

Stationary Phase Integrals, Quantum Toda Lattices, Flag Manifolds and the Mirror Conjecture

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July 8, 1996, Revised December 1, 1996

0. Introduction. Consider the differential operator

$$H = \frac{\hbar^2}{2} \sum_{i=0}^n \frac{\partial^2}{\partial t_i^2} - \sum_{i=1}^n e^{t_i - t_{i-1}} .$$

This operator is a quantization of the Hamiltonian of the Toda lattice on $n + 1$ identical particles with configuration coordinates t_0, \dots, t_n and with the exponential interaction potential $\exp(t_i - t_{i-1})$ of neighbors. The Toda lattice is known to be integrable on both classical and quantum levels: there exist commuting differential polynomials $D_m(\hbar\partial/\partial t, \exp t, \hbar)$, $m = 0, \dots, n$, which play the role of quantum conservation laws (i.e. $[H, D_0] = \dots = [H, D_n] = 0$) and whose symbols $D_m(p, \exp t, 0)$ form a complete set of Poisson-commuting first integrals of the classical Toda lattice. In this paper we study solutions $S(t)$ of the differential system $D_0 S = \dots = D_n S = 0$ whose characteristic Lagrangian variety L is the most degenerate invariant Lagrangian variety of the Toda lattice. According to [12, 13] this Lagrangian variety is the spectrum of the quantum cohomology algebra of the manifold of complete flags $0 \subset \mathbb{C}^1 \subset \dots \subset \mathbb{C}^{n+1}$. We represent solutions S by stationary phase integrals in $n(n+1)/2$ complex variables and point out the role these solutions play in the quantum cohomology theory. As we explain in the last section, our results prove the mirror conjecture [9] in the case of the flag manifolds.

*Research supported by NSF grant DMS-93-21915 and by Alfred P. Sloan Fellowship.

1. The Toda \mathcal{D} -module.

Denote D_0, \dots, D_n coefficients of the polynomial

$$\lambda^{n+1} + D_0\lambda^n + \dots + D_n =$$

$$= \det \begin{bmatrix} \lambda + p_0 & q_1 & 0 & 0 & \dots \\ -1 & \lambda + p_1 & q_2 & 0 & \dots \\ 0 & -1 & \lambda + p_3 & q_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & -1 & \lambda + p_n \end{bmatrix},$$

$D_0 = \sum p_i$, $D_1 = \sum_{i>j} p_i p_j + \sum q_i, \dots$. The polynomials $D_m(p, q)$ of p_0, \dots, p_n and $^1 q_1 = \exp(t_1 - t_0), \dots, q_n = \exp(t_n - t_{n-1})$ form a complete set of Poisson-commuting first integrals of the Toda lattice (see for instance [12]). Their quantizations

$$D_m(\hbar\partial/\partial t_0, \dots, \hbar\partial/\partial t_n, \exp(t_1 - t_0), \dots, \exp(t_n - t_{n-1})), \quad m = 0, \dots, n,$$

are defined unambiguously since any monomial in D_m containing q_i contains neither p_i nor p_{i-1} .

Theorem 1. $[H, D_0] = \dots = [H, D_n] = 0$.

Proof. The commutator $[H, \Delta]$ of the Hamiltonian operator H with the above determinant $\Delta = \lambda^{n+1} + D_0\lambda^n + \dots$ vanishes modulo \hbar^2 since symbols of D_m Poisson-commute with the symbol $\sum p_i^2/2 - \sum q_i$ of H . Also $[H, \Delta]$ does not contain any terms of order higher than \hbar^2 since for any 3 distinct i, j, k we have $\partial^3 q_l / \partial t_i \partial t_j \partial t_k = 0$. Computing symbols of the terms proportional to \hbar^2 we find that the contribution of Δq_i is equal to $-q_i \partial^2 \Delta(p, q) / \partial p_i \partial p_{i-1}$, and the contribution of $\hbar^2 \partial^2 \Delta / 2 \partial t_i^2$ is $q_i \partial^2 \Delta / 2 \partial p_i \partial p_{i-1} + q_{i+1} \partial^2 \Delta / 2 \partial p_i \partial p_{i+1}$. After summation over i all these contributions cancel out. \square

We will study solutions of the PDE system

$$D_0 S = \dots = D_n S = 0 \quad (\text{and hence } HS = 0),$$

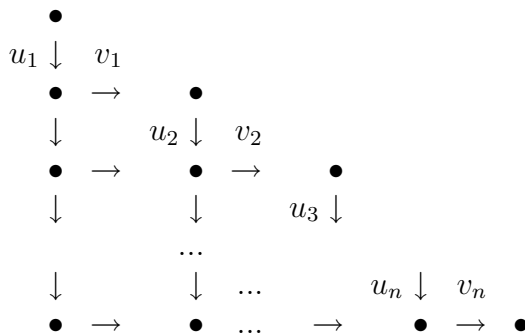
i.e. the solution sheaf of the left module $\mathcal{D}/\mathcal{D}(D_0, \dots, D_n)$ over the algebra \mathcal{D} of differential operators $D(\hbar\partial/\partial t, \exp t, \hbar)$ with Fourier-polynomial coefficients. We introduce the *symbol* $D(p, \exp t, 0)$ of the differential polynomial

¹Throughout Sections 1 – 3 we will systematically use the notation $\log q_i$ for $t_i - t_{i-1}$.

D and call the Lagrangian variety $L \subset T^*(\mathbb{C} - 0)^{n+1}$ given by the equations $D_0(p, q) = \dots = D_n(p, q) = 0$ the *characteristic* Lagrangian variety of this \mathcal{D} -module. According to [17] the Lagrangian variety L is nonsingular. The operator D_m is weighted-homogeneous of degree $m + 1$ with respect to the grading $\deg \hbar = 1$, $\deg q_i = 2$, $\deg t_i = 0$ and therefore L is also weighted-homogeneous with weights $\deg p_i = 1$, $\deg q_i = 2$. For generic t the fibers $L \cap T_{\exp t}^*$ of the projection $L \rightarrow (\mathbb{C} - 0)^{n+1}$ consist of $(n + 1)!$ distinct simple points. This follows from Sard's lemma, and also can be deduced by induction from the continued fraction formula for the determinant Δ which coincides with the numerator of the following rational function of λ :

$$\lambda + p_0 + \frac{q_1}{\dots} \cdot \frac{1}{\lambda + p_1 + \frac{q_n}{\dots + \frac{q_n}{\lambda + p_n}}} .$$

2. The stationary phase integrals. Consider the following “2 - dimensional Toda lattice” with $(n + 1)(n + 2)/2$ vertices and $n(n + 1)$ edges:



For each edge α of the lattice we introduce a factor Q_α and introduce another, excessive notation u_i (respectively v_i) for the factors Q_α corresponding to the vertical (respectively horizontal) edges next to the diagonal boundary of the lattice as shown on the diagram. We denote Y the affine algebraic variety in the $n(n + 1)$ -dimensional complex space with coordinates Q_α given by the $n(n - 1)/2$ equations $Q_\alpha Q_\beta = Q_\gamma Q_\delta$ making the diagram “commutative”:

one equation for each 1×1 cell

$$\begin{array}{ccc}
 & Q_\alpha & \\
 \bullet & \rightarrow & \bullet \\
 Q_\gamma \downarrow & & \downarrow Q_\beta \\
 \bullet & \rightarrow & \bullet \\
 & Q_\delta &
 \end{array} .$$

We put $q_1 = u_1 v_1, \dots, q_n = u_n v_n$ and denote Y_q the $n(n+1)$ -dimensional fibers of the map from Y to \mathbb{C}^n defined by these formulas.

If $q_i \neq 0$ for all $i = 1, \dots, n$, the relations between Q_α allow to express them via the set $\{T_\nu, \nu = (i, j), 0 \leq j \leq i \leq n\}$ of vertex variables:

$$Q_\alpha = \exp(T_{\nu_+(\alpha)} - T_{\nu_-(\alpha)}),$$

where $\nu_-(\alpha)$ and $\nu_+(\alpha)$ are respectively the indices of the source and target vertices of the edge α . In particular, $q_i = \exp(t_i - t_{i-1})$ where $t_0 = T_{00}, t_1 = T_{11}, \dots$, and the fiber Y_q is isomorphic to the $n(n+1)/2$ -dimensional complex torus with coordinates $\{\exp T_{ij}, 0 \leq j < i \leq n\}$.

On Y_q with $q \in (\mathbb{C} - 0)^n$ we introduce the holomorphic volume form

$$\omega_q = \wedge_{i=1}^n \wedge_{j=0}^{i-1} dT_{ij} ,$$

the holomorphic function \mathcal{F}_q obtained by restriction to Y_q of “the total Toda potential energy”

$$\mathcal{F} = \sum_{\text{edges } \alpha} Q_\alpha$$

and the *stationary phase integral*

$$S_\Gamma(t) = \int_{\Gamma \subset Y_q} e^{\mathcal{F}_q/\hbar} \omega_q .$$

In this definition, Γ represents a (possibly non-compact) cycle in Y_q of middle dimension such that the integral converges. For $\hbar > 0$ and generic q one can construct such a cycle by picking a non-degenerate critical point of \mathcal{F}_q and taking the union of descending gradient trajectories of the function $\text{Re } \mathcal{F}_q$ with respect to a suitable Riemannian metric on Y_q on the role of Γ . The stationary phase integral depends only on the homology class of the cycle in

the appropriate homology group \mathcal{H}_q which can be described as the inverse limit as $M \rightarrow \infty$ of the relative homology groups

$$H^{\dim c Y_q}(Y_q, \{\operatorname{Re} \mathcal{F}_q \leq -M\}).$$

The rank of the group \mathcal{H}_q is equal to the number of critical points of \mathcal{F}_q for generic q since all critical points of the real part of a holomorphic Morse function have the same Morse index. The Gauss-Manin parallel transport of cycles identifies the groups \mathcal{H}_q for close q but may give rise to a nontrivial global monodromy. The notation $S_\Gamma(t)$ emphasizes the multiple-valued character of the stationary phase integrals which therefore depend on the coordinates $\log q_i = t_i - t_{i-1}$ on the universal covering of the parameter space. The dependence of the integral on \hbar is suppressed in this notation.

Theorem 2. *The stationary phase integrals $S_\Gamma(t)$ satisfy the differential equations $D_0 S = D_1 S = \dots = D_n S = 0$.*

Proof. Application of the differential operator $\Delta = \lambda^{n+1} + D_0 \lambda^n + \dots$ to the stationary phase integral produces the amplitude factor $e^{-\mathcal{F}/\hbar} \Delta e^{\mathcal{F}/\hbar} = \det(\lambda + A)$ where

$$A = \begin{bmatrix} -u_1 & u_1 v_1 & 0 & \dots & & \\ -1 & v_1 - u_2 & u_2 v_2 & 0 & \dots & \\ 0 & -1 & v_2 - u_3 & u_3 v_3 & \dots & \\ & \cdot & \cdot & \cdot & & \\ & \dots & 0 & -1 & v_n & \end{bmatrix}.$$

We should show that this amplitude is congruent to λ^{n+1} modulo linear combinations of Lie derivatives along vector fields $Q^m \partial / \partial T_\nu$ tangent to Y_q , i.e. linear combinations of $\hbar \partial Q^m / \partial T_{ij} + Q^m \partial \mathcal{F} / \partial T_{ij}$ with $j < i$. Notice that $\partial \mathcal{F} / \partial T_\nu$ is the sum of Q_α over the (≤ 2) edges α ingoing the vertex ν minus such a sum over the outgoing edges.

We begin induction on n by noticing that for $n = 1$ the amplitude is equal to $\lambda^2 - \lambda \partial \mathcal{F} / \partial T_{21}$ and apply the induction hypothesis to the triangular lattice with the principle diagonal $i = j$ cut off.

Consider the differential operator Δ' defined as the determinant of the 3-diagonal $n \times n$ -matrix with $\lambda + \hbar \partial / \partial T_{i,i-1}$, $i = 1, \dots, n$, on the principal diagonal, -1 's under the diagonal and $v_i u_{i+1}$, $i = 1, \dots, n-1$, above the diagonal. Denote \mathcal{F}' the sum of all Q_α except u_i 's and v_i 's. By the induction hypothesis (and "commutativity" $Q_\alpha Q_\beta = Q_\gamma Q_\delta$ of the squares next to the diagonal

$i = j$) we may assume that the amplitude factor $\exp(-\mathcal{F}'/\hbar)\Delta'\exp(\mathcal{F}'/\hbar)$ is congruent to λ^n modulo Lie derivatives along $Q^m\partial/\partial T_{ij}$ with $|i - j| > 1$.

The vector field $Q^m\partial/\partial T_{i,i-1}$, with no edges adjacent to the vertex $\nu = (i, i-1)$ present in the monomial Q^m , produces the amplitude $Q^m\partial\mathcal{F}/\partial T_{i,i-1}$. Since the vertices $(i, i-1)$ do not have common edges, addition of such amplitudes allows to transform the amplitude factor $\exp(-\mathcal{F}'_q/\hbar)\Delta'\exp(\mathcal{F}'_q/\hbar)$ purely algebraically as if $\partial\mathcal{F}/\partial T_{i,i-1} = 0$. Using such transformations we can replace $\partial\mathcal{F}'/\partial T_{i,i-1}$ by $v_i - u_i$, and the induction hypothesis can be reformulated as the congruence to λ^{n+1} of the $(n+1)$ -determinant $\det(\lambda + B)$ where

$$B = \begin{bmatrix} v_1 - u_1 & v_1 u_2 & 0 & \dots & \dots & 0 \\ -1 & v_2 - u_2 & v_2 u_3 & 0 & \dots & 0 \\ 0 & -1 & v_3 - u_3 & v_3 u_4 & \dots & 0 \\ & \cdot & \cdot & \cdot & & 0 \\ & \dots & 0 & -1 & v_n - u_n & 0 \\ & & \dots & 0 & -1 & 0 \end{bmatrix}.$$

The matrices A and B admit the factorizations $A = UV$, $B = VU$ into the product of the following triangular matrices:

$$U = \begin{bmatrix} u_1 & 0 & \dots & & & \\ 1 & u_2 & 0 & \dots & & \\ 0 & 1 & u_3 & 0 & \dots & \\ & \cdot & \cdot & \cdot & & \\ & \dots & 0 & 1 & u_n & 0 \\ & & & 0 & 1 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} -1 & v_1 & 0 & \dots & & \\ 0 & -1 & v_2 & 0 & \dots & \\ 0 & 0 & -1 & v_3 & 0 & \dots \\ & \cdot & \cdot & \cdot & & \\ & & \dots & 0 & -1 & v_n \\ & & & \dots & 0 & -1 \end{bmatrix}.$$

Thus A is similar to B and has the same characteristic polynomial. \square

The family \mathcal{F}_q generates the following Lagrangian variety parametrized by critical points of the functions (and responsible for stationary phase approximations to the integrals $S_\Gamma(t)$):

$$\{(p, q) \mid \exists y \in Y_q : d_y \mathcal{F}_q = 0, p = \partial \mathcal{F}_q(y)/\partial t\}.$$

The identity $\det(\lambda + A) = \det(\lambda + B)$ also proves by induction the following

Corollary 1. *The Lagrangian variety generated by the family \mathcal{F}_q coincides with the invariant Lagrangian variety $L = \{(p, q) \mid D_0(p, q) = \dots = D_n(p, q) = 0\}$ of the Toda lattice.*

Notice that the equations $\partial\mathcal{F}/\partial T_\nu = 0$ of the critical points and the notations $p_i = \partial\mathcal{F}/\partial t_i$ can be interpreted, in the spirit of the elementary theory of linear electric circuits, as homological boundary conditions for the 1-chain $\sum Q_\alpha[\alpha]$ on the oriented graph. One can therefore describe the critical points by the relations $Q_\alpha = J_{\phi_+(\alpha)} - J_{\phi_-(\alpha)}$ introducing the 2-chain $\sum J_\phi[\phi]$, a linear combination of the $n(n+1)/2$ clockwise oriented 1×1 -cells of the lattice (the edge $\pm\alpha$ occurs in the boundary of the cells $\pm\phi_\pm(\alpha)$, the index $\phi = (i, j)$ runs $1 \leq j \leq i \leq n$). Additionally, $p_0 = -J_{11}, p_1 = J_{11} - J_{22}, p_2 = J_{22} - J_{33}, \dots, p_n = J_{nn}$. Now cancellations of all J_{ij} with $i > j$ in the total sum $\sum Q_\alpha$ prove

Corollary 2. *The generating function $\mathcal{F}_q(y_{crit})$ on the Lagrangian variety L equals $\sum_{i=1}^n 2J_{i,i} = -np_0 + (2-n)p_1 + (4-n)p_2 + \dots + np_n$.*

Remark. Corollary 2 can be also deduced (see [12]) from the weighted homogeneity of L .

3. Quantum cohomology of flag manifolds. The cohomology algebra $H^*(F)$ of the flag manifold $F = \{0 \subset \mathbb{C}^1 \subset \mathbb{C}^2 \subset \dots \subset \mathbb{C}^{n+1}\}$ is multiplicatively generated by the 1-st Chern classes p_i of the tautological line bundles with fibers $\mathbb{C}^{i+1}/\mathbb{C}^i$. A complete set of relations between the generators can be written in the form $(\lambda + p_0)(\lambda + p_1)\dots(\lambda + p_n) = \lambda^{n+1}$ equating the total Chern class of the sum of the tautological line bundles to that of the trivial bundle with the fiber \mathbb{C}^{n+1} .

On the Poincare-dual language of intersection indices $\langle \cdot, \cdot \rangle$ the structural constants $\langle ab, c \rangle$ of the cohomology algebra count (with signs) isolated common intersection points of the three cycles a, b, c in general position.

The quantum cohomology algebra of the flag manifold is defined as a deformation of the algebra $H^*(F)$ with structural constants $\langle a \circ b, c \rangle$ counting isolated holomorphic spheres $(\mathbb{C}P^1, 0, 1, \infty) \rightarrow (F, a, b, c)$ passing by the three marked points through the three cycles. A precise definition can be based on Gromov's compactness theorem and Kontsevich's concept [14] of stable holomorphic maps.

Let Σ denote a compact connected complex curve with at most double singular points. It is rational (i.e. $H^1(\Sigma, \mathcal{O}) = 0$) if and only if all irreducible components of Σ are spheres and the incidence graph of these components is a connected tree. Denote $x = (x_1, \dots, x_k)$ an ordered set of pairwise distinct non-singular marked points on Σ . Two holomorphic maps $(\Sigma, x) \rightarrow F, (\Sigma', x') \rightarrow F$ are called *equivalent* if they are identified by a

holomorphic isomorphism $(\Sigma, x) \rightarrow (\Sigma', x')$. A holomorphic map $(\Sigma, x) \rightarrow F$ is called *stable* if it does not admit non-trivial infinitesimal automorphisms. For rational Σ stability means that each irreducible component mapped to a point in F carries at least 3 marked or singular points.

The *degree* d of the map $\Sigma \rightarrow F$ is defined as the total 2-nd homology class of F it represents. According to [14, 3], equivalence classes of stable degree- d holomorphic map of rational curves with k marked points to the flag manifold F form a compact complex non-singular orbifold which we denote $F_{k,d}$. If non-empty, it has the dimension $\dim F + \langle -K_F, d \rangle + k - 3$ where $-K_F$ is an anti-canonical divisor of F .

Evaluation of maps $(\Sigma, x) \rightarrow F$ at x_i defines the *evaluation maps* $e_i : F_{k,d} \rightarrow F$, $i = 1, \dots, k$.

Examples. 1) The moduli space $F_{3,0}$ consists of classes of *constant* maps $(\mathbb{C}P^1, 0, 1, \infty) \rightarrow F$ and thus is isomorphic to F . The moduli spaces $F_{k,0}$ with $k < 3$ are not defined since constant rational maps with less than 3 marked points are unstable.

2) Forgetting the space \mathbb{C}^i in a flag defines the projection $F \rightarrow F_i$ to the partial flag manifold F_i with fibers isomorphic to $\mathbb{C}P^1$. Denote $a_i \in H_2(F)$ the homology class of the fiber. In fact any compact holomorphic curve in F of the degree a_i is one of the fibers. This identifies the moduli space F_{0,a_i} with the base F_i .

3) The evaluation map $e_1 : F_{1,a_i} \rightarrow F$ is an isomorphism.

4) The Borel-Weil representation theory for $SL_{n+1}(\mathbb{C})$ identifies the root lattice of type A_n with the Picard lattice $H^2(F)$ of the flag manifold. In $H_2(F)$ the classes a_1, \dots, a_n form the basis of simple coroots dual to the basis $J_1, \dots, J_n \in H^2(F)$ of *fundamental weights* $J_i = -p_0 + \dots - p_i$, i.e. Chern classes of the tautological bundles $\wedge^i(\mathbb{C}^i)^*$. The simplicial cone spanned in $H^2(F, \mathbb{R})$ by the fundamental weights is the Kähler cone of the flag manifold. This implies that the degree d of any compact holomorphic curve in F is a sum $d_1 a_1 + \dots + d_n a_n$ with all $d_i \geq 0$. The sum $\sum_{i>j} p_i - p_j = 2(J_1 + \dots + J_n)$ of positive roots represents the anti-canonical class of the flag manifold. Thus $\dim F_{3,d} = \dim F + 2(d_1 + \dots + d_n)$.

Let us introduce the grading in the algebra $\mathbb{C}[\Lambda] = \mathbb{C}[q_1, \dots, q_n]$ of the semi-group $\Lambda = \{d = \sum d_i a_i \in H_2(F) | d_i \geq 0\}$ by putting $\deg q^d = 4(d_1 + \dots + d_n)$. We extend $\mathbb{C}[\Lambda]$ -bilinearly the Poincare pairing $\langle A, B \rangle = \int_F A \wedge B$ in the De Rham cohomology of F to the graded $\mathbb{C}[\Lambda]$ -module $H^*(F, \mathbb{C}[\Lambda])$. Structural

constants of the $\mathbb{C}[\Lambda]$ -bilinear quantum multiplication \circ on $H^*(F, \mathbb{C}[L])$ are defined by

$$\forall A, B, C \in H^*(F) \quad \langle A \circ B, C \rangle = \sum_{d \in \Lambda} q^d \int_{F_{3,d}} e_1^*(A) \wedge e_2^*(B) \wedge e_3^*(C) .$$

One can show (see e. g. [3, 8]) that on $H^*(F, \mathbb{C}[\Lambda])$ the quantum multiplication defines the structure of a commutative associative graded Frobenius algebra with unity $1 \in H^*(F)$ (i.e. $\langle A \circ 1, C \rangle = \langle A, C \rangle$). Modulo (q_1, \dots, q_n) this structure coincides with the cup-product on $H^*(F)$. Recent developments in symplectic topology and theory of stable maps allow to extend the construction of the quantum cohomology algebra to arbitrary compact symplectic manifolds.

The spectrum of a quantum cohomology algebra can be naturally identified with certain Lagrangian variety — the characteristic Lagrangian variety of the *quantum cohomology \mathcal{D} -module*.

For $k > 0$ denote c the 1-st Chern class of the tautological line bundle over the moduli space $F_{k,d}$ with the fiber at the point $[(\Sigma, x) \rightarrow F]$ equal to the cotangent line $T_{x_1}^* \Sigma$ to the curve at the first marked point. Denote $pt = \sum_{i=0}^n p_i t_i = \sum (J_i - J_{i+1}) t_i$ the general 2-nd cohomology class of F and $dt = \sum_{i=1}^n d_i (t_i - t_{i-1})$ the value of this cohomology class on the homology class $d = \sum d_i a_i$. For each $A \in H^*(F)$ the vector-function $s_A(t)$ of t with values in $H^*(F, \mathbb{C})$ is defined by

$$\forall B \in H^*(F) \quad \langle s_A(t), B \rangle = \langle e^{pt/\hbar} A, B \rangle + \sum_{d \in \Lambda - 0} e^{dt} \int_{F_{2,d}} \frac{e_1^*(e^{pt/\hbar} A)}{\hbar - c} \wedge e_2^*(B) .$$

By the definition, s_A is a formal power series of $q_i = \exp(t_i - t_{i-1})$ with vector-coefficients which are polynomial in $\log q_i$ and \hbar^{-1} .

The vector series s_A satisfy the following linear differential equations with periodical coefficients (see e. g. [4, 5, 8]):

$$\hbar \frac{\partial}{\partial t_i} s_A = p_i \circ s_A, \quad i = 0, \dots, n,$$

where $p_i \circ$ are operators of quantum multiplication by p_i . In particular, the equations are compatible for any value of \hbar (i.e. $p_i \circ p_j = p_j \circ p_i$ and $\partial(p_i \circ) / \partial t_j = \partial(p_j \circ) / \partial t_i$), and the linear space of all solutions to this system

coincides with the space of all vector-functions s_A (of dimension $\text{rk } H^*(F)$ over, say, $\mathbb{C}((\hbar^{-1}))$).

By definition, the quantum cohomology \mathcal{D} -module is generated by the scalar functions $S_A(t) := \langle s_A(t), 1 \rangle$, i.e. coincides with \mathcal{D}/I where $I = \{D \in \mathcal{D} \mid DS_A = 0 \forall A \in H^*(F)\}$. It is easy to show (see for instance [8]) that if a homogeneous differential polynomial $D(\hbar\partial/\partial t, \exp t, \hbar)$ annihilates all the functions S_A then the relation $D(p, q, 0) = 0$ holds in the quantum cohomology algebra.

Examples. 5) By the very definition $\sum_i \partial S_A / \partial t_i = 0$ since $\sum p_i = 0$ in $H^*(F)$.

6) The relation $p_0^2 + \dots + p_n^2 = 0$ holds in the cohomology algebra $H^*(F)$ of the flag manifold. We claim that in the quantum cohomology algebra $p_0^{\circ 2} + \dots + p_n^{\circ 2} = 2q_1 + \dots + 2q_n$. Indeed, for the degree reasons $J_i \circ J_j$ must be equal to $J_i J_j$ plus a linear combination $\sum \lambda_m q_m$. The coefficient λ_m is equal to the number of holomorphic maps $\mathbb{C}P^1 \rightarrow F$ passing by $0, 1$ and ∞ through the generic divisors J_i, J_j and a given generic point in F . From Examples 2 and 4 we find therefore that $J_i \circ J_j = J_i J_j + \delta_{ij} q_i$ and thus $\sum p_i^{\circ 2} / 2 = \sum J_i^{\circ 2} - \sum J_i \circ J_{i-1} = \sum q_i$.

Theorem 3 ([13]). *The Hamiltonian operator $H = (\sum \hbar^2 \partial^2 / \partial t_i^2) / 2 - \sum \exp(t_i - t_{i-1})$ of the quantum Toda lattice annihilates the functions $S_A, A \in H^*(F)$.*

Proof. Application of the operator H to S_A yields

$$\langle H(p^\circ, q) s_A, 1 \rangle + \hbar \sum_i \left\langle \frac{\partial(p_i \circ)}{\partial t_i} s_A, 1 \right\rangle .$$

The first term vanishes due to Example 6. Since $\langle p_i \circ B, 1 \rangle = \langle B, p_i \circ 1 \rangle = \langle B, p_i \rangle$ is constant for any $B \in H^*(F)$, we have $\langle (\partial p_i \circ / \partial t_i) s_A, 1 \rangle = \langle s_A, (\partial p_i \circ / \partial t_i) 1 \rangle = 0$. \square

Corollary. $S_A = \langle e^{pt/\hbar} s, A \rangle$ where the coefficients $s^{(d)} \in H^*(F, \mathbb{C}(\hbar))$ of the vector-function $s = \sum_{d \in \Lambda} s^{(d)} q^d$ can be found recursively from

$$s^{(0)} = 1, \quad \hbar[\hbar(d, d) + \sum_{i=1}^n d_i J_i] s^{(d)} = \sum_{i: d_i > 0} s^{(d-a_i)} .$$

In particular, the formal series s converges everywhere.

Remark. Analogous computations for the flag manifold G/B of a semi-simple complex Lie group G of rank n give rise to the Hamiltonian $H = \sum (a_i, a_j) J_i J_j / 2 - \sum (a_i, a_i) q_i / 2$ of the Toda lattice corresponding to the system of simple coroots a_1, \dots, a_n Langlands - dual to the root system of G . The proof of the theorem below is a specialization to the case $G = SL_{n+1}(\mathbb{C})$ of the general results by B.Kim describing quantum cohomology \mathcal{D} -modules of flag manifolds G/B in terms of quantized Toda lattices.

Theorem 4. $D_0 S_A = D_1 S_A = \dots = D_n S_A = 0$ for all $A \in H^*(F)$.

Proof (see [13]). Since $HD_m S_A = D_m H S_A = 0$, the formal power series $S = \sum S^{(d)} q^d := D_m S_A$ with coefficients polynomial in $\log q$ (and \hbar^{-1}) satisfies the hypotheses of the following

Kim's Lemma. *If $S^{(0)} = 0$ and $HS = 0$ then $S = 0$.*

Proof (see [13]) uses only ellipticity of the operator H and polynomiality of its coefficients. Consider a non-zero term $S^{(d)} q^d$ of minimal degree and pick in it a non-zero monomial term $const \cdot (\log q)^m q^d$ of maximal degree $|m|$. Then this term occurs in HS with the coefficient $(d, d) \cdot const$ which is also non-zero for $d \neq 0$ since the symmetric form $(,)$ is positively definite. \square

Corollaries. (1) *The quantum cohomology algebra of the flag manifold F is canonically isomorphic to*

$$\mathbb{C}[p_0, \dots, p_n, q_1, \dots, q_n] / (D_0(p, q), \dots, D_n(p, q)) .$$

(2) *The total multiplicity of critical points of the functions \mathcal{F}_q in the generating family of the characteristic Lagrangian variety L is equal to $\text{rk } H^*(F) = (n+1)!$*

(3) $\text{rk } \mathcal{H}_q = \text{rk } H^*(F)$ if all $q_i \neq 0$.

(4) *The monodromy representation $\mathbb{Z}^{n+1} = \pi_1(\mathbb{C} - 0)^{n+1} \rightarrow \text{Aut}(\mathcal{H}_q)$ of the Gauss-Manin connection on \mathcal{H}_q is unipotent and equivalent to the action of \mathbb{Z}^{n+1} on $H^*(F, \mathbb{C})$ generated by the multiplication operators $A \mapsto \exp(2\pi i p_k)$, $k = 0, \dots, n$.*

(5) *For each $A \in H^*(F)$ the function S_A has the stationary phase representation $\int_{\Gamma \subset Y_q} \exp(F_q/\hbar) \omega_q$ with suitable $\Gamma = \Gamma(A) \in \mathcal{H}_q \otimes \mathbb{C}((\hbar^{-1}))$.*

(6) *Vice versa, the stationary phase integrals S_Γ admit the series expansions $\langle e^{pt/\hbar}, A(\Gamma) \rangle$.*

Remark. As it is shown in [12], the Poincare pairing \langle , \rangle on the quantum cohomology algebra $\mathbb{C}[p, q] / (D_0(p, q), \dots, D_n(p, q))$ can be described by the

residue formula

$$\langle A, B \rangle(q) = \frac{1}{(2\pi i)^{n+1}} \oint \frac{A(p, q)B(p, q)dp_0 \wedge \dots \wedge dp_n}{D_0(p, q)\dots D_n(p, q)}.$$

Comparison of the stationary phase approximations for the integrals S_Γ with the formal asymptotics

$$S_A(t) \sim \hbar^{\dim F/2} \sum_{p \in L \cap T_{\text{expt}}^*} C_p(A) \frac{e^{\sum 2J_i(p)/\hbar}}{\det^{1/2}(\partial D_i/\partial p_j)|_{(p, \text{expt})}}$$

for solutions of the quantum cohomology differential equations shows that the Jacobian $\det(\partial D_i/\partial p_j)$ computed at generic points $(p, q) \in L$ coincides, up to a constant factor, with the Hessian $\det(\partial^2 \mathcal{F}_q(T)/\partial T_\nu \partial T_\mu)$ of the function \mathcal{F}_q computed at the non-degenerate critical points corresponding to (p, q) . It would be interesting to find a direct proof of this identity between the two determinants of sizes $n + 1$ and $n(n + 1)/2$ respectively.

4. The mirror conjecture. By the mirror conjecture one usually means the profound equivalence (see e. g. [20, 15, 18]), discovered several years ago on the basis of string theory, between complex and symplectic geometry in Calabi – Yau manifolds (i.e. compact Kähler manifolds which admit non-vanishing holomorphic volume forms). In particular, the conjecture predicts that the quantum cohomology \mathcal{D} -module corresponding to a Calabi – Yau manifold X describes variations of periods of the holomorphic volume form on another Calabi – Yau manifold Y , which has the same dimension as X and whose Hodge diamond is mirror-symmetric to that of X .

In 1993 we suggested a generalization of the above correspondence beyond the class of Calabi – Yau manifolds.² Namely, we conjectured (see [9]) that the quantum cohomology \mathcal{D} -module of a compact symplectic $2N$ -dimensional manifold X is equivalent to the \mathcal{D} -module generated by stationary phase integrals $\int_{\Gamma \in Y_q} \exp(\mathcal{F}_q/\hbar)\omega_q$, where $(Y_q, \mathcal{F}_q, \omega_q)$ is a suitable algebraic family of (possibly non-compact) N -dimensional complex manifolds Y_q , holomorphic functions $\mathcal{F}_q : Y_q \rightarrow \mathbb{C}$ and non-vanishing holomorphic N -forms on Y_q .

²We should mention however that (earlier) E. Witten’s paper [19] can be also understood as a suggestion of a similar generalization. I am thankful to M. Atiyah who communicated to me this point of view usually ignored by successors of [19].

The following arguments spoke in favor of such a generalization.

1) In terms of the characteristic Lagrangian variety $L \subset T^*(H_2(X)^\vee)$ of the quantum cohomology \mathcal{D} -module the Poincare pairing $\langle \cdot, \cdot \rangle$ on the quantum cohomology algebra $\mathbb{C}[L]$ of the compact symplectic manifold X is given by the formula

$$\langle A, B \rangle(q) = \sum_{p \in T_q^* \cap L} \frac{A(p)B(p)}{\Delta(p)}$$

(where Δ is the restriction to the diagonal in $L \times L$ of the function representing Poincare-dual class of the diagonal in $X \times X$). This formula resembles the residue pairing

$$\frac{1}{(2\pi i)^N} \oint a(y)b(y) \frac{dy_1 \wedge \dots \wedge dy_N}{\frac{\partial \mathcal{F}_q}{\partial y_1} \dots \frac{\partial \mathcal{F}_q}{\partial y_N}} = \sum_{y_*: d_y \mathcal{F}_q(y_*)=0} \frac{a(y_*)b(y_*)}{\det(\partial^2 \mathcal{F}_q / \partial y_i \partial y_j)|_{y_*}}$$

in singularity theory, and in some examples, including complex projective spaces and Grassmannians, can be indeed replaced by some residue formula.

2) Solutions s_A to the differential equations $\hbar q_i \partial s_A / \partial q_i = p_i \circ s_A$ arising from the multiplication \circ in the quantum cohomology algebra of X admit asymptotical approximations

$$\langle s_A, B \rangle \sim \hbar^{N/2} \sum_{p \in T_q^* \cap L} C_p(A) e^{-K_X(p)/\hbar} \left[\frac{B(p)}{\sqrt{\Delta(p)}} + \mathcal{O}(\hbar) \right]$$

resembling the stationary phase approximations

$$\int_{\Gamma \subset Y_q} e^{\mathcal{F}_q(y)/\hbar} b(y) d^N y \sim \hbar^{N/2} \frac{e^{\mathcal{F}_q(y_*)/\hbar} b(y_*)}{\det^{1/2}(\partial^2 \mathcal{F}_q / \partial y_i \partial y_j)|_{y_*}}.$$

3) A by-product of our proof [11] for toric symplectic manifolds of Arnold's symplectic fixed point conjecture [1] was the following multiplicative structure in the Floer homology (it is at least morally equivalent to the quantum multiplication). A compact toric symplectic manifold X with Picard number n can be obtained as a symplectic reduction $X = \mathbb{C}^N // T^n$ of the linear space by the subtorus $T^n \subset T^N$ of the maximal torus on a generic level of the momentum map $\mathbb{C}^N \rightarrow \text{Lie}^* T^N \rightarrow \text{Lie}^* T^n$. Denote $(m_{ij})_{i=1}^n \quad \begin{matrix} n \\ j=1 \end{matrix}^N$ the matrix of the

projection $\text{Lie } {}^*T^N \rightarrow \text{Lie } {}^*T^n$. In the quantum cohomology algebra of X the classes Q_1, \dots, Q_N of coordinate hyperplane divisors satisfy the multiplicative relations $Q_1^{m_{i1}} \dots Q_N^{m_{iN}} = q_i, i = 1, \dots, n$, and on the other hand — can be expressed via some basis in $H^2(X)$ as $Q_j = p_1 m_{1j} + \dots + p_n m_{nj}, j = 1, \dots, N$. It is easy to see that the latter set of additive relations specifies the critical set of the function $\mathcal{F} = Q_1 + \dots + Q_N$ restricted to the $(N - n)$ -dimensional complex torus Y_q defined by the multiplicative relations. We arrive to the mirror family $(Y_q, \mathcal{F}_q, \omega_q)$ of the toric manifold X by introducing the holomorphic volume form $\omega_q = d \log Q_1 \wedge \dots \wedge d \log Q_N / d \log q_1 \wedge \dots \wedge d \log q_n$ on the torus Y_q .

4) In the case of $X = \mathbb{C}P^{N-1} = \mathbb{C}^N // T^1$ corresponding stationary phase integrals

$$\int_{\Gamma \subset \{Q_1 \dots Q_N = q\}} e^{(Q_1 + \dots + Q_N)/\hbar} \frac{dQ_1 \wedge \dots \wedge dQ_N}{dq}$$

satisfy *the same* differential equation $(\hbar q d/dq)^N S = qS$ as the series

$$s = e^{(p \log q)/\hbar} \sum_{d=0}^{\infty} \frac{q^d}{(p + \hbar)^N (p + 2\hbar)^N \dots (p + d\hbar)^N} \pmod{p^N}$$

generating the quantum cohomology \mathcal{D} -module of the complex projective space (see [10]). This confirms the mirror conjecture for $\mathbb{C}P^{N-1}$.

5) The above mirror family $(Y_q, \mathcal{F}_q, \omega_q)$ of a toric symplectic manifold X agrees with Batyrev's mirrors of Calabi – Yau anti-canonical hypersurfaces $X' \subset X$. According to [2] one can construct Calabi – Yau mirror manifolds Y' as anti-canonical divisors in the toric manifold obtained by dualization of the momentum polyhedron for X . In fact periods of holomorphic volume forms on Batyrev's mirrors can be expressed in terms of our mirrors $(Y_q, \mathcal{F}_q, \omega_q)$ as the formal Laplace transform

$$\int_{\gamma \subset Y_q \cap \mathcal{F}_q^{-1}(1)} \frac{\omega_q}{d\mathcal{F}_q}$$

of the stationary phase integrals. ³

³By the way, this observation suggests a formulation of a “quantum Lefschetz hyperplane section theorem” relating the quantum cohomology \mathcal{D} -module of a Fano manifold X with that of the anti-canonical hypersurfaces $X' \subset X$ via such a Laplace transform, i.e. without any mentioning mirror manifolds.

In [8] we proved the mirror conjecture for Fano ($K_X < 0$) and Calabi – Yau ($K_X = 0$) projective complete intersections. Namely, the solutions S_A of the quantum cohomology differential equation for X given in $\mathbb{C}P^{N-1}$ by r equations of degrees $l_1 + \dots + l_r \leq N$ are described by the following integrals $\int_{\Gamma \subset Y_q} \exp(\mathcal{F}_q/\hbar) \omega_q$: partition the variables Q_1, \dots, Q_N into $r + 1$ groups of lengths $l_0 + l_1 + \dots + l_r = N$ and denote G_0, \dots, G_r the total sums $\sum Q_\alpha$ in each group (for example, $G_0 = Q_1 + \dots + Q_{l_0}$, etc.). Then

$$Y_q = \{(Q_1, \dots, Q_N) \mid Q_1 \dots Q_N = q, G_1(Q) = \dots = G_r(Q) = 1\},$$

$$\mathcal{F}_q = G_0|_{Y_q}, \quad \omega_q = \frac{d \log Q_1 \wedge \dots \wedge d \log Q_N}{d \log q \wedge dG_1 \wedge \dots \wedge dG_r}.$$

These formulas include the traditional Calabi – Yau mirror phenomenon as a degenerate particular case with $l_0 = 0$ and $G_0 = 0$.

The very idea that the mirror phenomenon exists beyond the class of Calabi – Yau manifolds has not attracted much attention of specialists. One of the causes is that the results mentioned in this section do not go further than toric complete intersections. It was the actual purpose of the project [12] on flag manifolds started in 1993 by the author and B. Kim to improve this situation and enlarge the supply of examples confirming the conjecture. Although the discovered relation with Toda lattices has been analyzed in the literature on quantum cohomology of flag manifolds (see e. g. [17, 7, 13]), their mirrors have escaped us so far.

In the recent paper [6] T. Eguchi, K. Hori and C.-S. Xiong, independently on our lecture [9], have arrived to a similar idea of extending the mirror conjecture to Fano manifolds. They illustrate the idea with the examples of complex projective spaces, several rational surfaces, Grassmannians $G_{4,2}$ and $G_{5,2}$ and extrapolate formulas from the latter examples to general Grassmannians. These formulas served us as the starting point for the present paper; suitably modified, they give rise to a construction of mirrors for the flag manifolds. Indeed, the construction of $(Y_q, \mathcal{F}_q, \omega_q)$ in Section 2 provides a stationary phase integral representation for the solutions S_A of the differential equations defined by the quantum cohomology algebra of the flag manifold.

I would like to thank M. Kontsevich who brought the paper [6] to my attention and B. Kim for communicating his results [13] based on quantum Toda lattices. I am also thankful to B. Kostant who explained to me that

generalizations of the present paper to arbitrary semi-simple Lie algebras should intertwine mirror manifolds with Whittaker modules [16].

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