lemma can be taken for the definition of Adams' operations. For a permutation g with r cycles of lengths $m_1, \ldots, m_r$, it implies:

$$\operatorname{Tr}(g \mid V^{\otimes m}) = \Psi^{m_1}(V) \otimes \cdots \otimes \Psi^{m_r}(V).$$

`Proposition 4.` *Propositions 1,2,3 unambiguously determine the big J-function $\mathcal{J}$ in terms of stem and head correlators.*

`Proof.` Figure 1 can be viewed as a recursion relation that reconstructs $\mathcal{J}(t)$ by induction on degrees d of Novikov's monomials Q^d (in the sense of the natural partial ordering on the Mori cone). The key fact is that in genus 0, constant stable maps have no non-trivial automorphisms (and have > 2 marked or singular points). Consequently, arms which are not marked points, as well as legs, or stems with no legs attached, must have non-zero degrees. As a result, setting $t = 0$, one can reconstruct $\mathcal{J}(0)$ up to degree d from head and stem correlators, assuming that tails and arms are known in degrees $< d$, and then reconstruct the arm $\widetilde{T}(q)$ and tail $\delta t(q)$ (at $t = 0$) up to degree d from projections $\mathcal{J}(0)_1$ and $\mathcal{J}(0)_\zeta$ to $\mathcal{K}_+^{fake}$ and $\mathcal{K}_+^\zeta$ respectively.

It is essential here that even when the head has degree 0, it suffices to know the arms up to degree $< d$ (since at least 2 arms must be attached to the head). Also, when both the stem and the tail have degree 0, and there is only one leg attached, Proposition 3 recovers the information about the leg up to degree d from that of the arm up to degree $d/m < d$.

The previous procedure reconstructs $\widetilde{T}$ (the arm at $t = 0$), and hence the leg $T = \Psi^m(\widetilde{T})$ in all degrees. Now, starting with any (non-zero) input t, one can first determine $\widetilde{t}$ up to degree d from stem correlators, assuming that tails are known in degrees $< d$, and then recover $\mathcal{J}(t)$ (and hence arms and tails) up to degree d. ■

Thus, to complete the proof of the theorem, it remains to show that the tangent spaces from Proposition 2 coincide with the Lagrangian spaces $\mathcal{L}^\zeta = \nabla_\zeta \mathcal{T}^\zeta$ described in the adelic formulation of the theorem. This will be done in the next section.

8. Stems as stable maps to $X/\mathbb{Z}_m$

Let $\zeta \neq 1$ be a primitive mth root of 1, and let $\overline{\mathcal{M}}^{X,d}_{0,n+2}(\zeta)$ denote a *stem space*. It is formed by stems of degree d, considered as quotient maps by the symmetry of order m acting by ζ on the cotangent line at the horn of the covering curve. It is a Kawasaki stratum in $\overline{\mathcal{M}}^{X,md}_{0,mn+2}$.

Proposition 5. *The stem space $\overline{\mathcal{M}}^{X,d}_{0,n+2}(\zeta)$ is naturally identified with the moduli space $\overline{\mathcal{M}}^{X/\mathbb{Z}_m,d}_{0,n+2}(g,1,\dots,1,g^{-1})$ of stable maps to the orbifold $X/\mathbb{Z}_m$.*

Remark. This Proposition refers to the GW-theory of orbifold target spaces in the sense of Chen–Ruan [3] and Abramovich–Graber–Vistoli [1]. In particular, evaluations at marked points take values in the inertia orbifold, and notation of the moduli space indicates the *sectors*, i.e. components of the inertia orbifold where the evaluation maps land. In the case at hands the inertia orbifold is $X \times \mathbb{Z}_m$, and the string $(g,1,\dots,1,g^{-1})$, where g is the generator of $\mathbb{Z}_m$, designates (in a way independent of ζ) the sectors of the marked points.

Proof. The paper [16] by Jarvis–Kimura describes stable maps to the orbifold $point/\mathbb{Z}_m = B\mathbb{Z}_m$ in a way that can be easily adjusted to our case $X/\mathbb{Z}_m = X \times B\mathbb{Z}_m$. Namely, they are stable maps to X equipped with a principal $\mathbb{Z}_m$-cover over the complement to the set of marked and nodal points, possibly ramified over these points in a way *balanced* at the nodes (i.e. such that the holonomies around the node on the two branches of the curve are inverse to each other). The stem space is obtained when two marked points are assigned holonomies $g^{\pm 1}$ of maximal order, and all other marked points are unramified. ■

Thus, introducing the simplifying notation $\overline{\mathcal{M}} := \overline{\mathcal{M}}^{X,d}_{0,n+2}(\zeta)$, we identify stem correlators in the virtual Kawasaki formula [22] with integrals:

$$\left[\ \frac{\Phi}{1-q\zeta L^{1/m}}, T(L),\dots,T(L),\delta t(L)\ \right]^{X,d}_{0,n+2} =$$
$$\int_{[\overline{\mathcal{M}}]^{vir}} \operatorname{td}(T_{\overline{\mathcal{M}}})\operatorname{ch}\left(\frac{\operatorname{ev}_1^*\Phi\quad \operatorname{ev}_{n+2}^*\delta t(\zeta^{-1}L_{n+2}^{1/m})\quad \prod_{i=2}^{n+1}\operatorname{ev}_i^* T(L_i)}{(1-q\zeta L_1^{1/m})\ \operatorname{Tr}\left(\bigwedge^\bullet N^*_{\overline{\mathcal{M}}}\right)}\right),$$

Here $[\overline{\mathcal{M}}]^{vir}$ is the virtual fundamental cycle of the moduli space in GW-theory of $X/\mathbb{Z}_m$, $T_{\overline{\mathcal{M}}}$ is the virtual tangent bundle to $\overline{\mathcal{M}}$, and $N_{\overline{\mathcal{M}}}$ is the normal bundle to $\overline{\mathcal{M}}$ considered as a Kawasaki stratum in the appropriate moduli space of stable maps to X which are the m-fold covers of the stems. In several steps, we will express stem correlators in terms of cohomological GW-theory of X.

Let $(\mathcal{H}, \Omega)$ be the symplectic loop space of cohomological GW-theory of X:

$$\mathcal{H} = H((z)), \quad \Omega(f, g) = \mathrm{Res}_{z=0}(f(-z), g(z))\, dz, \quad (A, B) = \int_X AB.$$

Recall that the J-function of this theory is

$$\mathcal{J}_X^H(t) := -z + t(z) + \sum_a \phi_a \sum_{n,d} \frac{Q^d}{n!} \langle \frac{\phi^a}{-z-\psi}, t(\psi), \dots, t(\psi) \rangle_{0,n+1}^{X,d}.$$

For the purpose of applications to K-theory, we will ignore all odd-degree cohomology classes (that is, set all odd variables to 0), and respectively assume that H here denotes the even-dimensional part of cohomology.

Our first task is to express in terms of $\mathcal{J}_X^H$ the J-finction of the *orbifold* $X/\mathbb{Z}_m$. The answer is immediately extracted from the paper [16] by Jarvis–Kimura: one only needs to replace the ground field $\mathbb{Q}$ with the group ring $\mathbb{Q}[\mathbb{Z}_m]$ (in fact, the center of the group ring, but our group is abelian). In other words, to parameterize the cone $\mathcal{L}_{X/\mathbb{Z}_m}^H$, one needs to replace in the above formula for $\mathcal{J}_X^H$ the variable t with $\sum_{h \in \mathbb{Z}_m} t^{(h)} h$, where each $t^{(h)} \in H[[z]]$. The resulting J-function takes values in $H \otimes_{\mathbb{Q}} \mathbb{Q}[\mathbb{Z}_m]$. The components corresponding to different group elements are referred to as "sectors". The Poincaré pairing on X becomes divided by m (since the fundamental class $[X/\mathbb{Z}_m] = [X]/m$) and coupled with the usual inner product on the group ring: $(h, h') = 0$ for $h' \neq h^{-1}$, and $(h, h^{-1}) = 1/m$ (so that sector h pairs non-trivially only with sector h^{-1}).

For the purpose of expressing stem correlators, we need only one type of correlators for $X/\mathbb{Z}_m$. It is obtained by setting $\sum_h t^{(h)} h = t \cdot g^0 + (\delta t) g^{-1}$, and differentiating the resulting J-function one time in the direction of δt at the point $\delta t = 0$. Thus, this is a tangent vector to the cone $\mathcal{L}_{X/\mathbb{Z}_m}$, but we also need to keep track of its applications point (obtained by setting $\delta t = 0$ before differentiation), and so we give names to both, the application point: $\mathcal{J}_{X/\mathbb{Z}_m}^H(t)$, belonging to sector g^0, and the tangent vector $\delta \mathcal{J}_{X/\mathbb{Z}_m}^H(t)$, belonging to sector g^{-1}. Thus, we have the following Proposition.

Proposition 6. *$\mathcal{J}^H_{X/\mathbb{Z}_m}(t) = \mathcal{J}^H_X(t)$, and*

$$\delta \mathcal{J}^H_{X/\mathbb{Z}_m} = \delta t(z) + \sum_a \phi_a \sum_{n,d} \langle \frac{\phi^a}{-z-\psi}, t(\psi), \dots, t(\psi), \delta t(\psi)\rangle^{X,d}_{0,n+2}.$$

Remark. The Poincaré pairing on the identity sector differs from the usual one by the factor $1/m$. As a result, the basis Poincaré-dual to ϕ_a is $m\phi^a =: \widetilde{\phi}^a$. This change would show in the definition of the J-function of $X/\mathbb{Z}_m$. However, correlators in the orbifold theory also differ from the usual ones by the factor $1/m$, and these two changes cancel out.

The sum $T_{\overline{\mathcal{M}}} \oplus N_{\overline{\mathcal{M}}}$ is the restriction to $\overline{\mathcal{M}}$ of the virtual tangent bundle to the moduli space of stable maps of degree md with $mn+2$ marked points. According to [4], in the Grothendieck group $K^0(\overline{\mathcal{M}})$, this tangent bundle is represented by push-forward from the universal family $\widetilde{\pi} : \widetilde{\mathcal{C}} \to \overline{\mathcal{M}}$:[7]

$$\widetilde{\pi}_* \operatorname{ev}^* T_X + \widetilde{\pi}_*(1 - L^{-1}) + \left(-\widetilde{\pi}_* \widetilde{i}_* \mathcal{O}_{\widetilde{Z}}\right)^\vee,$$

where L stands for the universal cotangent line at the "current" ($mn+$3-rd) marked point of the universal curve, $\widetilde{i} : \widetilde{Z} \to \widetilde{C}$ is the embedding of the nodal locus, and $\vee$ means dualization. This decomposes the virtual bundle into the sum of three parts, respectively responsible for: (i) deformations of maps to X of a fixed complex curve, (ii) deformations of complex structure and/or configuration of marked points, and (iii) bifurcations of the curve's combinatorics through smoothing at the nodes.

Part (i) is the *index bundle*

$$\operatorname{Ind}(T_X) := \widetilde{\pi}_* \widetilde{\operatorname{ev}}^*(T_X).$$

Here we use the following notation: maps $\pi : \mathcal{C} \to \overline{\mathcal{M}}$ and $\operatorname{ev} : \mathcal{C} \to X/\mathbb{Z}_m$ form the universal stable map diagram, while $\widetilde{\pi} : \widetilde{\mathcal{C}} \to \overline{\mathcal{M}}$ and $\widetilde{\operatorname{ev}} : \widetilde{\mathcal{C}} \to X$ are their $\mathbb{Z}_m$-equivariant lifts to the family of ramified $\mathbb{Z}_m$-covers.

We need to extract from the index bundle the eigenspace of the generator, g, of the group $\mathbb{Z}_m$, with the eigenvalue ζ^{-k}. For this, we begin with the $\mathbb{Z}_m$-module $\mathbb{C}$ where g acts by ζ^k, denote $\mathbb{C}_{\zeta^k}$ the corresponding line bundle over $B\mathbb{Z}_m$, and take $\left(\operatorname{Ind}(T_X) \otimes \mathbb{C}_{\zeta^k}\right)^{\mathbb{Z}_m}$. This (trivial) result can be expressed in terms of orbifold GW-theory of $X/\mathbb{Z}_m = X \times B\mathbb{Z}_m$ as $\pi_* \operatorname{ev}^* \left(T_X \otimes \mathbb{C}_{\zeta^k}\right)$. Namely, as we mentioned

[7] In [4], we find $T_X - 1$ in place of T_X, but in genus 0, $\widetilde{\pi}_*(1) = 1$.

in Section 4, the K-theoretic push-forward operation on global quotients considered as orbifolds automatically extracts the invariant part of sheaf cohomology. Thus,

$$\mathrm{Tr}(\mathrm{Ind}(T_X)) = \oplus_{k=0}^{m-1} \zeta^{-k} \pi_* \,\mathrm{ev}^* \left(T_X \otimes \mathbb{C}_{\zeta^k} \right).$$

Recall that an invertible characteristic class of complex vector bundles is determined by an invertible formal series in one variable, the 1st Chern class $x = c_1(l)$ of the universal line bundle. Alongside the usual Todd class td, we introduce *moving* Todd classes (aka equivariant K-theoretic inverse Euler classes), one for each $\lambda \neq 1$:

$$\mathrm{td}(l) = \frac{x}{1-e^{-x}}, \quad \mathrm{td}_\lambda(l) = \frac{1}{1-\lambda e^{-x}}.$$

The contribution of $\mathrm{Tr}(\mathrm{Ind}(T_X))$ into our integral over $\overline{\mathcal{M}}$ reads:

$$(*) \qquad \mathrm{td}\left(\pi_* \,\mathrm{ev}^*(T_X)\right) \prod_{k=1}^{m-1} \mathrm{td}_{\zeta^k} \left(\pi_* \,\mathrm{ev}^* (T_X \otimes \mathbb{C}_{\zeta^k}) \right).$$

Introduce $\mathcal{J}^{tw}_{X/\mathbb{Z}_m}$ and $\delta\mathcal{J}^{tw}_{X/\mathbb{Z}_m}$ as *twisted* counterparts of $\mathcal{J}^{H}_{X/\mathbb{Z}_m}$ and $\delta\mathcal{J}^{H}_{X/\mathbb{Z}_m}$. Namely, following [6], one defines GW-invariants *twisted* by a chosen bundle, E, over the target space, and a chosen multiplicative characteristic class, S, by systematically replacing virtual fundamental cycles of moduli spaces of stable maps with their cap-products (such as $[\overline{\mathcal{M}}]^{vir} \cap S(\mathrm{Ind}(E))$ in our case) with the chosen characteristic class of the corresponding index bundle.

`Proposition 7.` *Denote by* $\square$ *and* $\square_\zeta$ *the Euler–Maclaurin asymptotics of the infinite products*

$$\square \sim \prod_{\text{Chern roots } x \text{ of } T_X} \prod_{r=1}^{\infty} \frac{x-rz}{1-e^{-mx+mrz}},$$

$$\square_\zeta \sim \prod_{\text{Chern roots } x \text{ of } T_X} \prod_{r=1}^{\infty} \frac{x-rz}{1-\zeta^{-r} e^{-x+rz/m}}.$$

Then $\mathcal{J}^{tw}_{X/\mathbb{Z}_m}$ *lies in the overruled Lagrangian cone* $\square\mathcal{L}^H_X$, *and* $\delta\mathcal{J}^{tw}_{X/\mathbb{Z}_m}$ *lies in the transformed tangent space* $\square_\zeta \mathcal{T}_{\square^{-1}\mathcal{J}^{tw}_{X/\mathbb{Z}_m}} \mathcal{L}^H_X$.

`Proof.` The Quantum Riemann–Roch Theorem of [6], which expresses twisted GW-invariants in terms of *un*twisted ones, was generalized to the case of orbifold target spaces by Hsian-Hua Tseng [24]. The proposition is obtained by direct applications of the Quantum RR Theorem of [24] to each of the twisting data $E = T_X \otimes \mathbb{C}_{\zeta^k}$, $S = \mathrm{td}_{\zeta^k}$.

For $k=0$, the Euler–Maclaurin asymptotics (for both $\mathcal{J}^{tw}$ and $\delta\mathcal{J}^{tw}$) come from the product

$$\prod_{r=1}^{\infty}\frac{x-rz}{1-e^{-x+rz}},$$

and for $k\neq 0$, from

$$\prod_{r=1}^{\infty}\frac{1}{1-\zeta^k e^{-x+rz}}$$

for $\mathcal{J}^{tw}$, and

$$\prod_{r=1}^{\infty}\frac{1}{1-\zeta^k e^{-x+rz-kz/m}}$$

for $\delta\mathcal{J}^{tw}$. The extra factor $e^{-kz/m}$ in the denominator comes from the way how in the orbifold HRR theorem of [5], the logarithm (k/m in our case) of the eigenvalue ($e^{2piik/m}$), by which the symmetry g acts on the twisting bundle ($T_X\otimes\mathbb{C}_{\zeta^k}$), enters the "Bernoulli polynomial" ingredient of the formula. Namely, $\frac{e^{kt/m}}{e^t-1}$, where $t=z\partial_x$, formally expands as $\sum_{r=1}^{\infty} e^{(k/m)z\partial_x - rz\partial_x}$.

Multiplying out the products over $k=0,\dots,m-1$, and using $\prod_{k=0}^{m-1}(1-\zeta^k u)=1-u^m$ and $\zeta^m=1$ to simplify, we obtain the required results. ■

Part (ii) of the bundle $T_{\overline{\mathcal{M}}}\oplus N_{\overline{\mathcal{M}}}$ comes from deformations of the complex structure and marked points. It can be described as the K-theoretic push-forward $\widetilde{\pi}_*(1-L^{-1})$ along the universal curve $\widetilde{\pi}:\widetilde{\mathcal{C}}\to\overline{\mathcal{M}}$ (think of $H^1(\Sigma,T_\Sigma)$). To express the trace Tr of it, one need to consider push-forwards of $L^{-1}\otimes\mathbb{C}_{\zeta^k}$ and appropriately twisted GW-invariants of the orbifold $X/\mathbb{Z}_m$. More precisely, we need the twisting classes to have the form:

$$\operatorname{td}\left(\pi_*(1-L^{-1})\right)\prod_{k=1}^{m-1}\operatorname{td}_{\zeta^k}\left(\pi_*[(1-L^{-1})\operatorname{ev}^*(\mathbb{C}_{\zeta^k})]\right).$$

The general problem of computing GW-invariants of orbifolds twisted by characteristic classes of the form

$$\prod_\alpha S_\alpha\left(\pi_*[(L^{-1}-1)\operatorname{ev}^*(E_\alpha)]\right)$$

is solved in [23] (see also Chapter 2 of thesis [21]). The answer is described as the change of the dilaton shift.[8] Namely, if $-z=c_1(L^{-1})$, and S_α denote the twisting multiplicative characteristic class, then the dilaton shift changes from $-z$ to $-z\prod_\alpha S_\alpha(L^{-1}E_\alpha)$. In our situation,

[8]Generalizing the case of manifold target spaces discussed in [4].

$\alpha = 0, \dots, m-1$, $S_0 = \mathrm{td}^{-1}$, $S_k = \mathrm{td}_\zeta^{-1}$ for $k \neq 0$, and $E_k = \mathbb{C}_{\zeta^k}$. Respectively, the new dilaton shift is

$$-z\frac{(1-e^z)}{(-z)}\prod_{k=1}^{m-1}(1-\zeta^k e^z) = 1 - e^{mz}.$$

Thus, the *dilaton shift changes from* $-z$ *to* $1-e^{mz}$.

Parts (i) and (ii) together form the part of the virtual tangent bundle to $\overline{\mathcal{M}}_{0,mn+2}^{md,X}$ (albeit restricted to $\overline{\mathcal{M}}$) *logarithmic* with respect to the nodal divisor. What remains is part (iii), supported on the nodal divisor, which consists of one-dimensional summands (one per node), the *smoothing mode* of the glued curve at the node. Contributions of part (iii) into the ratio $\mathrm{td}(T_{\overline{\mathcal{M}}})/\operatorname{ch}(\operatorname{Tr}\bigwedge^\bullet(N^*_{\overline{\mathcal{M}}})$ in the virtual Kawasaki formula is described in terms of yet another kind of twisted GW-invariants of the orbifold $X/\mathbb{Z}_m$, where the twisting classes are supported at the nodal locus. The effect of such twisting on GW-invariants can be found by a recursive procedure based on *un*gluing the curves at the nodes. As it is seen in [4], this does not change the overruled Lagrangian cones, but affects generating functions through a *change of polarization.* Referring to [23] (or [21]) for the generalization to orbifold target spaces needed here, we state the results.

Let $\overline{\mathcal{M}}$ denote a moduli space of stable maps to the orbifold $X/\mathbb{Z}_m$, and $\pi : \mathcal{C} \to \overline{\mathcal{M}}$ the projection of the universal family of such stable maps. Let $Z = \cup_{h\in\mathbb{Z}_m} Z_h$ be the decomposition of the nodal stratum $Z \subset \mathcal{C}$ into the disjoint union according to the ramification type of the node, and $i : Z_h \to \mathcal{C}$ denote the embedding. Let $S_{h,a}$ be invertible multiplicative characteristic classes, and $E_{h,\alpha}$ arbitrary orbibundles over $X/\mathbb{Z}_m$, where $h \in \mathbb{Z}_m$, $\alpha = 1, \dots, K_h$. The twisting in question is obtained by systematically including into the integrands of GW-theory of $X/\mathbb{Z}_m$ the factors

$$\prod_{h\in\mathbb{Z}_m}\prod_{\alpha=1}^{K_h} S_{h,\alpha}\left(\pi_*[i_*\mathcal{O}_{Z_h}\otimes \mathrm{ev}^* E_{h,\alpha}]\right).$$

According to the results of [23] (Theorem 1.10.3 in [21]), the effect of such twisting is completely accounted by a change of polarization in the symplectic loop space of GW-theory of $X/\mathbb{Z}_m$, described separately for each sector. Namely, for the sector corresponding to $h \in \mathbb{Z}_m$, define a power series $u_h(z) = z + a_2z^2 + a_3z^3 + \cdots$ by

$$\frac{z}{u_h(z)} = \prod_{\alpha=1}^{K_h} S_{h.a}^{-1}\left(E_{h,a}\otimes L\right), \quad \text{where } c_1(L) := z.$$

Define the Laurent series $v_{h,k}, k = 0, 1, 2, \ldots$, by

$$\frac{1}{u_h(-\psi - z)} = \sum_{k\geq 0} (u_h(\psi))^k \, v_{h,k}(z),$$

which is the expansion of the L.H.S. in the region $|\psi| < |z|$. Then, as one can check, $\phi_a z^k, \phi^a v_{h,k}(z)$, $a = 1, \ldots, \dim H$, $k = 0, 1, 2, \ldots$, form a (topological) Darboux basis in the sector h of the symplectic loop spaces of the GW-theory of $X/\mathbb{Z}_m$. The genus-0 descendant potential of the twisted theory is expressed from that of untwisted one by taking the overruled Lagrangian cone of the untwisted theory for the graph of differential of a function in the Lagrangian polarization associated with this basis. Note that the positive space polarization, which is spanned by $\{\phi_a z^k\}$, stays the same as in the untwisted theory, while the negative space, which is spanned by $\{\phi^a v_{h,k}(z)\}$, differs from that of untwisted theory, which is spanned by $\{\phi^a z^{-1-k}\}$.

`Remarks.` (1) The standard polarization $\mathcal{H}_\pm$ of the symplectic loop space of quantum cohomology theory of a manifold is obtained by the same formalism:

$$\frac{1}{-\psi - z} = \sum_{k\geq 0} \frac{\psi^k}{(-z)^{k+1}},$$

and $\mathcal{H}_-$ is spanned by $\phi^a(-z)^{-1-k}$, $k = 0, 1, 2, \ldots$.

(2) As it was mentioned in Section 5, in fake K-theory one obtains a Darboux basis from $z/u(z) = \mathrm{td}(L)$, and respectively the expansion:

$$\frac{1}{1 - e^{\psi+z}} = \sum_{k\geq 0} (e^\psi - 1)^k \frac{e^{kz}}{(1 - e^z)^{k+1}}.$$

Consequently, $\mathcal{K}_+^{\text{fake}}$ and $\mathcal{K}_-^{\text{fake}}$ are spanned respectively by $\Phi_a(q-1)^k$ and $\Phi^a q^k/(1-q)^{k+1}$, $a = 1, \ldots, \dim K$, $k = 0, 1, 2, \ldots$.

In stem theory, there are two types of nodes (Figure 2). When a stem acquires an unramified node (as shown in the top picture), the covering curve carries a $\mathbb{Z}_m$-symmetric m-tuple of nodes. The smoothing bundle has dimension m and carries a regular representation of $\mathbb{Z}_m$. When a stem degenerates into a chain of two components glued at a balanced ramification point of order m (the bottom picture), the smoothing mode is one-dimensional and carries the trivial representation of $\mathbb{Z}_m$. Contributions of these smoothing modes into the ratio $\mathrm{td}(T_{\overline{\mathcal{M}}})/\,\mathrm{ch}(\mathrm{Tr} \bigwedge^\bullet (N^*_{\overline{\mathcal{M}}})$ is accounted by the following twisting factors in the integrals over $\overline{\mathcal{M}}$, considered as orbifold-theoretic GW-invariants:

$$\mathrm{td}\left(-\pi_* i_* \mathcal{O}_{Z_g}\right)^\vee \mathrm{td}\left(-\pi_* i_* \mathcal{O}_{Z_1}\right)^\vee \prod_{k=1}^{m-1} \mathrm{td}_{\zeta^k}\left(-\pi_*(\mathrm{ev}^* \mathbb{C}_{\zeta^k} \otimes i_* \mathcal{O}_{Z_1})\right)^\vee,$$

where Z_1 stands for the unramified nodal locus, and Z_g for the ramified one. This twisting results in the change of polarizations. In the g-ramified sector, the new polarization is determined from the expansion of

$$\frac{1}{1-e^{(\psi+z)/m}} = \frac{1}{1-q^{1/m}L^{1/m}}.$$

Here the factor $1/m$ occurs because what was denoted L in the GW-theory of $X/\mathbb{Z}_m$ is the universal cotangent line at the ramification point to the quotient curve, which is $L^{1/m}$ in our earlier notations of stem spaces (where L stands for the universal cotangent line to the covering curve). In the unramified sector, the new polarization is found from

$$\frac{z}{u(z)} = \mathrm{td}(L) \prod_{k=1}^{m-1} \mathrm{td}_{\zeta^k}(\mathbb{C}_{\zeta^k} \otimes L) = \frac{z}{1-e^{-z}} \prod_{k=1}^{m-1} \frac{1}{1-\zeta^k e^{-z}} = \frac{z}{1-e^{-mz}},$$

and consequently the expansion of

$$\frac{1}{1-e^{m\psi+mz}} = \frac{1}{1-q^m L^m}.$$

We conclude that the *negative space of polarizations in the ramified and unramified sectors are spanned respectively by*

$$\Phi^a q^{k/m}/(1-q^{1/m})^{k+1} \text{ and } \Phi^a q^{mk}/(1-q^m)^{k+1} = \Phi^a\, \Psi^m\left(q^k/(1-q)^{k+1}\right).$$

`Remark.` The occurrence of Adams' operation Ψ^m here is not surprising. The smoothing modes at m cyclically permuted copies on an unramified node of the stem curve form an m-dimensional space carrying the regular representation of $\mathbb{Z}_m$. The trace Tr of the bundle formed by these modes is, according to Lemma of the previous section, $\Psi^m(L_- \otimes L_+)$ (in notation of Figure 2, the top picture).

It remains to apply the above results to those generating functions of stem theory which occur in the virtual Kawasaki formula. Introduce a generating function, $\delta\mathcal{J}^{st}_{X/\mathbb{Z}_m}$, of stem theory as the image under the Chern character map $\mathrm{ch}: \mathcal{K}^{\mathrm{fake}} \to \mathcal{H}$ of

$$\delta t(q^{1/m}) + \sum_{a,n,d} \Phi_a \frac{Q^d}{n!}\left[\frac{\Phi^a}{1-q^{1/m}L^{1/m}}, T(L), \dots, T(L), \delta t(L^{1/m})\right]^{X,d}_{0,n+2}.$$

Replacing $q^{1/m}$ with ζq and Q^d with Q^{md}, we would obtain the sum of correlators of stem theory as they appeared in Section 7. On the

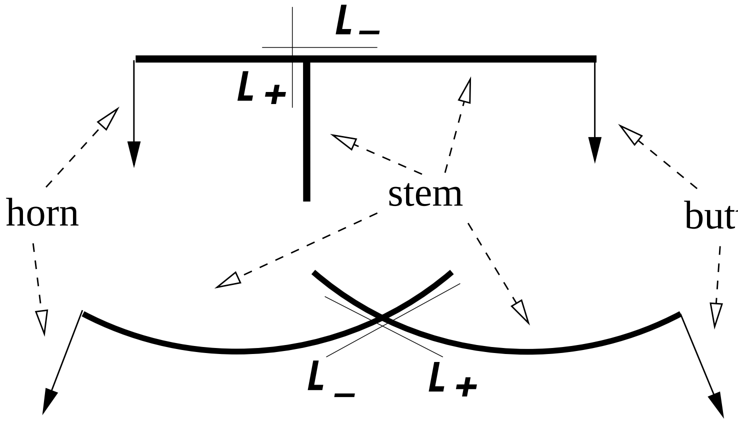

Figure 2. Two types of stem nodes.

other hand, the interpretation of stem correlators as GW-invariants of $X/\mathbb{Z}_m$ twisted in three different ways (corresponding to parts (i), (ii), (iii) of the tangent space), and the previous results on the effects of these twistings, provide the following description of $\delta\mathcal{J}^{st}X/\mathbb{Z}_m$ in terms of GW-theory of X.

Proposition 8. *$\delta\mathcal{J}^{st}_{X/\mathbb{Z}_m}(\delta t, T)$ lies in $\square_\zeta\square^{-1}\mathcal{T}_{\mathcal{J}^{tw}_{X/\mathbb{Z}_m}}\square\mathcal{L}^H_X$, where the input T is related to the application point $\mathcal{J}^{tw}_{X/\mathbb{Z}_m}$ by the projection $[\cdots]_+$ along the negative space of the polarization of the unramified sector:*

$$\operatorname{ch}(1 - q^m + T(q)) = \left[\mathcal{J}^{tw}_{X/\mathbb{Z}_m}\right]_+ .$$

Proof. According to Proposition 7, $\mathcal{J}^{tw}_{X/\mathbb{Z}_m}$ lies in the cone $\square\mathcal{L}^H$, and $\delta\mathcal{J}^{tw}_{X/\mathbb{Z}_m}$ lies in the space $\square_\zeta\square^{-1}\mathcal{T}$, where $\mathcal{T}$ is the tangent space to $\square\mathcal{L}^H$ at the point $\mathcal{J}^{tw}_{X/\mathbb{Z}_m}$. It follows from the previous discussion that $\delta\mathcal{J}^{st}_{X/\mathbb{Z}_m}$, being obtained from $\delta\mathcal{J}^{tw}_{X/\mathbb{Z}_m}$ by changing dilaton shift and polarizations only, lies in the same space. Changing the content of the horn in the definition of $\delta\mathcal{J}^{tw}_{X/\mathbb{Z}_m}$ from $\widetilde{\phi}^a/(-z-\psi) = \phi^a/(-z/m-\psi/m)$ to $\phi^a/(1-e^{z/m+\psi/m})$ is equivalent to applying to the same space the polarization associated with the g-ramified sector. However, the new dilaton shift and polarization in the unramified sector both affect the way the input T of $\delta\mathcal{J}^{st}_{X/\mathbb{Z}_m}$ is computed in terms of $\mathcal{J}^{tw}_{X/\mathbb{Z}_m}$. Namely, $\operatorname{ch}(T) = [\mathcal{J}^{tw}_{X/\mathbb{Z}_m}]_+ - (1-e^{mz})$. ■

To put the next proposition into context, let us recall that the stem correlators of Section 7, in order to represent the expansion $\mathcal{J}(t)_\zeta$ of the true K-theoretic J-function $\mathcal{J}$ at $q=\zeta^{-1}$, need to be computed at a specific input T, the *leg*, which is characterized in a rather complex way. Namely, the expansion $\mathcal{J}(0)_1$ of the value of $\mathcal{J}$ at the input $t=0$ lies in the cone $\mathcal{L}^{\text{fake}}$ of fake quantum K-theory of X (Proposition 1). The contribution $\widetilde{T}$, i.e. the *arm* corresponding to $t=0$, is obtained as the input point of $\mathcal{J}(0)_1$, i.e. by applying the projection $(\dots)_+$ along the negative space of polarization (described in Remark 2) and the dilaton shift of fake quantum K-theory:

$$1-q+\widetilde{T}(q)=\left(\mathcal{J}(0)_1\right)_+ .$$

According to Proposition 3, $T=\Psi^m(\widetilde{T})$ (where Adams' operation acts also on q and Q).

On the other hand, Proposition 7 locates the stem generating function $\delta\mathcal{J}^{st}_{X/\mathbb{Z}_m}$ in terms of the tangent space $\mathcal{T}_{\mathcal{J}^{tw}_{X/\mathbb{Z}_m}}\square\mathcal{L}^H_X$. Furthermore, according to the Quantum HRR Theorem stated in Section 5, $\mathcal{L}^H_X=\triangle^{-1}\operatorname{ch}\left(\mathcal{L}^{\text{fake}}\right)$, and hence the cone and its tangent space in question are the images under $\square\triangle^{-1}\operatorname{ch}$ of $\mathcal{L}^{\text{fake}}$ and of a certain tangent space to it. The following proposition implies that when the input T is the leg, the requisite tangent space is exactly $\mathcal{T}_{\mathcal{J}(0)_1}\mathcal{L}^{\text{fake}}$.

`Proposition 9.` $\operatorname{ch}^{-1}(\square\mathcal{L}^H_X)=\Psi^m(\mathcal{L}^{\text{fake}})$, *where the Adams operation* $\Psi^m:\mathcal{K}^{\text{fake}}\to\mathcal{K}^{\text{fake}}$ *acts on* q *by* $\Psi^m(q):=q^m$.

`Proof.` From the QHRR theorem of Section 5 and Proposition 7, we have:

$$\triangle^{-1}\operatorname{ch}(\mathcal{J}^{\text{fake}}_X)=\mathcal{J}^H_X=\square^{-1}\mathcal{J}^{tw}_{X/\mathbb{Z}_m}.$$

We intentionally neglect to specify the arguments, since they are determined by the argument, t, of $\mathcal{J}^H_X$, by polarizations, and by the transformations $\triangle$ and $\square$ themselves. The Adams operation Ψ^m acts on cohomology classes via the Chern isomorphism:

$$\operatorname{ch}\left(\Psi^m(\operatorname{ch}^{-1}a)\right)=m^{\deg(a)/2}a.$$

The J-function $t\mapsto\mathcal{J}^H_X(t)$ has degree 2 with respect to the grading, defined by the usual grading in cohomology, $\deg z=2$, $\deg Q^d=2\int_d c_1(T_X)$, and $\deg t=2$. The latter means that in the expression $t=\sum_{k,\alpha}t_k^\alpha\phi_\alpha z^k$ the variable t_k^α is assigned degree $2-\deg\phi_\alpha-2k$. Therefore, writing $\mathcal{J}^H_X/(-z)=\sum_d J_dQ^d$, and rescaling the variables by $\widetilde{t}_k^\alpha=m^{1-\deg\phi_\alpha/2-k}t_k^\alpha$, we find

$$m^{-1}\Psi^m(\mathcal{J}^H_X(t))=\sum_d m^{-\deg Q^d}J_d(\widetilde{t})Q^d=e^{-(\log m)\ c_1(T_X)/z}\mathcal{J}^H_X(\widetilde{t}).$$

The second equality is an instance of the genus-0 *divisor equation* (see [6]). Thus, Proposition 9 would follow from the identity

$$\Box = m^{\frac{1}{2}\dim_{\mathbb{C}} X}\Psi^m(\triangle)\ e^{-(\log m)\ c_1(T_X)/z}.$$

To establish it, note that both $\triangle$ and $\Box$ are the Euler–Maclaurin asymptotics of infinite products

$$\prod_{\text{Chern roots } x \text{ of } T_X}\ \prod_{r=1}^{\infty} S(x-rz),$$

where S is respectively

$$\frac{x}{1-e^{-x}} \quad \text{and} \quad \frac{x}{1-e^{-mx}} = m^{-1}\ \Psi^m\left(\frac{x}{1-e^{-x}}\right).$$

The factor m^{-1} contributes into the asymptotics in the form

$$\prod_{\text{Chern roots } x \text{ of } T_X} e^{-(\log m)\ x/z} m^{1/2} = e^{-(\log m)\ c_1(T_X)/z} m^{\frac{1}{2}\dim_{\mathbb{C}} X}. \qquad \blacksquare$$

`Remark.` Most steps of our arguments apply to the case of *twisted* cohomological GW-invariants. The previous proof, however, employs the grading in cohomology, and does not work therefore for twisted GW-invariants unless the twisted virtual fundamental classes are homogeneous, and the degrees of Novikov's variables are adusted accordingly. Still, these assumptions are correct in the case of twisting by (equivariant) Euler classes. This is why our main theorem applies to such, Euler-twisted theories. We will use this fact in some applications given in the last section.

`Corollary.` *$\delta\mathcal{J}^{st}_{X/\mathbb{Z}_m}(\delta t, T)$ lies in the space $\Box_\zeta \triangle^{-1}\mathcal{T}_{\mathcal{J}^{\text{fake}}(\widetilde{T})}\mathcal{L}^{\text{fake}}$, where $T = \Psi^m(\widetilde{T})$.*

`Proof.` According to Proposition 8, $\delta\mathcal{J}^{st}_{X/\mathbb{Z}_m}$ lies in the space $\Box_\zeta\Box^{-1}\mathcal{T}_{\mathcal{J}^{tw}_{X/\mathbb{Z}_m}}\Box\mathcal{L}^H$, where the input T of $\delta\mathcal{J}^{st}_{X/\mathbb{Z}_m}$ is determined by $T = [\mathcal{J}^{tw}_{X/\mathbb{Z}_m}]_+ - (1-q^m)$. By Proposition 9, $\mathcal{J}^{tw}_{X/\mathbb{Z}_m} = \Psi^m(\mathcal{J}^{\text{fake}})$, and the input of $\mathcal{J}^{\text{fake}}$ is determined as $\widetilde{T} = \left(\mathcal{J}^{\text{fake}}\right)_+ - (1-q)$. Here $(\cdots)_+$ refers to the projection to $\mathcal{K}^{\text{fake}}_+$ along $\mathcal{K}^{\text{fake}}_-$, i.e. the polarization described in Remark 2, while the projection $[\cdots]_+$ refers to the polarization in the unramified sector. The latter polarization is obtained from the former by the Adams operation: $\Psi^m : \mathcal{K}^{\text{fake}} \to \mathcal{K}^{\text{fake}}$, and the relation between dilaton shifts is the same: $1-q^m = \Psi^m(1-q)$. Therefore $T = \Psi^m(\widetilde{T})$, and the tangent space $\mathcal{T}_{\mathcal{J}^{tw}_{X/\mathbb{Z}_m}}\Box\mathcal{L}^H$ can be described as $\Box\triangle^{-1}\mathcal{T}_{\mathcal{J}^{\text{fake}}(\widetilde{T})}\mathcal{L}^{\text{fake}}$. $\blacksquare$

We note that

$$\Box_\zeta \triangle^{-1} \sim \prod_{\text{Chern roots } x \text{ of } T_X} \prod_{r=1}^{\infty} \frac{1-q^r e^{-x}}{1-\zeta^{-r} q^{r/m} e^{-x}}.$$

One obtain ∇_ζ by replacing in this formula $q^{1/m}$ with $q\zeta$ and computing the Euler–Maclaurin asymptotics of the result as $q\zeta \to 1$.

According to Proposition 2, the expansion $\mathcal{J}(t)_\zeta$ near $q = \zeta^{-1}$ of the true K-theoretic J-function is expressed in terms of correlators of stem theory (as they appeared in Section 7), computed at the input T equal to the leg contribution. More precisely, $\mathcal{J}(t)_\zeta$ is obtained from $\delta \mathcal{J}^{st}_{X/\mathbb{Z}_m}$, defined as

$$\delta t(q^{1/m}) + \sum_{a,n,d} \Phi_a \frac{Q^d}{n!} \left[\frac{\Phi^a}{1-q^{1/m}L^{1/m}}, T(L), \dots, T(L), \delta t(L^{1/m}) \right]_{0,n+2}^{X,d},$$

by changing $q^{1/m} \to q\zeta$ (including such change in δt) and $Q^d \to Q^{md}$ (excluding such a change in δt).

Combining these facts with Corollary, we conclude that $\nabla_\zeta^{-1} \mathcal{J}(t)_\zeta$, after the change $q \mapsto q/\zeta$, falls into the subspace $\mathcal{T}$ of $\mathcal{K}^{\text{fake}}$ which is obtained from the tangent space $\mathcal{T}_{\mathcal{J}(0)_1} \mathcal{L}^{\text{fake}}$ by the changes $q^{1/m} \mapsto q$ and $Q^d \mapsto Q^{md}$.

This completes the proof of the Hirzebruch–Riemann–Roch Theorem in true quantum K-theory.

9. Floer's S^1-equivariant K-theory, and $\mathcal{D}_q$-modules

In this section, we show that tangent spaces to the overruled Lagrangian cone $\mathcal{L}$ of quantum K-theory on X carry a natural structure of modules over a certain algebra $\mathcal{D}_q$ of finite-difference operators with respect to Novikov's variables. This structure, although manifest in some examples (see [14]) and predictable on heuristic grounds of S^1-equivariant Floer theory [8, 9], has been missing so far in the realm of K-theoretic GW-invariants. We first recall the heuristics, and then derive the $\mathcal{D}_q$-invariance of the tangent spaces to $\mathcal{L}$ from the divisor equation in quantum cohomology theory and our HRR Theorem in quantum K-theory.

Let X be a compact symplectic (or Kähler) target space, which for simplicity is assumed simply-connected in this preliminary discussion, so that $\pi_2(X) = H_2(X)$. Let $k = \operatorname{rk} H_2(X)$, let $d = (d_1, \dots, d_k)$ be integer coordinates on $H_2(X, \mathbb{Q})$, and $\omega_1, \dots, \omega_k$ be closed 2-forms on X with integer periods, representing the corresponding basis of $H^2(X, \mathbb{R})$.

On the space L_0X of contractible parametrized loops $S^1 \to X$, as well as on its universal cover $\widetilde{L_0X}$, one defines closed 2-forms Ω_a, that to two vector fields ξ and η along a given loop associates the value

$$\Omega_a(\xi,\eta) := \oint \omega_a(\xi(t),\eta(t))\ dt.$$

A point $\gamma \in \widetilde{L_0X}$ is a loop in X together with a homotopy type of a disk $u : D^2 \to X$ attached to it. One defines the *action functionals* $H_a : \widetilde{L_0X} \to \mathbb{R}$ by evaluating the 2-forms ω_a on such disks:

$$H_a(\gamma) := \int_{D^2} u^*\omega_a.$$

Consider the action of S^1 on $\widetilde{L_0X}$, defined by the rotation of loops, and let V denote the velocity vector field of this action. It is well-known that V is Ω_a-hamiltonian with the Hamilton function H_a, i.e.:

$$i_V\Omega_a + dH_a = 0, \quad a = 1,\dots,k.$$

Denote by z the generator of the coefficient ring $H^*(BS^1)$ of S^1-equivariant cohomology theory. The S^1-equivariant de Rham complex (of $\widetilde{L_0X}$ in our case) consists of S^1-invariant differential forms with coefficients in $\mathbb{R}[z]$, and is equipped with the differential $D := d + zi_V$. Then

$$p_a := \Omega_a + zH_a, \qquad a = 1,\dots,k,$$

are degree-2 S^1-equivariantly closed elements of the complex: $Dp_a = 0$. This is a standard fact that usually accompanies the formula of Duistermaat–Heckman.

Furthermore, the lattice $\pi_2(X)$ acts by deck transformations on the universal covering $\widetilde{L_0X} \to L_0X$. Namely, an element $d \in \pi_2(X)$ acts on $\gamma \in \widetilde{L_0X}$ by replacing the homotopy type $[u]$ of the disk with $[u]+d$. We denote by $Q^d = Q_1^{d_1}\cdots Q_k^{d_k}$ the operation of pulling-back differential forms by this deck transformation. It is an observation from [8, 9] that the operations Q_a and the operations of exterior multiplication by p_a do not commute:

$$p_aQ_b - Q_bp_a = -zQ_a\delta_{ab}.$$

These are commutation relations between generators of the algebra of differential operators on the k-dimensional torus:

$$[-z\partial_{\tau_a}, e^{\tau_b}] = -ze^{\tau_a}\delta_{ab}.$$

Likewise, if P_a denotes the S^1-equivariant line bundle on $\widetilde{L_0X}$ whose Chern character is e^{-p_a}, then tensoring vector bundles by P_a and pulling back vector bundles by Q_a do not commute:

$$P_a Q_b = q Q_a P_b \delta_{ab}.$$

These are commutation relations in the algebra of finite-difference operators, generated by multiplications and translations:

$$Q_a \mapsto e^{\tau_a}, \quad P_a \mapsto e^{z\partial_{\tau_a}} = q^{\partial_{\tau_a}}, \quad \text{where} \quad q = e^z.$$

Thinking of these operations acting on S^1-equivariant Floer theory of the loop space, one arrives at the conclusion that S^1-equivariant Floer cohomology (K-theory) should carry the structure of a module over the algebra of differential (respectively finite-difference) operators. Here is how this heuristic prediction materializes in GW-theory.

`Proposition 10.` *Let $\mathcal{D}$ denote the algebra of differential operators generated by $p_a, a = 1, \dots, k$, and Q^d, with d lying in the Mori cone of X. Define a representation of $\mathcal{D}$ on the symplectic loop space $\mathcal{H} = H^*(X, \mathbb{C}[[Q]]) \otimes \mathbb{C}((z))$ using the operators $p_a - zQ_a\partial_{Q_a}$ (where p_a acts by multiplication in the classical cohomology algebra of X) and Q^d (acting by multiplication in the Novikov ring). Then tangent spaces to the overruled Lagrangian cone $\mathcal{L}^H \subset \mathcal{H}$ of cohomological GW-theory on X are $\mathcal{D}$-invariant.*

`Proof.` Invariance with respect to multiplication by Q^d is tautological since the Novikov ring $\mathbb{Q}[[Q]]$ (which contains the semigroup algebra of the Mori cone: we assume that $d_a = \int_d p_a \geq 0$ for all a and all d in the Mori cone) is considered as the ground ring of scalars. To prove invariance with respect to operators $p_a - zQ_a\partial_{Q_a}$, recall from [13] that tangent spaces to $\mathcal{L}^H$ have the form $S_\tau^{-1}\mathcal{H}_+$, where $H \ni \tau \mapsto S_\tau(z)$ is a matrix power series in $1/z$ whose matrix entries are the following cohomological GW-invariants:

$$S_a^b = \delta_a^b + \sum_{l,d} \frac{Q^d}{l!} \sum_\mu \langle \phi_a, \tau, \dots, \tau, \frac{\phi^b}{z - \psi} \rangle_{0,n+2}^{X,d}.$$

The matrix S_τ lies in the *twisted loop group*, i.e. $S_\tau^{-1}(z) = S_\tau^*(-z)$ (where by "$*$" we denote transposition with respect to the Poincaré pairing). Let ∂_{τ_a} denote the differentiation in τ in the direction of the degree 2 cohomology class p_a. According to the *divisor equation*,

$$zQ_a\partial_{Q_a} S_\tau(z) + S_\tau(z) p_a = z\partial_{\tau_a} S_\tau(z).$$

In fact $z\partial_{\tau_a} S = p_a \bullet S$, where $\bullet$ stands for quantum cup-product. (This follows from the property of $\mathcal{L}^H$ to be overruled.) Transposing, we get:

$$(p_a - zQ_a\partial_{Q_a}) S_\tau^{-1}(z) = -z\partial_{\tau_a} S_\tau^{-1}(z) = S_\tau^{-1}(z)(p_a \bullet).$$

Also, if $\tau = \sum_\mu \tau_\mu \phi_\mu \in H$, then for any μ

$$z\partial_{\tau_\mu} S_\tau(z) = (\phi_\mu \bullet) S_\tau(z), \quad \text{and hence} \quad -z\partial_{\tau_\mu} S_\tau^{-1}(z) = S_\tau^{-1}(\phi_\mu \bullet).$$

Thus, if $\tau = \sum \tau_\mu(Q)\phi_\mu$ and $h \in \mathcal{H}_+$, so that $f(z,Q) = S_\tau^{-1}(z)h(z,Q)$ lies in $\mathcal{T}_\tau$, then

$$(p_a - zQ_a\partial_{Q_a})f = S_\tau^{-1}(z)\left[(p_a\bullet) - zQ_a\partial_{Q_a} - z\sum_\mu (\phi_\mu\bullet)Q_a\partial_{Q_a}\tau_\mu\right] h.$$

Since $\mathcal{H}_+$ is invariant under the operator in brackets, the result follows. ■

`Remarks.` (1) Each ruling space $z\mathcal{T}_\tau$, and therefore the whole cone $\mathcal{L}^H$, is $\mathcal{D}$-invariant, too.

(2) Symbols of differential operators annihilating all columns of S provide relations between operators $p_a\bullet$ in the quantum cohomology algebra of X (see [11]).

`Corollary 1.` *Tangent and ruling spaces of $\mathcal{L}^{fake}$ are $\mathcal{D}$-invariant.*

`Proof.` In the QHRR formula $\mathrm{ch}(\mathcal{L}^{\mathrm{fake}}) = \triangle\mathcal{L}^H$ of Section 5, the operator $\triangle$ commutes with $\mathcal{D}$, since it does not involve Novikov's variables, and since the operators (which do occur in $\triangle$) of multiplication in the classical cohomology algebra of X commute with p_a. ■

`Lemma.` *The subspace $\mathcal{T} \subset \mathcal{K}^{\mathrm{fake}}$ obtained from $\mathcal{T}_{\mathcal{J}(0)_1}\mathcal{L}^{\mathrm{fake}}$ by the change $z \mapsto mz, Q \mapsto Q^m$, is $\mathcal{D}$-invariant.*

`Proof.` The tangent space in question is $\triangle(z)S^{-1}_{\tau^{(0)}(Q)}(z,Q)\mathcal{H}_+$ for some $\tau^{(0)} = \sum_\mu \tau_\mu^{(0)}\phi_\mu \in H$. (Recall that $\mathcal{H}_+ = H[[z]]$, and $H = H^*(X, \mathbb{C}[[Q]])$.) The space $\mathcal{T}$ is therefore $\triangle(mz)S^{-1}_{\tau^{(0)}(Q^m)}(mz, Q^m)\mathcal{H}_+$, where $\mathcal{H}_+$ is $\mathcal{D}$-invariant, and $\triangle$ commutes with $\mathcal{D}$. Since $zQ_a\partial_{Q_a} = mzQ_a^m\partial_{Q_a^m}$, we find that the divisor equation still holds in the form:

$$(p_a - zQ_a\partial_{Q_a})S_\tau^{-1}(mz, Q^m) = S_\tau^{-1}(mz, Q^m)(p_a\bullet_{(\tau, Q^m)}),$$

where the last subscript indicates that the matrix elements of $p_a\bullet$ depend on τ and Q^m. The result now follows as in Proposition 10. ■

`Corollary 2.` *Let ζ be a primitive mth root of unity. Then the factor $\mathcal{L}^\zeta = \nabla_\zeta \mathcal{T}^\zeta$ of the adelic cone $\widehat{\mathcal{L}}$ is $\mathcal{D}$-invariant.*

`Proof.` Recall that the space $\mathcal{T}^\zeta$ is related to $\mathcal{T}$ by the change $q = \zeta e^z$, and the action of z in the operator $p_a - zQ_a\partial_{Q_a}$ should be understood in the sense of this identification. The result follows from Lemma since ∇_ζ commutes with $\mathcal{D}$ (like $\triangle$, in Corollary 1). ■

`Theorem.` *Let $\mathcal{D}_q$ denote the algebra of finite-difference operators, generated by integer powers of $P_a, a = 1, \dots, k$, and Q^d, with d lying in the Mori cone of X. Define a representation of $\mathcal{D}_q$ on the symplectic loop space $\mathcal{K}$, using the operators $P_a q^{Q_a \partial_{Q_a}}$ (where P_a acts by multiplication in $K^0(X)$ by the line bundle with the Chern character e^{-p_a}) together with the operators of multiplication by Q^d in the Novikov ring. Then tangent (and ruling) spaces to the overruled Lagrangian cone $\mathcal{L} \subset \mathcal{K}$ of true quantum K-theory on X are $\mathcal{D}_q$-invariant.*

`Proof.` Thanks to the adelic characterization of the cone $\mathcal{L}$ and its ruling spaces, given by Theorem of Section 6 and its Corollary, this is an immediate consequence of the following Lemma.

`Lemma.` *The adelic cone $\widehat{\mathcal{L}}$ is $\mathcal{D}_q$-invariant.*

`Proof.` It is obvious that the factors $\mathcal{L}^\zeta$ are $\mathcal{D}$-invariant for ζ other than roots of unity, since in this case $\mathcal{L}^\zeta = \mathcal{K}_+^{\text{fake}}$. For $\zeta = 1$, it follows from Corollary 1 that the family of operators $e^{\epsilon(zQ_a\partial_{Q_a} - p_a)}$ preserves $\mathcal{L}^{\text{fake}}$, and so does the operator with $\epsilon = 1$, which coincides with $P_a q^{Q_a \partial_{Q_a}}$. When $\zeta \neq 1$ is a primitive mth root of unity, the family of operators $e^{\epsilon(zQ_a\partial_{Q_a} - p_a)}$ preserves $\mathcal{L}^\zeta$ by Corollary 2. However, at $\epsilon = 1$, the operator of the family differs from $P_a q^{Q_a \partial_{Q_a}}$ (because $q = \zeta e^z$) by the factor $\zeta^{Q_a \partial_{Q_a}}$, which acts as $Q_a \mapsto Q_a \zeta$. It is essential that this extra factor commutes with $S^{-1}_{\tau^{(0)}(Q^m)}(mz, Q^m)$ (due to $\zeta^m = 1$). Since it also preserves $\mathcal{H}_+$, the result follows. ■

`Example.` It is known[9] [14] that for $X = \mathbb{C}P^{n-1}$,

$$\mathcal{J}(0) = (1-q) \sum_{d=0}^{\infty} \frac{Q^d}{(1-Pq)^n \cdots (1-Pq^d)^n},$$

where $P \in K^0(\mathbb{C}^{n-1})$ represents the Hopf line bundle. It follows (from the string equation) that $(\mathcal{J}(0)/(1-q)$ lies in the tangent space $\mathcal{T}_{\mathcal{J}(0)}\mathcal{L}$. Applying powers T^r of the translation operator $T := Pq^{Q\partial_Q}$, we conclude that, for all integer r, the same tangent space contains

$$P^r \sum_{d=0}^{\infty} \frac{Q^d q^{rd}}{(1-Pq)^n \cdots (1-Pq^d)^n}.$$

In fact, $\mathcal{J}(0)$ satisfies the nth order finite-difference equation $D^n \mathcal{J}(0) = Q\mathcal{J}(0)$, where $D := 1 - T$. Therefore the $\mathcal{D}_q$-module generated by $\mathcal{J}(0)/(1-q)$ is spanned over the Novikov ring by $T^r \mathcal{J}(0)/(1-q)$ with

[9] This result is derived from birational isomorphisms between some genus-0 moduli spaces of stable maps to $\mathbb{C}P^{n-1} \times \mathbb{C}P^1$ and toric compactifications of spaces of maps $\mathbb{C}P^1 \to \mathbb{C}P^{n-1}$.

$r = 0, \dots, n-1$. The projections of these elements to $\mathcal{K}_+$ are $P^r, r = 0, \dots, n-1$, which span the ring $K^0(\mathbb{C}P^{n-1}) = \mathbb{Z}[P, P^{-1}]/(1-P)^n$. The K-theoretic Poincaré pairing on this ring is given by the residue formula:

$$(\Phi(P), \Phi'(P)) = -\operatorname{Res}_{P=1} \frac{\Phi(P)\Phi'(P)}{(1-P)^n}\frac{dP}{P}.$$

By computing the pairings with the above series we actually evaluate K-theoretic GW-invariants:

$$(\Phi(P), T^r \mathcal{J}(0)/(1-q)) = \sum_d Q^d \langle \frac{\Phi(P)}{1-qL}, P^r \rangle_{0,2}^{X,d}, \quad r = 0, \dots, n-1.$$

Thus, we started with known values of all $\langle \Phi L^k, 1\rangle_{0,2}^{X,d}$ and computed all $\langle \Phi L^k, \Phi' \rangle_{0,2}^{X,d}$ (and hence, by virtue of general properties of genus-0 GW-invariants, all $\langle \Phi L^k, \Phi' L^l \rangle_{0,2}^{X,d}$) using the $\mathcal{D}_q$-module structure alone.

10. Quantum K-theory of projective complete intersections

`Theorem.` *Let X be a complete intersection in the projective space $\mathbb{C}P^{n-1}$, $n > 4$, given by $k(\geq 0)$ equations of degrees $l_1, \dots, l_k > 1$, such that $l_1^2 + \cdots + l_k^2 \leq n$. Then the following series represents a point in the overruled Lagrangian cone of true quantum K-theory of X:*

$$I_X := (1-q) \sum_{d \geq 0} \frac{\prod_{j=1}^{k} \prod_{r=0}^{l_j d} (1 - P^{l_j} q^r)}{\prod_{r=1}^{d} (1 - Pq^r)^n} Q^d.$$

More precisely, $I_X = \nu_* \mathcal{J}_X(0)$, where $\nu_* : K^0(X) \to K^0(\mathbb{C}P^{n-1})$ is the K-theoretic push-forward induced by the embedding $\nu : X \to \mathbb{C}P^{n-1}$, and $\mathcal{J}_X(0)$ is the value of the J-function of true quantum K-theory of X at the input $t = 0$.

`Remarks.` (1) To clarify this formulation, we remind that P represents the Hopf line bundle in $K^0(\mathbb{C}P^{n-1})$. By Lefschetz' hyperplane section theorem, the inclusion $X \subset \mathbb{C}P^{n-1}$ induces an isomorphism $H_2(X, \mathbb{Q}) \to H_2(\mathbb{C}P^{n-1}, \mathbb{Q})$, whenever $2 \leq n - k - 2$. The latter holds true under our numerical restrictions on l_j and n. Consequently, the degrees of holomorphic curves in X are represented in I_X by their degrees d in the ambient projective space.

(2) When $\sum l_j^2 \leq n$, we also have $\sum l_j < n-2$ (strictly, unless $k = 1$, $l_1 = 2$, while $n = 4$). Since we assumed $n > 4$, we have for each $d > 0$:

$$1 + \sum \frac{l_j d(l_j d + 1)}{2} < n \frac{d(d+1)}{2}.$$

This means that the coefficient of I_X at Q^d is a reduced rational function of q. Thus, the projection of I_X to $\mathcal{K}_+$ is $1-q$, i.e. I_X corresponds to the input value $t = 0$.

(3) Note that the example $n = 4, k = 1, l_1 = 2$ of the conic $\mathbb{C}P^1 \times \mathbb{C}P^1 \subset \mathbb{C}P^3$ is exceptional in the sense of both previous remarks. It would be interesting to analyze the role of the series I_X in quantum K-theory of the conic.

`Corollary.` *For all* $s \in \mathbb{Z}$

$$\sum_d Q^d \langle \frac{\nu^*\Phi(P)}{1-qL}, \nu^* P^s \rangle_{0,2}^{X,d} = (\Phi(P), T^s I_X/(1-q)),$$

where $(\cdot,\cdot)$ *is the K-theoretic Poincaré pairing on* $K^0(\mathbb{C}P^{n-1})$*, and*

$$T^s I_X/(1-q) = P^s \sum_{d\geq 0} \frac{\prod_{j=1}^k \prod_{r=0}^{l_j d}(1-P^{l_j}q^r)}{\prod_{r=1}^d (1-Pq^r)^n} Q^d q^{sd}.$$

When $k = 0$, it is known from [14], that the formula for I_X represents the value $\mathcal{J}(0)$ of the K-theoretic J-function of the projective space. We will begin our proof of the theorem, however, with re-deriving this fact (and without the restriction $n > 4$, of course) from the main theorem of this paper. After that we explain how to adjust the argument to the case of projective complete intersections.

To prove the theorem for $X = \mathbb{C}P^{n-1}$, we will show that expansions of the series I near $q = \zeta^{-1}$ pass the tests required by the Quantum HRR Theorem of Section 6.

The technique we use goes back to the method developed in [6] for the proof of the "Quantum Lefschetz Principle." Let us first outline the method in its generalized form introduced in [5].

Suppose we are given a point (e.g. $\mathcal{J}_X^H(0)$) on an overruled Lagrangian cone (such as $\mathcal{L}_X^H$, for instance). Consider a pseudo-differential operator in the Novikov's variables in the form

$$\exp\left\{ \frac{\Phi_{-1}(zQ\partial_Q)}{z} + \sum_{k\geq 0} \Phi_k(zQ\partial_Q) z^k \right\}.$$

Here $zQ_i\partial_{Q_i}$ is supposed to act (as in the previous section) by $-p_i + zQ_i\partial/\partial Q_i$, where p_i is the degree 2 class corresponding to $Q_i\partial_{Q_i}$. It follows from Lemma in [6] (in the proof of the quantum Lefschetz theorem) that by *applying the operator to a point on the overruled Lagrangian cone one obtains a point also lying on the cone.* More precisely, as we already know from the previous section, ruling spaces to the cone are $\mathcal{D}$-modules with respect to Novikov's variables. Therefore the terms of the exponent with $k \geq 0$ are only capable of adding to a point on the cone a vector from the same ruling space. However, the action of the term Φ_{-1}, generally speaking, moves the point to another ruling space. For example, the action $\exp\{z^{-1}\sum_i z\tau_i Q_i\partial_{Q_i}\}$ changes a point $J = \sum J_d Q^d$ to $J(\tau) = e^{-p\tau/z}\sum J_d e^{d\tau} Q^d$, which lies on the same cone due to the divisor equation. Furthermore, the action of $\exp\{\Phi_{-1}(zQ\partial_Q)/z\}$ is equivalent to that of $\exp\{\Phi_{-1}(z\partial/\partial\tau)/z\}$, which in its turn, modulo relations in the $\mathcal{D}$-module generated by the J-function, and modulo higher order terms in z, is equavalent to the translation in the space H of parameters of ruling spaces (by the vector, expressible under some simplifying assumptions as $\Phi_{-1}(p\bullet)$, i.e. the value of the function Φ_{-1} computed in the quantum cohomology algebra).

Next, given a point J on an overruled Lagrangian cone $\mathcal{L}$, one constructs a point on the rotated cone $e^{-\Phi(-p,z)/z}\mathcal{L}$, where $e^{\Phi(-p,z)/z}$ is the Euler-Maclaurin asymptotics of an infinite product $\prod_{r=-\infty}^{0} S(-p+rz)$ as follows. We have $S(zQ\partial_Q + rz)Q^d = S(-p+p(d)z+rz)$, where $p(d)$ denotes the value $\int_d p$ on d of the degree 2 class p. Therefore

$$e^{\Phi(zQ\partial_Q,z)/z}Q^d = e^{\Phi(-p,z)/z}Q^d\frac{\prod_{r=-\infty}^{0} S(-p+rz)}{\prod_{r=-\infty}^{p(d)} S(-p+rz)}.$$

The fraction on the right is known as the *modifying factor M_d*. Rewriting $J = \sum_d J_d Q^d$, we conclude that since $e^{\Phi(zQ\partial_Q,z)/z}J$ lies in the cone $\mathcal{L}$, *the modified series $\sum_d J_d Q^d M_d$ lies on the rotataed cone $e^{-\Phi(-p,z)/z}\mathcal{L}$.*

Returning now to our problem for $X = \mathbb{C}P^{n-1}$), we befing with a point on the cone $\mathcal{L}_X^H$ (see [9]; we will tend to omit the subscript X in this example):

$$\mathcal{J}^H(0) = -z\sum_{d\geq 0}\frac{Q^d}{\prod_{r=1}^{d}(p-rz)^n}.$$

Here p is the hyperplane class in $H^2(\mathbb{C}P^{n-1}$. We employ the above method to construct a point on $\mathcal{L}^{fake}$. Recall from Section 5 that $\mathcal{L}^{\text{fake}} = \text{ch}^{-1}\triangle\mathcal{L}^H$, where

$$\log\triangle \sim \sum_{r=1}^{\infty}\sum_x s(x-rz), \quad s(u) := \log\frac{u}{1-e^{-u}},$$

and x runs Chern roots of $T_{\mathbb{C}P^{n-1}}$. We claim that in fact

$$\log\triangle \sim \sum_{r=1}^{\infty}\left(n\ s(p-rz) - s(-rz)\right).$$

Indeed, $T_{\mathbb{C}P^{n-1}} = nP^{-1} - 1$, and the construction of the operator $\log \triangle$ from (Chern roots of) a bundle is additive. Note that the last summand does not affect the way $\triangle$ acts on $\mathcal{L}^H$, since, being overruled, the cone $\mathcal{L}^H$ is invariant under multiplication by functions of z.

As the method requires, replacing p in $\log \triangle$ with $-zQ\partial_Q$ and applying the resulting operator to Q^d, we find the modifying factor

$$M_d = \prod_{r=1}^{d} e^{n\ s(p-rz)} = \prod_{r=1}^{d} \frac{(p-rz)^n}{(1-e^{-p+rz})^n}.$$

Thus

$$I^{\text{fake}} := -z \sum_{d\geq 0} \frac{Q^d}{\prod_{r=1}^{d}(1-e^{-p+rz})^n}$$

lies in $\mathcal{L}^{\text{fake}}$, the overruled Lagrangian cone of fake quantum K-theory of $\mathbb{C}P^{n-1}$. Multiplying this by $(1-e^z)/(-z)$ (which is a scalar z-series and thus preserves the overruled cone), and replacing $e^{-p} = \operatorname{ch}(P)$ with P, and e^z with q, we obtain the same expression as I_X from Theorem for $X = \mathbb{C}P^{n-1}$:

$$I = (1-q) \sum_{d\geq 0} \frac{Q^d}{(1-Pq)^n(1-Pq^2)^n \cdots (1-Pq^d)^n}.$$

We have proved therefore, that *the expansion of* I *near* $q=1$ *lies in* $\mathcal{L}^{\text{fake}}$ as required.

To analyze the expansion of I near $q = \zeta^{-1}$ where ζ is an m-th root of 1, we begin again with the point $\mathcal{J}^H(0)$ in $\mathcal{L}^H$ and generate a point I^{tw} on the cone $\mathcal{L}^{tw}_{X/\mathbb{Z}_m}$ and a tangent vector δI^{tw} to this cone at this point, applying the above method to the twisting operators $\square$ and $\square_\zeta$ from Proposition 7. Again, the description of the tangent bundle $T\mathbb{C}P^{n-1} = nP^{-1} - 1$ allows us to replace Chern roots of in the definition of $\square$ and $\square_\zeta$ with n copies of p:

$$\square \sim \prod_{r=1}^{\infty} \frac{(p-rz)^n}{(1-e^{-mp+mrz})^n}, \quad \square_\zeta \sim \prod_{r=1}^{\infty} \frac{(p-rz)^n}{(1-\zeta^{-r}e^{-p+rz/m})^n}.$$

Replacing p with $-zQ\partial_Q$ and applying the operators to Q^d we find the modifying factors (alternatively one can read them off the formulation in [5] of the orbifold Qauntum Lefschetz Theorem specialized to the case of $X/\mathbb{Z}_m$ and the sectors g^0 and g^{-1}) and respectively

$$I^{tw} = -z \sum_{d\geq 0} \frac{Q^d}{\prod_{r=1}^{d}(1-e^{-mp+mrz})^n},$$

$$\delta I^{tw} = \sum_{d\geq 0} \frac{Q^d}{\prod_{r=1}^{md}(1-\zeta^{-r}e^{-p+rz/m})^n}.$$

Multiplying I^{tw} with $(1-e^{mz})/(-z)$ (which leaves it on the cone $\mathcal{L}^{tw}_{X/\mathbb{Z}^m}$) and replacing e^{-p} with P, e^z with q, and Q^d with Q^{md}, we obtain

$$(1-q^m)\sum_{d\geq 0} \frac{Q^{md}}{(1-P^mq^m)^n(1-P^mq^{2m})^n\cdots(1-P^mq^{md})^n},$$

which is exactly $\Psi^m(I)$. This provides the compatibility check: the series I considered as a point in the loop space $\mathcal{K}$ of true K-theory, projects to $\mathcal{K}_+$ to $1-q$, i.e. corresponds to the input $T=0$. Thus the application point I^{tw} of the tangent vector δI^{tw} would pass the test for zero input. What is left is to check that the expansion of I at $q=\zeta^{-1}$ lies in the tangent space to the cone $\mathcal{L}^{tw}$ at this point.

To this end, we perform in δI^{tw} the appropriate change of notation: $e^{z/m}=\zeta q$, $e^{-p}=P$, $Q\mapsto Q^d$, and obtain

$$\widetilde{I}_\zeta := \sum_{d\geq 0} \frac{Q^{md}}{(1-Pq)^n(1-Pq^2)^n\cdots(1-Pq^{md})^n}.$$

This should be understood as a Laurent series expansion near $q=\zeta^{-1}$ and compared with such expansion I_ζ for

$$I = \sum_{d\geq 0} \frac{Q^{d}}{(1-Pq)^n(1-Pq^2)^n\cdots(1-Pq^{d})^n}.$$

We see that Q^d-terms with d multiple to m agree, but all other terms present in I_ζ are missing in $\widetilde{I}_\zeta$. Nevertheless we deduce from this that I_ζ lies in the same tangent space to $\mathcal{L}^{tw}$ as $\widetilde{I}_\zeta$ (i.e. in $\nabla_\zeta\mathcal{T}$). namely, introduce the operator

$$D := \sum_{\delta=0}^{m-1} Q^\delta \frac{1}{\prod_{r=1}^{\delta}(1-q^{Q\partial_Q}q^r)^n}.$$

It should be understood as an expansion near $q=\zeta^{-1}$, and it is important that within the given range $0<r\leq\delta<m$ of the indices δ and r the denominators have no zeroes at $q=\zeta^{-1}$, and thus D is a power series in $zQ\partial_Q$ ($z=\log q$). Since tangent spaces to $\mathcal{L}^{tw}$ are D-modules in Novikov's variables, we conclude that $D\widetilde{I}_\zeta$ lies in the same tangent space as $\widetilde{I}_\zeta$. It remains to note that $D\widetilde{I}_\zeta$ coincides with I_ζ.

What we have established about the series I means that the decomposition of it into elementary fractions obeys the recursion relations of Section 7, with the leg contribution obtained by Adams' operation Ψ^m from the arm contribution, corresponding to the input point $t(q) = [I]_+ - (1-q)$. Since the projection $[\dots]_+$ of I to $\mathcal{K}_+$ is $1-q$, we find that $t=0$ as required, and hence $I = \mathcal{J}(0)$. $\square$

`Remark.` With the exception of the last property $[I]_+ = 1-q$, this seemingly sophisticated argument is in fact general enough to work for q-hypergeometric series I_X that can be associated to any symplectic toric manifold X as follows. Let X be obtained by symplectic reduction $X = \mathbb{C}^n//T^k$ by the action of the subtorus $T^k \subset T^n$ of the maximal torus, the embedding being determined (in some basis of $\pi_1(T^k)$) by the integer matrix (m_{ij}), $i = 1, \dots, k$, $j = 1, \dots, n$ (see [8, 12] for more details). Let $Q^d = Q_1^{d_1} \cdots Q_k^{d_k}$ represent a point in the Mori cone of X in coordinates $(d_1, \dots, d_k)$ on $H_2(X)$ corresponding to the chosen basis of $\pi_1(T^k)$, and P_i^{-1}, $i = 1, \dots, k$, denote the line bundles over X whose 1st Chern classes form the dual basis of $H^2(X)$. In this notation:

$$I_X = \sum_d Q^d \prod_{j=1}^n \frac{\prod_{r=-\infty}^0 (1 - q^r \prod_{i=1}^k P_i^{m_{ij}})}{\prod_{r=-\infty}^{\sum_i d_i m_{ij}} (1 - q^r \prod_{i=1}^k P_i^{m_{ij}})}.$$

The property $[I_X]_+ = 1-q$, however, does not hold unless X is a product of complex projective spaces. It would be interesting to find out if nevertheless $I_X \in \mathcal{L}_X$.

The above computation will also work for the series I_X corresponding to projective complete intersection described in the theorem. However, there is a catch here, related to the fact that cohomology and K-theory of X may not be entirely describable in terms of the ambient projective space, and thus the information gained about I_X won't yet allow to make a legitimate application of our Quantum HRR Theorem. More specifically, our computation would only be concerned with the properties of $\nu_*(I_X)$ expressed in terms of $\nu_*(I^{\text{fake}})$, and the latter may not even lie on $\mathcal{L}^{\text{fake}}$.

In order to bypass the difficulty, we introduce a model of quantum K-theory of a *supermanifold* ΠE, interpolating between those of X and $\mathbb{C}P^{n-1}$. Let E be the total space of the sum of the line bundles over $\mathbb{C}P^{n-1}$ of degrees $l_1, \dots, l_k$, while Π indicates the fiberwise parity change. By definition, genus-0 moduli spaces of stable maps to ΠE are the same as to $\mathbb{C}P^{n-1}$, but the virtual structure sheaf is changed, by tensoring the structure sheaf $\mathcal{O}_{0,r,d}^{vir}$ with *the S^1-equivariant K-theoretic Euler class of the bundle $E_{0,r,d}$ (i.e. the Koszul complex of the dual, $E_{0,r,d}^*$).* Here $E_{0,r,d}$ stands for the bundle $\pi_* \operatorname{ev}^* E$ whose fiber over a stable map $f : \Sigma \to \mathbb{C}P^{n-1}$ is $H^0(\Sigma, f^*E)$. The circle S^1 is made to act by multiplication by unitary scalars fiberwise on E, and hence on $E_{0,r,d}$. Respectively, correlators of quantum K-theory of ΠE take values in the representation ring $\mathbb{C}[S^1] = \mathbb{C}[\Lambda, \Lambda^{-1}]$. Their algebraic-geometrical meaning (instead of holomorphic Euler characteristics of a sheaf) is the trace of S^1 on the sheaf cohomology. The ring $K^0(\Pi E)$ coincides with $K^0(\mathbb{C}P^{n-1})\otimes\mathbb{C}[S^1]$, and is equipped with the K-theoretic Poincaré pairing

$$(\Phi, \Phi')_{\Pi E} = -\operatorname{Res}_{P=1} \Phi(P)\Phi'(P)\frac{\prod_{j=1}^{k}(1-P^{l_j}\Lambda)}{(1-P)^n}\frac{dP}{P}.$$

This pairing becomes non-degenerate if division by $1-\Lambda$ is allowed. After this localization, the resulting quantum K-theory of the supermanifold ΠE satisfies all the axioms of genus-0 quantum K-theory.

Furthermore, the Quantum HRR Theorem of Section 6 and its proof given in Sections 7 and 8 work verbatim for true quantum K-theory of ΠE.[10]

Thus, applying the same technology as in the case of $X = \mathbb{C}P^{n-1}$, we establish that *under the numerical assumptions of Theorem, we have* $\mathcal{J}_{\Pi E}(0) = I_{\Pi E}$*, where*

$$I_{\Pi E} := (1-q)\sum_{d\geq 0} Q^d \frac{\prod_{j=1}^{k}\prod_{r=1}^{l_j d}(1-P^{l_j}\Lambda q^r)}{\prod_{r=1}^{d}(1-Pq^r)^n}.$$

Here are a few formulas that elucidate this claim:

$$\mathcal{J}^H_{\Pi E}(0) = -z\sum_{d\geq 0} Q^d \frac{\prod_{j=1}^{k}\prod_{r=1}^{l_j d}(\lambda + l_j p - rz)}{\prod_{r=1}^{d}(p-rz)^n},$$

where λ is the 1st Chern class of the universal S^1-bundle Λ^{-1} (i.e. $\operatorname{ch}(\Lambda) = e^{-\lambda}$);

$$\widetilde{I}_\zeta := \sum_{d\geq 0} Q^{md} \frac{\prod_{j=1}^{k}\prod_{r=1}^{ml_j d}(1-\Lambda P^{l_j} q^r)}{\prod_{r=1}^{md}(1-Pq^r)^n};$$

$$D := \sum_{\delta=0}^{m-1} Q^\delta \frac{\prod_{j=1}^{k}\prod_{r=1}^{l_j\delta}(1-\Lambda q^{l_j Q\partial_Q} q^r)}{\prod_{r=1}^{\delta}(1-q^{Q\partial_Q}q^r)^n}.$$

Once the equality $\mathcal{J}_{\Pi E}(0) = I_{\Pi E}$ is proved, to establish the equality $\nu_*\mathcal{J}_X(0) = I_X$, it remains to notice that for all $s\in\mathbb{Z}$

$$(\nu^* P^s, \mathcal{J}_X(0))_X = (P^s, \mathcal{J}_{\Pi E}(0))_{\Pi E}\,|_{\Lambda=1}.$$

[10] Note that we are not using any geometric fixed point localization with respect to S^1, so that all moduli spaces, Kawasaki strata, etc. remain the same, and only the meaning and values of the correlators are modified appropriately.

Indeed, when X is given in $\mathbb{C}P^{n-1}$ by a section of E, the moduli space $X_{0,r,d}$ is given in $(\mathbb{C}P^{n-1})_{0,r,d}$ by the corresponding section of the bundle $E_{0,r,d}$, and (according to [21, 22]) the virtual structure sheaf of $X_{0,r,d}$ is described in $K^0((\mathbb{C}P^{n-1})_{0,r,d})$ by tensoring the virtual structure sheaf of $(\mathbb{C}P^{n-1})_{0,r,d}$ with the K-theoretic Euler class of $E_{0,r,d}$, albeit, the non-equivariant one, and hence the specialization to $\Lambda = 1$.

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